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4-dimensional locally strongly convex homogeneous affine hypersurfaces

Abdelouahab Chikh Salah and Luc Vrancken

Abstract. We study 4 dimensional locally strongly convex, locally homogeneous, hypersurfaces whose affine shape operator has two distinct principal curvatures. In case that one of the eigenvalues has dimension 1 these hypersurfaces have been previously studied in [4], [6] and [2] in which a classification of such submanifolds was obtained in dimension 4 and 5 under the additional assumption that the multiplicity of one of the eigenvalues is 1. In this paper we complete the classification in dimension 4 by considering the case that the multiplicity of both eigenvalues is 2.

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1. Introduction

In this paper we study locally strongly convex, locally homogeneous, hypersurfaces in the affine space $\mathbb{R}^{n+1}$. In case that the dimension is 2 these hypersurface were first studied in the book of Guggenheimer [5]. Their result was completed in the paper of Nomizu and Sasaki [12] thus giving a complete classification in dimension 2.

Starting from dimension 2 only partial results exist so far. Apart from the results of [14] and [9] most of these results deal with the case that the affine hypersurface is locally strongly convex, i.e. the induced affine metric is a positive definite metric. In that case such hypersurfaces have been studied in the paper of Sasaki [15] in the case that the hypersurface is an affine sphere and in a series of papers by Dillen and Vrancken [1] giving a complete classification in dimension 3.

The only other results are those of [2], [6] which give a classification in dimension 5, provided that the affine hypersurface is locally strongly convex and quasi umbilical. This means that the affine shape operator has only two distinct eigenvalues and that the multiplicity of one of those two eigenvalues is 1. In dimension 4 this result state

Theorem 1.1. *([4]) Let M^4 be a locally strongly convex, locally homogeneous, proper quasiumbilical affine hypersurface in $\mathbb{R}^5$. Then M is equivalent to the*

convex part of one of the following hypersurfaces:

$$\left(y-\frac{1}{2}(x^2+z^2+u^2)\right)^5 w^2=1,$$
$$\left(y-\frac{1}{2}(x^2+u^2)\right)^4\left(z-\frac{1}{2}w^2\right)^3=1,$$
$$\left(y-\frac{1}{2}x^2\right)^3 z^2u^2w^2=1,$$
$$\left(y-\frac{1}{2}x^2-\frac{1}{2}(w^2/z+u^2/z)\right)^5 z^4=1,$$
$$\left(y-\frac{1}{2}x^2-\frac{1}{2}w^2/z\right)^4 z^3u^2=1,$$
$$\left(y-\frac{1}{2}x^2\right)\left(z^2-(w^2+u^2)\right)=1,$$

where (x,y,z,w,u) *are the coordinates of* $\mathbb{R}^5$

Here, we investigate locally strongly convex affine hypersurface of dimension 4 whose affine shape operator has two distinct eigenvalues. In view of the previous result, in order to obtain a complete classification it is sufficient to deal with the case that the multiplicity of both eigenvalues is two. By combining the theorems 3.1, 3.2 and 3.3, we show that there exist just three such hypersurfaces and we prove the following theorem:

Theorem 1.2. *Let* M^4 *be a locally strongly convex, locally homogeneous, affine hypersurface in* $\mathbb{R}^5$*. Assume that* M *has two distinct eigenvalues, both of multiplicity* 2*. Then* M *is equivalent to the convex part of one of the following hypersurfaces:*

$$\left(x_1-x_4^2\right)^3\left(x_2-x_5^2\right)^3 x_3^2=1, \tag{1.1}$$
$$x_2^3\left(x_1-(x_3^2+x_4^2)-\frac{x_5^2}{x_2}\right)^5=1, \tag{1.2}$$
$$2x_2x_3x_4-x_4^2-x_1\left(x_3^2-2x_5\right)-2x_2^2x_5=1, \tag{1.3}$$

where (x_1,x_2,x_3,x_4,x_5) *are the coordinates of* $\mathbb{R}^5$.

Note in [3] several constructions were introduced which permitted to introduce several classes of affine homegeneous locally homogeneous hypersurfaces. For example all the examples of [2] and [6] can be recovered using these constructions. The same is true for the first two examples in our main theorem. However the final example introduces a new class of affine homogeneous hypersurfaces which is not obtained as one of the Calabi constructions.

Finally note that there is a conjecture of Dillen-Vrancken, together with an additional version of Niebergall-Ryan (for isoparametric hypersurfaces, [10]), see [8], stating that for an positive definite affine homogeneous affine

hypersurface or a positive definite isoparametric affine hypersurface the affine shape operator S has at most one non zero eigenvalue.

2. Preliminaries

Let M^n be a differentiable manifold in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D and a parallel volume element ω and let ξ be an arbitrary local transversal vector field to M^n. For any vector fields $X, Y, X_1, \ldots, X_n$ we can write

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \tag{2.1}$$

$$\theta(X_1, \ldots, X_n) = \omega(X_1, \ldots, X_n, \xi), \tag{2.2}$$

thus defining on M an affine connection ∇, a symmetric (0,2)-type tensor h, called the second fundamental form, and the volume element θ. We say that M is nondegenerate if h is nondegenerate and this condition is independent of the choice of transversal vector field ξ. In this case, it is known (see [13]) that there is a unique choice (up the sign) of transversal vector field such that the induced volume element θ satisfy the following condition:

$$\nabla\theta = 0, \tag{2.3}$$

$$\theta = \omega_h, \tag{2.4}$$

where ω_h is the metric volume element induced by h. We call ∇ the induced affine connection, ξ the affine normal and h the affine metric. Of course this choice remains invariant under equiaffine transformations.

By combining (2.3) and (2.4), we obtain the apolarity condition which states that $\nabla\omega_h = 0$. A nondegenerate hypersurface equipped with this transversal vector field is called a Blaschke hypersurface. Throughout this paper, we will always assume that M is a Blaschke hypersurface. If h is positive (or negative) definite, the hypersurface is called locally strongly convex. Notice that if h is negative definite, we can always replace ξ by $-\xi$, thus making the new affine metric positive definite. therefore, if we say that M is locally strongly convex, we will always assume ξ is chosen so that h is positive definite.

Condition (2.3) implies that $D_X\xi$ is tangent to M^n for any tangent vector X to M. Hence, we can define a (1,1)-tensor field S called the affine shape operator by

$$D_X\xi = -SX. \tag{2.5}$$

M is called an affine sphere if for all X, $SX = \lambda X$. We define the affine mean curvature H by $H = (1/n)trace(S)$.

