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The Minkowski-Lorentz space for Computer Aided Design purposes

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Abstract

This document deals with the Computer Aided Geometric Design with a short presentation of the Minkowski-Lorentz space. This space generalizes to \mathbb{R}^5 the one used in the relativity theory. The Minkowski-Lorentz space offers a more intuitive writing of a sphere given by a point, a normal vector at the point and its curvature. It also eases the use of canal surfaces thus represented by curves. The quadratic computation in \mathbb{R}^3 becomes linear in that space. The use of spheres, canal surfaces and their particular case known as Dupin cyclides is illustrated in a schematic seahorse. The seahorse applies the G^1 connection in the Minkowski-Lorentz space.

Oriented spheres and Pencils

An oriented sphere S with centre Ω and radius $r > 0$ satisfies the relationship $\overrightarrow{\Omega M} = \rho \vec{N}$ with the rule $\rho = r$ (resp. $\rho = -r$) if the unit normal vector \vec{N} to the sphere at point M is getting outside (resp. inside). The power of the point M to the sphere S is defined by $\chi_S(M) = \Omega M^2 - \rho^2$. The set of points solution of $\lambda_1 \chi_{S_1}(M) + \lambda_2 \chi_{S_2}(M) = 0$ is called the spheres pencil defined by S_1 et S_2 .

There kinds of pencils exist, a circle based pencil (Fig. 1(a)), a tangent spheres pencil (Fig. 1(b)) a limited points pencil (Fig. 1(c)).

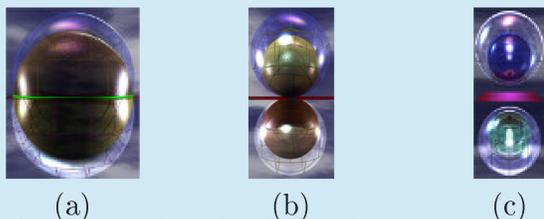


Figure 1: The 3 kinds of spheres pencils in \mathbb{R}^3

The Minkowski-Lorentz space

The quadratic form of Lorentz is defined on the basis $(\vec{e}_0; \vec{e}_1; \vec{e}_2; \vec{e}_3; \vec{e}_\infty)$ by $Q_{4,1}(x_0, x, y, z, x_\infty) = x^2 + y^2 + z^2 - 2x_0 x_\infty$. The light cone C_l satisfies the equation $x^2 + y^2 + z^2 - 2x_0 x_\infty = 0$ in the frame $(O_5; \vec{e}_0; \vec{e}_1; \vec{e}_2; \vec{e}_3; \vec{e}_\infty)$. The unit sphere Λ^4 with centre O_5 in \mathbb{R}^5 is given by :

$$\Lambda^4 = \left\{ \sigma \in \mathbb{R}^5 \mid Q_{4,1}(\overrightarrow{O_5 \sigma}) = \overrightarrow{O_5 \sigma}^2 = 1 \right\}$$

It represents the oriented spheres and planes of \mathbb{R}^3 . A sphere or a plane S is represented by a point σ of \mathbb{R}^5 .

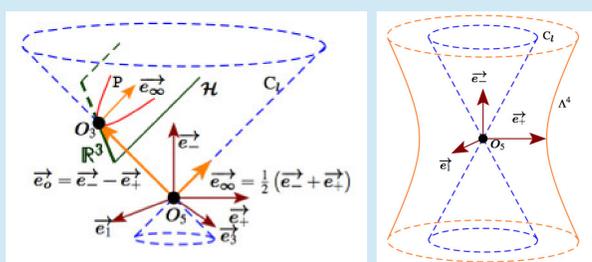


Figure 2: The Minkowski-Lorentz space

Linear pencils of spheres on Λ^4

On the unit sphere Λ^4 any pencil of sphere is represented by the intersection $\mathcal{C} = \Lambda^4 \cap \mathcal{P}$ of a plane called 2-plane \mathcal{P} passing through O_5 . \mathcal{C} is a unit circle seen differently depending on the type of plane.