The following fundamental equations of Gauss, Codazzi and Ricci are given by (see [13])

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \quad \text{(Equation of Gauss)} \tag{2.6}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \quad \text{(Equation of Codazzi of } h\text{)} \tag{2.7}$$

$$(\nabla_X S)Y = (\nabla_Y S)X, \quad \text{(Equation of Codazzi of } S\text{)} \tag{2.8}$$

$$h(X,SY) = h(SX,Y). \quad \text{(Equation of Ricci)} \tag{2.9}$$

Since $\dim(M) \geq 2$, if M is an affine sphere, it fallows from (2.8) that λ is constant. If $\lambda \neq 0$, we say that M is proper affine sphere and if $\lambda = 0$, we call M an improper affine sphere. In case that M is locally strongly convex, it follows from (2.8) that S is diagonalisable. From (2.7) it follows that the cubic form $C(X,Y,Z) = (\nabla h)(X,Y,Z)$ is symmetric in X,Y,Z. The theorem of Berwald states that C vanishes identically if and only if M is an open part of nondegenerate quadric.

Let $\hat{\nabla}$ denote the Levi Civita connection of the affine metric h. The difference tensor K is defined by

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,$$

for vector fields X and Y on M. Notice that, since both connections are torsion free, K is symmetric in X and Y. We also write $K_X Y = K(X,Y)$. From [13], we have that

$$h(K_X Y, Z) = -\frac{1}{2}C(X,Y,Z). \tag{2.10}$$

Notice also that the apolarity condition together with (2.10) implies that $\operatorname{Tr} K_X = 0$ for every tangent vector X. The basic equations can also be expressed in terms of the Levi Civita connection and the difference tensor. In particular, we have the Codazzi equation for K which states that

$$\begin{aligned}&\hat{\nabla}K(X,Y,Z) - \hat{\nabla}K(Y,X,Z)\\ &\quad = \frac{1}{2}h(Y,Z)SX - \frac{1}{2}h(X,Z)SY + \frac{1}{2}h(SX,Z)Y - \frac{1}{2}h(SY,Z)X.\end{aligned} \tag{2.11}$$

A nondegenerate hypersurface M of the equiaffine space $\mathbb{R}^{n+1}$ is called locally homogeneous if for all points p and q of M, there exists a neighborhood U_p of p in M, and an equiaffine transformation A of $\mathbb{R}^{n+1}$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p in M, then M is called homogeneous.

3. Main results

In the remainder of this paper, M^4 will always denote a locally strongly convex, locally homogeneous Blaschke hypersurface of $\mathbb{R}^5$. Since M is locally strongly convex, it follows from the Ricci equation that the affine shape operator is diagonalizable. In view of the previous results we will restrict ourselves here to the case that M has tow distinct eigenvalues both of multiplicity 2.

Let a point $p \in M$. We construct a tangent basis e_1, e_2, e_3, e_4 at the point p and λ_1, λ_2 by the eigenvalues of the shape operator S, such that

$Se_1 = \lambda_1 e_1, \quad Se_2 = \lambda_1 e_2, \quad Se_3 = \lambda_2 e_3, \quad Se_4 = \lambda_2 e_4$. Then, since λ_1 and λ_2 are different numbers, and as M is homogeneous the eigenvalues of the shape operator are constant it follows from [11] that we can extend these vectors to local vector fields E_1, E_2, E_3, E_4, such that

$$SE_1 = \lambda_1 E_1, \quad SE_2 = \lambda_1 E_2, \quad SE_3 = \lambda_2 E_3, \quad SE_4 = \lambda_2 E_4.$$

We define functions Γ^k_{ij} such that the connection ∇ is given by

$$\nabla_{E_i} E_j = \sum_{k=1}^{4} \Gamma^k_{ij} E_k \qquad for\ i = 1, \dots, 4, \quad j = 1, \dots, 4. \tag{3.1}$$

Analysing the Codazzi equation for S (2.8) it follows that:

$$\Gamma^1_{13}, \Gamma^2_{13}, \Gamma^1_{14}, \Gamma^2_{14}, \Gamma^1_{23}, \Gamma^2_{23}, \Gamma^1_{24}, \Gamma^2_{24}, \Gamma^3_{31}, \Gamma^4_{31}, \Gamma^3_{32}, \Gamma^4_{32}, \Gamma^3_{41}, \Gamma^4_{41}, \Gamma^3_{42}, \Gamma^4_{42}$$

vanish and that

$$\Gamma^3_{21} = \Gamma^3_{12}, \quad \Gamma^4_{21} = \Gamma^4_{12}, \quad \Gamma^1_{43} = \Gamma^1_{34}, \quad \Gamma^2_{43} = \Gamma^2_{34}.$$

From the Codazzi equation for h (2.7) we have

$$\begin{aligned}
&\Gamma^3_{11} = 2\Gamma^1_{31}, \quad \Gamma^4_{11} = 2\Gamma^1_{41}, \quad \Gamma^3_{22} = 2\Gamma^2_{32}, \quad \Gamma^4_{22} = 2\Gamma^2_{42}, \quad \Gamma^1_{33} = 2\Gamma^3_{13},\\
&\Gamma^2_{33} = 2\Gamma^3_{23}, \quad \Gamma^1_{44} = 2\Gamma^4_{14}, \quad \Gamma^2_{44} = 2\Gamma^4_{24},\\
&\Gamma^3_{14} = \Gamma^1_{43} - \Gamma^4_{13}, \quad \Gamma^3_{24} = \Gamma^2_{43} - \Gamma^4_{23}, \quad \Gamma^1_{32} = \Gamma^3_{12} - \Gamma^2_{31}, \quad \Gamma^1_{42} = \Gamma^4_{12} - \Gamma^2_{41},\\
&\Gamma^1_{12} = 2\Gamma^1_{21} - \Gamma^2_{11}, \quad \Gamma^1_{22} = 2\Gamma^2_{12} - \Gamma^2_{21}, \quad \Gamma^3_{34} = 2\Gamma^3_{43} - \Gamma^4_{33}, \quad \Gamma^3_{44} = 2\Gamma^4_{34} - \Gamma^4_{43}.
\end{aligned}$$

The apolarity condition

$$\begin{aligned}
\Gamma^3_{13} &= -\Gamma^1_{11} - \Gamma^2_{12} - \Gamma^4_{14}, \qquad & \Gamma^3_{23} &= -\Gamma^1_{21} - \Gamma^2_{22} - \Gamma^4_{24},\\
\Gamma^2_{32} &= -\Gamma^1_{31} - \Gamma^3_{33} - \Gamma^4_{34}, \qquad & \Gamma^2_{42} &= -\Gamma^1_{41} - \Gamma^3_{43} - \Gamma^4_{44}.
\end{aligned}$$

Hence the only remaining equations are those obtained from the Gauss equation. Here, in case the frame is completely defined in an equiaffine invariant way we can use the fact that M is affine homogeneous implies that all connection coefficients are constant. This is of course not the case when the frame is not uniquely defined. Therefore before exploiting the Gauss equations we first look at the remaining degrees of freedom.