- If \mathcal{P} is a space-like plane that is $\forall \vec{u} \in \vec{\mathcal{P}}, \vec{u}^2 > 0$ then \mathcal{C} is drawn as an ellipse (Fig.3.(a)). The set \mathcal{C} represents a based circle sphere pencil where all spheres get a common circle.
- If \mathcal{P} is a light-like plane that is $\forall \vec{u} \in \vec{\mathcal{P}}, \vec{u}^2 = 0$ and \mathcal{P} is parallel to a hyperplane tangent at C_l . (Fig.3.(b)) Then the set \mathcal{C} is drawn as two straight lines symmetric wrt O_5 . All spheres in the pencil are tangent at a point.
- If \mathcal{P} is a time-like plane that is $\forall \vec{u} \in \vec{\mathcal{P}}, \vec{u}^2 < 0$ then \mathcal{C} is drawn as a hyperbola and forms a limited points pencil. (Fig.3.(c)) These points are obtained from the light directions of \mathcal{P} .

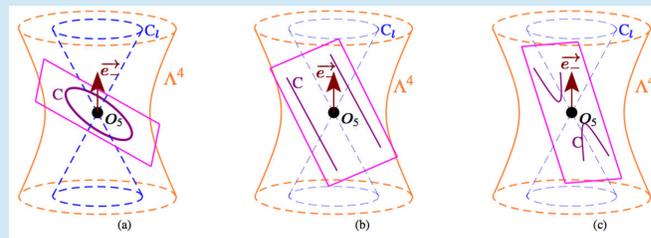


Figure 3: The representation of the three spheres pencil types on Λ^4

Canal surfaces on Λ^4

The envelop of a one-parameter set of oriented spheres in \mathbb{R}^3 defines a canal surface. The cones and the Dupin cyclides are known examples of canal surfaces of degree 2. On Λ^4 , any curve $t \rightarrow \sigma(t)$ represents a canal surface. Its characteristic circles are obtained by the intersection of 2 particular spheres (Fig 4).

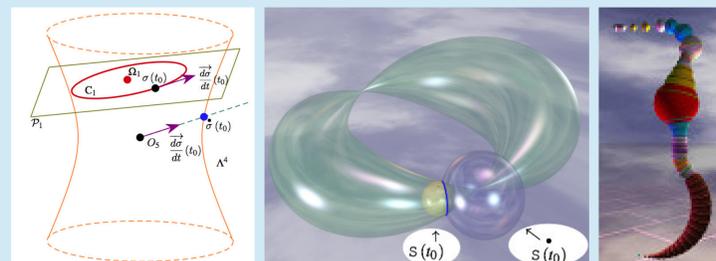


Figure 4: A Dupin cyclide on Λ^4 (left) on \mathbb{R}^3 (centre) and seahorse (right)

On Λ^4 the circle C_1 represents a Dupin cyclide. The tangent vector at the curve on point $\sigma(t_0)$ is given by $\frac{d\sigma}{dt}(t_0)$. The characteristic circle of the Dupin cyclide is provided by the intersection of the two spheres $S(t_0)$ and $S(\dot{\sigma}(t_0))$. These spheres are represented in Λ^4 by $\sigma(t_0)$ and $\dot{\sigma}(t_0)$. The last sphere is obtained by the intersection between the half line $\left[O_5; \frac{d\sigma}{dt}(t_0) \right)$ and Λ^4 . The figure 5 shows two cyclides from Λ^4 to \mathbb{R}^3 .



Figure 5: The same representation, on Λ^4 of a Dupin cyclide and of a circular cone: the implementation is the same with or without the point at infinity of \mathbb{R}^3 (M_2 is sent to the infinity), the modeling is the same as envelope of spheres or planes (Dupin cyclide or circular cone).

Conclusion : The Minkowski-Lorentz space offers a new way to handle curves and surfaces for CAD purposes making the computation easier. Algorithms for G^1 joins, not given here, are used to sketch a seahorse as example. **References :**

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