The fact that we have, the eigenvalues have multiplicity two, means that there are rotations possible in $\mathcal{E}_1 = span\{E_1, E_2\}$ and $\mathcal{E}_2 = span\{E_3, E_4\}$ which are the two eigenspaces relative to λ_1 and λ_2 respectively. These rotations are given by

$$\begin{cases} \tilde{E}_1 = \cos t\, E_1 + \sin t\, E_2, \\ \tilde{E}_2 = -\sin t\, E_1 + \cos t\, E_2, \end{cases} \quad and \quad \begin{cases} \tilde{E}_3 = \cos s\, E_3 + \sin s\, E_4, \\ \tilde{E}_4 = -\sin s\, E_3 + \cos s\, E_4, \end{cases} . \tag{3.2}$$

However we see that the vectors T_1, T_2 defined by

$$\begin{cases} T_1 = K(\tilde{E}_1, \tilde{E}_1) + K(\tilde{E}_2, \tilde{E}_2) = K(E_1, E_1) + K(E_2, E_2), \\ T_2 = K(\tilde{E}_3, \tilde{E}_3) + K(\tilde{E}_4, \tilde{E}_4) = K(E_3, E_3) + K(E_4, E_4), \end{cases} \tag{3.3}$$

are independent under such rotations and therefore defined in an equiaffine invariant way. Moreover if we write $T_i = V_i + W_i$, where $V_i \in \mathcal{E}_1$ and $W_i \in \mathcal{E}_2$

for $i = 1, 2$ both components are defined in an affine invariant way. Of course the apolarity condition gives

$$\begin{cases} V_1 + V_2 = 0, \\ W_1 + W_2 = 0. \end{cases}$$

If necessary up to interchanging the two eigenspaces, we have to investigate the following 3 cases.

$$\text{Type 1 :} \qquad \begin{cases} V_1 \neq 0, \\ W_2 \neq 0, \end{cases} \tag{3.4}$$

$$\text{Type 2 :} \qquad \begin{cases} V_1 \neq 0, \\ W_2 = 0, \end{cases} \tag{3.5}$$

$$\text{Type 3 :} \qquad \begin{cases} V_1 = 0, \\ W_2 = 0. \end{cases} \tag{3.6}$$

We will study all types, one after the other. Note that in the first case, the frame is uniquely determined, as we may assume that E_1 is in the direction of V_1 and E_3 is in the direction of W_2. In the second case we can still choose E_1 in the direction of V_1 and we retain the rotation freedom in the bundle $\mathcal{E}_2$. In the third case we retain the rotation freedom in both bundles. In the next sections we will show how to introduce also in those cases a unique frame, in order to be able to apply the next lemma.

Note that in all the cases, by (3.3) we still have that

$$\Gamma^2_{22} = -\Gamma^1_{21},$$
$$\Gamma^4_{44} = -\Gamma^3_{43}.$$

Then by evaluating the Gauss equation we get by a straightforward calculation that

Lemma 3.1. *Let M be a locally strongly convex, affine homogeneous hypersurface. Assume that M has two distinct eigenvalues both of multiplicity 2. Suppose that the frame is chosen in a unique affine invariant way such that all the previous formulas remain valid. Then all the connection coefficients are constant and therefore the Gauss equations imply that*

$G1211$ $\Gamma^1_{11}(\Gamma^2_{11} - \Gamma^1_{21}) - \Gamma^2_{12}(2\Gamma^2_{11} + \Gamma^1_{21}) + 3\Gamma^1_{21}\Gamma^2_{21} = 0,$

$G1212$ $(\Gamma^2_{21})^2 - \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}(\Gamma^2_{11} + \Gamma^1_{21}) + \lambda_1 = 0,$

$G1213$ $-\Gamma^3_{12}(2\Gamma^1_{11} + 2\Gamma^2_{12} + \Gamma^4_{14} - 2\Gamma^2_{21}) + \Gamma^4_{12}(\Gamma^1_{34} - \Gamma^4_{13}) - 2(\Gamma^4_{24}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})$
$+ \Gamma^2_{11}(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (\Gamma^4_{23} - \Gamma^2_{34})\Gamma^2_{42}) = 0,$

$G1214$ $\Gamma^3_{12}\Gamma^4_{13} - \Gamma^4_{12}(\Gamma^1_{11} + \Gamma^2_{12} - \Gamma^4_{14} - 2\Gamma^2_{21}) + 2\Gamma^4_{23}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) \quad + 2(\Gamma^4_{24} - 2\Gamma^2_{11})\Gamma^2_{42} = 0,$

$G1221$ $-(\Gamma^2_{11} - 3\Gamma^1_{21})(\Gamma^2_{11} - 2\Gamma^1_{21}) + (2\Gamma^2_{12} - \Gamma^2_{21})(\Gamma^1_{11} - 2\Gamma^2_{12} + \Gamma^2_{21}) - \lambda_1 = 0,$

$G1222$ $3\Gamma^2_{11}\Gamma^2_{12} - 3\Gamma^1_{21}\Gamma^2_{21} = 0,$

$G1223$ $\Gamma^3_{12}(2\Gamma^2_{11} - 4\Gamma^1_{21} + \Gamma^4_{24}) + \Gamma^4_{12}(\Gamma^4_{23} - \Gamma^2_{34}) - 2(\Gamma^1_{11}\Gamma^2_{32} + \Gamma^4_{14}\Gamma^2_{32} - 2\Gamma^2_{21}\Gamma^2_{32} - \Gamma^2_{21}\Gamma^3_{33}$
$- \Gamma^2_{21}\Gamma^4_{34} + \Gamma^2_{12}(5\Gamma^2_{32} + 2(\Gamma^3_{33} + \Gamma^4_{34})) + (\Gamma^4_{13} - \Gamma^1_{34})\Gamma^2_{42}) = 0,$

$G1224$ $-\Gamma^3_{12}\Gamma^4_{23} + \Gamma^4_{12}(2\Gamma^2_{11} - 4\Gamma^1_{21} - \Gamma^4_{24}) + 2\Gamma^4_{13}\Gamma^2_{32} + 2(-4\Gamma^2_{12} + \Gamma^4_{14} + 2\Gamma^2_{21})\Gamma^2_{42} = 0,$

$G1234$ $(\Gamma^1_{11} + 2\Gamma^4_{14} + \Gamma^2_{21})\Gamma^4_{23} + \Gamma^4_{13}(\Gamma^2_{11} - \Gamma^1_{21} - 2\Gamma^4_{24}) = 0,$

$G1243$ $(\Gamma^1_{11} + 2(\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21})(\Gamma^4_{23} - \Gamma^2_{34}) - (\Gamma^2_{11} - \Gamma^1_{21} + 2\Gamma^4_{24})(\Gamma^4_{13} - \Gamma^1_{34}) = 0,$

$G1244$ $\Gamma^4_{14}(\Gamma^2_{11} - \Gamma^1_{21}) + (\Gamma^2_{21} - \Gamma^2_{12})\Gamma^4_{24} - \Gamma^4_{23}\Gamma^1_{34} + \Gamma^4_{13}\Gamma^2_{34} = 0,$

$G1311$ $-\Gamma^2_{11}\Gamma^3_{12} + 3\Gamma^1_{21}\Gamma^2_{31} - (6\Gamma^1_{11} + 5(\Gamma^2_{12} + \Gamma^4_{14}))(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (\Gamma^4_{13} + 2\Gamma^1_{34})\Gamma^2_{42} = 0,$

$G1312$ $(2\Gamma^2_{12} + \Gamma^4_{14} + \Gamma^2_{21})\Gamma^2_{31} - 4\Gamma^4_{24}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^2_{11}(3\Gamma^2_{32} + 2(\Gamma^3_{33} + \Gamma^4_{34})) - \Gamma^4_{13}\Gamma^2_{41}$
$+ 2\Gamma^2_{34}\Gamma^2_{42} = 0,$

$G1313$ $2(\Gamma^3_{12}\Gamma^2_{31} + (\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})(2\Gamma^2_{32} + 3\Gamma^3_{33} + 2\Gamma^4_{34}) - \Gamma^2_{42}(\Gamma^4_{33} - 2\Gamma^3_{43})) + \lambda_2 = 0,$

$G1314$ $2(\Gamma^4_{12}\Gamma^2_{31} + \Gamma^4_{33}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (2(\Gamma^2_{32} + \Gamma^3_{33}) + 3\Gamma^4_{34})\Gamma^2_{42}) = 0,$

$G1321$ $\Gamma^3_{12}(4\Gamma^1_{11} + 2\Gamma^2_{12} + 3\Gamma^4_{14}) - (2\Gamma^1_{11} - 2\Gamma^2_{12} + \Gamma^4_{14} + \Gamma^2_{21})\Gamma^2_{31} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{32}$
$- \Gamma^4_{12}(\Gamma^4_{13} + \Gamma^1_{34}) + \Gamma^4_{13}\Gamma^2_{41} = 0,$

$G1322$ $\Gamma^3_{12}(\Gamma^2_{11} + 2\Gamma^4_{24}) - 3\Gamma^1_{21}\Gamma^2_{31} + (\Gamma^1_{11} + \Gamma^4_{14})\Gamma^2_{32} - \Gamma^4_{12}\Gamma^2_{34} - \Gamma^2_{12}(\Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^4_{13}\Gamma^2_{42} = 0,$

$G1323$ $2\Gamma^2_{31}(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^3_{12}(2\Gamma^2_{32} + 4\Gamma^3_{33} + 3\Gamma^4_{34}) + \Gamma^4_{12}(\Gamma^4_{33} - 2\Gamma^3_{43}) = 0,$

$G1324$ $-\Gamma^4_{12}(\Gamma^3_{33} + 2\Gamma^4_{34}) + 4\Gamma^2_{31}\Gamma^2_{42} - \Gamma^3_{12}(\Gamma^4_{33} + 2\Gamma^2_{42}) = 0,$

$G1331$ $-2((\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(3\Gamma^1_{11} + 2(\Gamma^2_{12} + \Gamma^4_{14})) - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^4_{24} + \Gamma^4_{13}\Gamma^1_{34}) - \lambda_1 = 0,$

$G1332$ $-2(\Gamma^2_{11}(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14}) + (2\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{24} + \Gamma^4_{13}\Gamma^2_{34}) = 0,$

$G1333$ $-\Gamma^4_{24}(2\Gamma^3_{12} + \Gamma^2_{31}) + \Gamma^4_{33}\Gamma^1_{34} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(5\Gamma^2_{32} + 6\Gamma^3_{33} + 5\Gamma^4_{34}) - 3\Gamma^4_{13}\Gamma^3_{43} = 0,$

$G1341$ $-2\Gamma^4_{13}(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) + (4\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^1_{34} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{34} = 0,$

$G1342$ $-4\Gamma^4_{13}\Gamma^4_{24} + (\Gamma^2_{11} + 2\Gamma^4_{24})\Gamma^1_{34} + (\Gamma^1_{11} + 2\Gamma^2_{12})\Gamma^2_{34} = 0,$

$G1344$ $\Gamma^4_{24}\Gamma^2_{31} - \Gamma^4_{14}(\Gamma^2_{32} + \Gamma^3_{33}) + \Gamma^4_{12}\Gamma^2_{34} + (\Gamma^1_{11} + \Gamma^2_{12})\Gamma^4_{34} - \Gamma^1_{34}(\Gamma^4_{33} + 2\Gamma^2_{42}) + 3\Gamma^4_{13}\Gamma^3_{43} = 0,$

$G1411$ $-\Gamma^2_{11}\Gamma^4_{12} - (\Gamma^4_{13} - 3\Gamma^1_{34})(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + 3\Gamma^1_{21}\Gamma^2_{41} - (\Gamma^1_{11} - 5\Gamma^4_{14})\Gamma^2_{42} = 0,$

$G1412$ $\Gamma^2_{31}(\Gamma^4_{13} - \Gamma^1_{34}) + 2\Gamma^2_{34}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + (-\Gamma^1_{11} + \Gamma^2_{12} - \Gamma^4_{14} + \Gamma^2_{21})\Gamma^2_{41}$
$+ (4\Gamma^4_{24} - 3\Gamma^2_{11})\Gamma^2_{42} = 0,$

$G1413$ $2(\Gamma^3_{12}\Gamma^2_{41} + \Gamma^2_{32}(2\Gamma^2_{42} + \Gamma^3_{43}) + \Gamma^3_{33}(2\Gamma^2_{42} + \Gamma^3_{43}) + \Gamma^4_{34}(4\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{42}\Gamma^4_{43}) = 0,$

$G1414$ $4(\Gamma^2_{42})^2 - 2\Gamma^3_{43}\Gamma^2_{42} + 2\Gamma^4_{12}\Gamma^2_{41} + 2(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})\Gamma^4_{43} + \lambda_2 = 0,$

$G1422$ $\Gamma^4_{12}(\Gamma^2_{11} - 2\Gamma^4_{24}) + \Gamma^2_{32}(\Gamma^4_{13} - \Gamma^1_{34}) - \Gamma^3_{12}\Gamma^2_{34} - 3\Gamma^1_{21}\Gamma^2_{41} - (\Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42} = 0,$

$G1423$ $2(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})\Gamma^2_{41} - \Gamma^3_{12}\Gamma^3_{43} + \Gamma^4_{12}(\Gamma^4_{43} - 2(\Gamma^2_{32} + \Gamma^3_{33} + 2\Gamma^4_{34})) = 0,$

$G1424$ $4\Gamma^2_{41}\Gamma^2_{42} + \Gamma^4_{12}(\Gamma^3_{43} - 2\Gamma^2_{42}) - \Gamma^3_{12}\Gamma^4_{43} = 0$

$G1431$ $-2\Gamma^4_{13}(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) + (4\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^1_{34} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^2_{34} = 0,$

$G1433$ $\Gamma^3_{12}\Gamma^2_{34} + \Gamma^4_{13}(\Gamma^3_{33} - 2\Gamma^4_{34}) - \Gamma^4_{24}\Gamma^2_{41} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42} - \Gamma^4_{14}\Gamma^3_{43}$
$+ \Gamma^1_{34}(-2\Gamma^2_{32} - 3\Gamma^3_{33} - 2\Gamma^4_{34} + \Gamma^4_{43}) = 0,$

$G1441$ $2((\Gamma^1_{11} - 2\Gamma^4_{14})\Gamma^4_{14} - (\Gamma^2_{11} - 2\Gamma^1_{21})\Gamma^4_{24} + (\Gamma^4_{13} - \Gamma^1_{34})\Gamma^1_{34}) - \lambda_1 = 0,$

$G1442$ $2(\Gamma^2_{11}\Gamma^4_{14} + (\Gamma^2_{12} - 2\Gamma^4_{14})\Gamma^4_{24} + (\Gamma^4_{13} - \Gamma^1_{34})\Gamma^2_{34}) = 0,$

$G1444$ $3\Gamma^4_{13}\Gamma^4_{34} + \Gamma^4_{24}(2\Gamma^4_{12} + \Gamma^2_{41}) + \Gamma^4_{14}(\Gamma^3_{43} - 5\Gamma^2_{42}) - \Gamma^1_{34}(\Gamma^4_{34} + \Gamma^4_{43}) = 0,$

$G2311$ $\Gamma^3_{12}(3\Gamma^1_{11} + 2(\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21}) - (\Gamma^1_{11} - 2\Gamma^2_{12})\Gamma^2_{31} + \Gamma^1_{21}\Gamma^2_{32} - \Gamma^4_{12}\Gamma^1_{34}$
$- \Gamma^4_{24}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) + \Gamma^4_{23}\Gamma^2_{42} = 0,$

$G2312$ $\Gamma^3_{12}(\Gamma^2_{11} + 2\Gamma^4_{24}) + (-\Gamma^2_{11} - 2\Gamma^1_{21} + \Gamma^4_{24})\Gamma^2_{31} - \Gamma^4_{12}\Gamma^2_{34} - \Gamma^2_{21}(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^4_{23}\Gamma^2_{41} = 0,$

$G2313$ $2\Gamma^2_{31}(2\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - \Gamma^3_{12}(2\Gamma^2_{32} + 4\Gamma^3_{33} + 3\Gamma^4_{34}) + \Gamma^4_{12}(\Gamma^4_{33} - 2\Gamma^3_{43}) = 0,$

$G2314$ $-\Gamma^4_{12}(\Gamma^3_{33} + 2\Gamma^4_{34}) + 4\Gamma^2_{31}\Gamma^2_{42} - \Gamma^3_{12}(\Gamma^4_{33} + 2\Gamma^2_{42}) = 0,$

$G2322$ $\Gamma^3_{12}(\Gamma^2_{12} + \Gamma^2_{21}) - 3\Gamma^2_{12}\Gamma^2_{31} - (\Gamma^1_{21} - 5\Gamma^4_{24})\Gamma^2_{32} - (\Gamma^4_{23} + 2\Gamma^2_{34})\Gamma^2_{42} = 0,$

$G2323$ $2((\Gamma^3_{12})^2) - \Gamma^2_{31}\Gamma^3_{12} + \Gamma^2_{32}(2\Gamma^2_{32} - \Gamma^3_{33}) + \Gamma^2_{42}(\Gamma^4_{33} - 2\Gamma^3_{43})) + \lambda_2 = 0,$

$G2331$ $-2((\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^1_{21} + (2(\Gamma^1_{11} + 2\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21})\Gamma^4_{24} + \Gamma^4_{23}\Gamma^1_{34}) = 0,$

$G2332$ $-2(2(\Gamma^4_{24})^2 - \Gamma^1_{21}\Gamma^4_{24} + (\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{21} + \Gamma^4_{23}\Gamma^2_{34}) - \lambda_1 = 0,$

$G2333$ $-(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})(3\Gamma^3_{12} - \Gamma^2_{31}) + \Gamma^4_{24}(\Gamma^3_{33} - 5\Gamma^2_{32}) + \Gamma^4_{33}\Gamma^2_{34} - 3\Gamma^4_{23}\Gamma^3_{43} = 0,$

$$G2342 \quad \Gamma^2_{21}\Gamma^1_{34} - \Gamma^1_{21}\Gamma^2_{34} + 2\Gamma^4_{24}(\Gamma^2_{34} - 2\Gamma^4_{23}) = 0,$$

$$G2343 \quad -\Gamma^3_{12}(\Gamma^4_{13} - 2\Gamma^1_{34}) + \Gamma^2_{31}(\Gamma^4_{13} - \Gamma^1_{34}) + \Gamma^2_{34}(3\Gamma^2_{32} - \Gamma^3_{33} + \Gamma^4_{34}) + \Gamma^4_{24}(\Gamma^4_{33} - 2\Gamma^3_{43}) + \Gamma^4_{23}(-\Gamma^2_{32} + \Gamma^3_{33} - 3\Gamma^4_{34} + \Gamma^4_{43}) = 0,$$

$$G2411 \quad \Gamma^4_{12}(\Gamma^1_{11} - 2\Gamma^4_{14} - \Gamma^2_{21}) - \Gamma^3_{12}\Gamma^1_{34} - (\Gamma^4_{23} - \Gamma^2_{34})(\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34}) - (\Gamma^1_{11} - 2\Gamma^2_{12})\Gamma^2_{41} + (\Gamma^1_{21} + \Gamma^4_{24})\Gamma^2_{42} = 0,$$

$$G2422 \quad \Gamma^4_{12}(\Gamma^2_{12} + \Gamma^2_{21}) + \Gamma^2_{32}(\Gamma^4_{23} - 3\Gamma^2_{34}) - 3\Gamma^2_{12}\Gamma^2_{41} - (\Gamma^1_{21} + 5\Gamma^4_{24})\Gamma^2_{42} = 0,$$

$$G2424 \quad 2((\Gamma^4_{12})^2) - \Gamma^2_{41}\Gamma^4_{12} + \Gamma^2_{42}(2\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{32}\Gamma^4_{43}) + \lambda_2 = 0,$$

$$G2431 \quad -2(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{23} + \Gamma^1_{21}\Gamma^1_{34} + (2(\Gamma^1_{11} + 2\Gamma^2_{12} + \Gamma^4_{14}) - \Gamma^2_{21})\Gamma^2_{34} = 0,$$

$$G2441 \quad 2(\Gamma^4_{14}(\Gamma^1_{21} - 2\Gamma^4_{24}) + (2\Gamma^2_{12} - \Gamma^2_{21})\Gamma^4_{24} + \Gamma^1_{34}(\Gamma^4_{23} - \Gamma^2_{34})) = 0,$$

$$G2442 \quad 2(\Gamma^4_{14}\Gamma^2_{21} - \Gamma^4_{24}(\Gamma^1_{21} + 2\Gamma^4_{24}) + (\Gamma^4_{23} - \Gamma^2_{34})\Gamma^2_{34}) - \lambda_1 = 0,$$

$$G2444 \quad 3\Gamma^4_{23}\Gamma^4_{34} + \Gamma^4_{14}(3\Gamma^4_{12} - \Gamma^2_{41}) + \Gamma^4_{24}(5\Gamma^2_{42} + \Gamma^3_{43}) - \Gamma^2_{34}(\Gamma^4_{34} + \Gamma^4_{43}) = 0,$$

$$G3411 \quad (\Gamma^3_{12} - \Gamma^2_{31})\Gamma^2_{41} + \Gamma^2_{31}(\Gamma^2_{41} - \Gamma^4_{12}) - (\Gamma^2_{32} + \Gamma^3_{33} + \Gamma^4_{34})(\Gamma^4_{33} - \Gamma^3_{43}) + \Gamma^2_{42}(\Gamma^4_{34} - \Gamma^4_{43}) = 0,$$

$$G3431 \quad -2(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{33} + (\Gamma^3_{12} - \Gamma^2_{31})\Gamma^2_{34} + 2\Gamma^4_{24}(\Gamma^4_{12} - \Gamma^2_{41}) - 2(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^4_{14})\Gamma^2_{42} - \Gamma^1_{34}(\Gamma^2_{32} + 2(\Gamma^3_{33} + \Gamma^4_{34} - \Gamma^4_{43})) = 0,$$

$$G3433 \quad \Gamma^3_{33}(\Gamma^4_{33} - \Gamma^3_{43}) - \Gamma^4_{34}(2\Gamma^4_{33} + \Gamma^3_{43}) + 3\Gamma^3_{43}\Gamma^4_{43} = 0,$$

$$G3434 \quad (\Gamma^4_{43})^2 - \Gamma^3_{33}\Gamma^4_{43} + \Gamma^4_{33}(\Gamma^4_{33} + \Gamma^3_{43}) + \lambda_2 = 0,$$

$$G3441 \quad 2\Gamma^4_{24}(\Gamma^3_{12} - \Gamma^2_{31}) + \Gamma^2_{34}(\Gamma^2_{41} - \Gamma^4_{12}) + \Gamma^1_{34}(2\Gamma^4_{33} + \Gamma^2_{42} - 4\Gamma^3_{43}) - 2\Gamma^4_{14}(\Gamma^2_{32} + \Gamma^3_{33} + 5\Gamma^4_{34} - 2\Gamma^4_{43}) - 2(\Gamma^1_{11} + \Gamma^2_{12})(2\Gamma^4_{34} - \Gamma^4_{43}) = 0,$$

$$G3442 \quad 2\Gamma^4_{14}\Gamma^2_{31} - \Gamma^1_{34}\Gamma^2_{41} + \Gamma^2_{34}(2\Gamma^4_{33} - \Gamma^2_{42} - 4\Gamma^3_{43}) + 2\Gamma^4_{24}(\Gamma^2_{32} - 4\Gamma^4_{34} + 2\Gamma^4_{43}) = 0,$$

$$G3443 \quad -(\Gamma^4_{33} - 3\Gamma^3_{43})(\Gamma^4_{33} - 2\Gamma^3_{43}) + (2\Gamma^4_{34} - \Gamma^4_{43})(\Gamma^3_{33} - 2\Gamma^4_{34} + \Gamma^4_{43}) - \lambda_2 = 0.$$

Note that in the previous lemma $Gijk\ell$ means the component of Gauss equation (2.6) for $X = E_i, Y = E_j, Z = E_k$ in the direction of E_ℓ.

3.1. Hypersurfaces of type 1

As mentioned before the frame is completely determined and therefore we can apply the previous lemma. We will show that this case leads to a contradiction. As M is of Type 1, we have that

$$\Gamma^1_{11} + \Gamma^2_{12} \neq 0, \qquad \Gamma^3_{33} + \Gamma^4_{34} \neq 0. \tag{3.7}$$

Adding (G1211) and (G1222), we get that

$$(\Gamma^1_{11} + \Gamma^2_{12})(\Gamma^2_{11} - \Gamma^1_{21}) = 0.$$

Using (3.7) we deduce that $\Gamma^2_{11} = \Gamma^1_{21}$. Similarly we obtain from adding (G3433) and (G3444) that $\Gamma^4_{33} = \Gamma^3_{43}$. Adding (G1221) and (G1212) we see that

$$2(\Gamma^1_{11} - 2\Gamma^2_{12})(\Gamma^2_{12} - \Gamma^2_{21}) = 0.$$

Similarly from (G3443) and (G3434) we get that

$$2(\Gamma^3_{33} - 2\Gamma^4_{34})(\Gamma^4_{34} - \Gamma^4_{43}) = 0.$$

So if necessary upto interchanging the vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ there are 3 subcases to consider.

Case I : $\Gamma^2_{12} \neq \Gamma^2_{21}$ and $\Gamma^4_{34} \neq \Gamma^4_{43}$. Hence we must have that $\Gamma^2_{12} = \frac{1}{2}\Gamma^1_{11}$ and $\Gamma^4_{34} = \frac{1}{2}\Gamma^3_{33}$. Moreover from (G1211) and (G3433) it follows that $\Gamma^1_{21} = \Gamma^3_{43} = 0$.

Next we use (G1341) and the difference of (G1441) and (G1331). This gives us

$$2\Gamma^4_{14}(\Gamma^1_{34} - 2\Gamma^4_{13}) + \Gamma^1_{11}(\frac{11}{2}\Gamma^1_{34} - 3\Gamma^4_{13}) = 0,$$
$$-2(\Gamma^1_{34})^2 + 4\Gamma^4_{13}\Gamma^1_{34} + 4\Gamma^1_{11}(3\Gamma^1_{11} + 4\Gamma^4_{14}) = 0.$$

From (3.7) we see that $\Gamma^1_{11} \neq 0$ and $\Gamma^3_{33} \neq 0$, and therefore it follows from the previous equations that supposing that $\Gamma^1_{11} \neq -\frac{4}{3}\Gamma^4_{14}$, solving the first equations for Γ^4_{13} and substituting this result in the second equation leads to a contradiction. Therefore we must have that $\Gamma^1_{34} = 0$ and $\Gamma^4_{14} = -\frac{3}{4}\Gamma^1_{11}$. Similarly from the symmetry of the problem in this case we deduce that also $\Gamma^3_{12} = 0$ and $\Gamma^2_{32} = -\frac{3}{4}\Gamma^3_{33}$.

Using now (G1331) and (G1313) we immediately see that

$$\lambda_1 = -\tfrac{15}{4}(\Gamma^1_{11})^2,$$
$$\lambda_2 = -\tfrac{15}{4}(\Gamma^3_{33})^2.$$

The equations (G1311) and (G1424) now become

$$\Gamma^4_{13}\Gamma^2_{42} - \frac{57}{16}\Gamma^1_{11}\Gamma^3_{33} = 0,$$
$$-2(\Gamma^4_{12} - 2\Gamma^2_{41})\Gamma^2_{42} = 0,$$

which implies that $\Gamma^2_{42} \neq 0$ and $\Gamma^4_{12} = 2\Gamma^2_{41}$. Similarly we get interchanging the two vector bundles that also $\Gamma^4_{24} \neq 0$ and $\Gamma^2_{34} = 2\Gamma^4_{23}$.

The equations (G4113 and (G2331)) now immediately imply that

$$\Gamma^4_{43} = \tfrac{5}{2}\Gamma^3_{33}, \qquad \Gamma^2_{21} = \tfrac{5}{2}\Gamma^1_{11}.$$

A contradiction now follows from (G1414).

With similar computations, we can also prove the non-existence of solutions in the cases II and III, which are respectively given by :

$$\textbf{Case II} : \Gamma^2_{12} = \Gamma^2_{21}, \qquad \Gamma^4_{34} = \Gamma^4_{43}.$$
$$\textbf{Case III} : \Gamma^2_{12} \neq \Gamma^2_{21}, \qquad \Gamma^2_{12} = \frac{1}{2}\Gamma^1_{11}, \qquad \Gamma^4_{34} = \Gamma^4_{43}.$$

Hence there is not hypersurface in $\mathbb{R}^5$ of type 1.

3.2. Hypersurfaces of type 2

In this case we have that $V_1 \neq 0$ and $W_2 = 0$. We choose E_1 in the direction of V_1. This fixes the rotation freedom in $\mathcal{E}_1$. As W_2 vanishes, we have that $\Gamma^4_{34} = -\Gamma^3_{33}$. In order to determine a frame in $\mathcal{E}_2$ we consider the following subcases:

1. The restriction of the difference tensor K to the space $\mathcal{E}_2$ does not vanish identically. Applying a rotation we can choose in this case the vector field E_3 such that $h(K(E_3, E_3), E_4) = 0$. So we have that $\Gamma^3_{43} = 0$ and $\Gamma^3_{33} \neq 0$.

2. The restriction of the difference tensor K to the space $\mathcal{E}_2$ vanishes identically. However $K(\mathcal{E}_1, \mathcal{E}_1) \not\subset \mathcal{E}_1$. In that case we can fix the vector field E_3 such that one of the following holds:
   (a) $h(K(E_1, E_1), E_3) \neq 0$ and $h(K(E_1, E_1), E_4) = 0$,
   (b) $h(K(E_1, E_2), E_3) \neq 0$ and
   $h(K(E_1, E_1), E_3) = h(K(E_1, E_1), E_4) = h(K(E_1, E_2), E_4) = 0$,
   (c) $h(K(E_2, E_2), E_3) \neq 0$ and
   $h(K(E_1, E_1), E_3) = h(K(E_1, E_1), E_4) = h(K(E_1, E_2), E_3) = 0$
   and $h(K(E_1, E_2), E_4) = h(K(E_2, E_2), E_4) = 0$,
   which leads to an immediate contradiction as $W_1 = 0$.
3. We have that $h(K(\mathcal{E}_1, \mathcal{E}_1), \mathcal{E}_2) = h(K(\mathcal{E}_2, \mathcal{E}_2), \mathcal{E}_2) = 0$. In this case we can consider the following subcases:
   (a) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can pick E_3 and E_4 in the direction of the corresponding eigenspaces,
   (b) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ is a multiple of the identity, but the symmetric operator K_{E_2} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can again pick E_3 and E_4 in the direction of the corresponding eigenspaces,
   (c) both the symmetric operators K_{E_1} and K_{E_2} restricted to $\mathcal{E}_2$ are a multiple of the identity,

So we see that except in the last case, the frame is uniquely determined and we can apply the Lemma 3.1.

Type 2.1 We have that $\Gamma^3_{43} = 0$ and $\Gamma^3_{33} \neq 0 \neq \Gamma^1_{11} + \Gamma^2_{12}$. From $(G3433)$ it immediately follows that $\Gamma^4_{33} = 0$. Adding now $(G3443)$ and $(G3434)$ we get

$$-6\Gamma^3_{33}(\Gamma^3_{33} + \Gamma^4_{43}) = 0,$$

which implies that $\Gamma^4_{43} = -\Gamma^3_{33}$. Now $(G1424)$ becomes

$$\Gamma^3_{12}\Gamma^3_{33} + 2\Gamma^1_{41}(\Gamma^4_{12} - 2\Gamma^2_{41}) = 0.$$

So there exist a constant β_1 such that

$$\Gamma^4_{12} = \beta_1\Gamma^3_{33} + 2\Gamma^2_{41},$$
$$\Gamma^3_{12} = -2\beta_1\Gamma^1_{41}.$$

Moreover from $(G3434)$ we deduce that $\lambda_2 = -2(\Gamma^3_{33})^2$ and from $(G1423)$ we get that $\Gamma^2_{41} = -\frac{\beta_1}{2}(2\Gamma^1_{31} + \Gamma^3_{33})$. Seeing the equation

$$G1413 \qquad 2(1 + \beta_1^2)(2\Gamma^1_{31} + \Gamma^3_{33})\Gamma^1_{41} = 0,$$

we have two small subcases to consider
1. if $\Gamma^1_{41} = 0$ then from solving

$$\begin{cases} G1313 & 2(\Gamma^1_{31} - \Gamma^3_{33})(2\Gamma^1_{31} + \Gamma^3_{33}) = 0, \\ G2323 & 2(2\Gamma^1_{31} - \Gamma^3_{33})(\Gamma^1_{31} + \Gamma^3_{33}) = 0, \end{cases}$$

we have that $\Gamma^1_{31} = 0$ and $\Gamma^3_{33} = 0$, which is a contradiction with condition of this case.
2. if $\Gamma^1_{41} \neq 0$ then $\Gamma^1_{31} = -\frac{1}{2}\Gamma^3_{33}$. From the equation

$$G2423 \qquad -4\left(\beta_1^2+1\right)\Gamma^3_{33}\Gamma^1_{41} = 0,$$

we again find that $\Gamma^3_{33} = 0$, which is a contradiction with our assumptions.

Type 2.2.a We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{41} = 0$ and $\Gamma^1_{31} \neq 0 \neq \Gamma^1_{11} + \Gamma^2_{12}$. By adding $G1211$ and $G1222$ it follows that $\Gamma^2_{11} = \Gamma^1_{21}$. From $G1423$ we get that

$$\Gamma^4_{12}(2\Gamma^1_{31} + \Gamma^4_{43}) - 4\Gamma^1_{31}\Gamma^2_{41} = 0.$$

As $\Gamma^1_{31} \neq 0$, there exists a constant β_1 such that

$$\Gamma^4_{43} = (4\beta_1 - 2)\Gamma^1_{31},$$
$$\Gamma^2_{41} = \beta_1\Gamma^4_{12}.$$

Similarly it follows from $G1323$ that we can write

$$\Gamma^3_{12} = \beta_2\Gamma^4_{33} + 2\Gamma^2_{31},$$
$$\Gamma^4_{12} = -2\beta_2\Gamma^1_{31}.$$

The equations (($G1314$) and ($G1414$)) now immediately imply respectively that

$$\Gamma^4_{33} = -2\beta_2\Gamma^2_{31}, \qquad \lambda_2 = -4(1 + 2\beta_1(-1+\beta_2^2))\Gamma^1_{31}.$$

Now the equations ($G1313$) and ($G2424$) reduces to

$$\begin{cases} -4\left(\beta_2^2-1\right)\left(2\beta_1(\Gamma^1_{31})^2 + (\Gamma^2_{31})^2\right) = 0, \\ -8(-1+2\beta_1)\left(\beta_2^2-1\right)(\Gamma^1_{31})^2 = 0. \end{cases}$$

Hence $\beta_2^2 = 1$. Note that if necessary by replacing e_3 with $-e_3$ we may assume that $\beta_2 = -1$. The equation ($G2322$) then become

$$(\Gamma^1_{21} - 5\Gamma^4_{24})\Gamma^1_{31} - 3\Gamma^2_{12}\Gamma^2_{31} = 0.$$

So there exists a constant β_3 such that

$$\Gamma^2_{31} = \beta_3\Gamma^1_{31}, \quad \Gamma^1_{21} = 3\beta_3\Gamma^2_{12} + 5\Gamma^4_{24}.$$

The equation of ($G1433$) becomes

$$-2\beta_1\Gamma^1_{31}(\Gamma^4_{24} - 2\Gamma^1_{43}) = 0.$$

If $\Gamma^4_{24} \neq 2\Gamma^1_{43}$, then $\beta_1 = 0$, and ($G3421$) gives $\beta_3 = 0$. The equations (($G3442$) and ($G2311$)) now immediately imply that $\Gamma^4_{24} = 2\Gamma^1_{43} = 0$ which is a contradiction. Therefore we always have $\Gamma^4_{24} = 2\Gamma^1_{43}$.
The equations (($G1444$), ($G1344$), ($G2444$), ($G2333$), ($G1322$) and ($G1441$)) immediately imply respectively

$$\Gamma^1_{34} = 0, \quad \Gamma^4_{14} = -2\Gamma^2_{34}, \quad \Gamma^2_{34} = 0, \quad \beta_3 = 0, \quad \Gamma^1_{11} = 0, \quad \lambda_1 = 0.$$

As $\Gamma^1_{11} = 0$ we must have that $\Gamma^2_{12} \neq 0$. A contradiction now follows from

$$G1331 : \quad -4(\Gamma^2_{12})^2 = 0.$$

Type 2.2.b We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = 0$. The conditions in this case now imply that $\Gamma^3_{12} \neq 0$ and $\Gamma^4_{12} = 0$. Using $(G1414)$ we get that $\lambda_2 = 0$. Adding $(G1313)$ and $(G2323)$, we get $(\Gamma^3_{12})^2 = 0$, which is again a contradiction.

Type 2.3.a We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = \Gamma^3_{12} = \Gamma^4_{12} = 0$. By the choice of E_3 and E_4 we also have that

$$\begin{cases} \Gamma^1_{34} = 0, \\ 2\Gamma^4_{14} + \Gamma^1_{11} + \Gamma^2_{12} \neq 0 \neq \Gamma^1_{11} + \Gamma^2_{12}. \end{cases} \tag{3.8}$$

From equations $(G1313)$ and $(G3434)$ we immediately see that

$$\lambda_2 = 0, \quad \Gamma^4_{33} = 0, \quad \Gamma^4_{43} = 0.$$

Adding $(G2322)$ and $(G2311)$, we get that

$$-(\Gamma^1_{11} + \Gamma^2_{12})\Gamma^2_{31} = 0.$$

So we deduce that $\Gamma^2_{31} = 0$. Similarly we obtain from adding $(G2422)$ and $(G2411)$ that $\Gamma^2_{41} = 0$. Similarly from adding $(G1211)$ and $(G1222)$ we obtain $\Gamma^2_{11} = \Gamma^1_{21}$. From (G1212) we obtain that

$$\lambda_1 = -2(\Gamma^1_{21})^2 + (\Gamma^1_{11} - \Gamma^2_{21})\Gamma^2_{21}.$$

From the equation $(G1211)$

$$3\Gamma^1_{21}(-\Gamma^2_{12} + \Gamma^2_{21}) = 0,$$

we obtain two cases,

Case 1: We assume that $\Gamma^2_{21} \neq \Gamma^2_{12}$ then $\Gamma^1_{21} = 0$. Moreover, from $(G1212)$ we deduce that $\Gamma^2_{12} = \frac{1}{2}\Gamma^1_{11}$.
Using the condition (3.8), from $(G1341)$ we immediately see that $\Gamma^4_{13} = 0$. The equation $(G1342)$ yields $\Gamma^2_{43} = 0$. Using $(G2431)$ we obtain $\Gamma^4_{23} = 0$. By subtracting equations $(G1332)$ and $(G1442)$ we get $\Gamma^4_{24} = 0$. We are now left with the equations $(G1331)$, $(G1441)$, $(G2332)$ and $(G2442)$ which state that

$$\begin{aligned} &(\Gamma^1_{11} - 2\Gamma^4_{14} - \Gamma^2_{21})(2\Gamma^4_{14} - \Gamma^2_{21}) = 0, \\ &(-4\Gamma^1_{11} - 2\Gamma^4_{14} + \Gamma^2_{21})(3\Gamma^1_{11} + 2\Gamma^4_{14} + \Gamma^2_{21}) = 0, \\ &\Gamma^2_{21}(-4\Gamma^1_{11} - 2\Gamma^4_{14} + \Gamma^2_{21}) = 0, \\ &\Gamma^2_{21}(-\Gamma^1_{11} + 2\Gamma^4_{14} + \Gamma^2_{21}) = 0. \end{aligned}$$

Solving the above system we get a contradiction with

$$2\Gamma^4_{14} + \Gamma^1_{11} + \Gamma^2_{12} = 2\Gamma^4_{14} + \tfrac{3}{2}\Gamma^1_{11} \neq 0.$$

Case 2: Assume that $\Gamma^2_{21} = \Gamma^2_{12}$. The equation $(G1234)$ is given by

$$(\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})\Gamma^4_{23} - 2\Gamma^4_{13}\Gamma^4_{24} = 0.$$

We assume that $\{\Gamma^4_{23}, \Gamma^4_{24}\}$ are not both vanishing. Then there exist a constant α such that

$$\Gamma^4_{13} = \alpha\Gamma^4_{23}, \qquad \Gamma^4_{14} = \alpha\Gamma^4_{24} - \frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12}).$$

The condition (3.8) becomes $2\alpha\Gamma^4_{24} \neq 0$. The equations $(G1243)$, $(G1431)$ and $(G3444)$ yield that $\Gamma^2_{43} = 0$, $\Gamma^4_{23} = 0$ and $\Gamma^2_{34} = 0$.
Subtracting $(G2332)$ and $(G2442)$ we see that

$$4\Gamma^4_{24}(\Gamma^1_{21} - \alpha\Gamma^2_{12}) = 0.$$

Using (3.8) we deduce that $\Gamma^1_{21} = \alpha_1\Gamma^2_{12}$. Similarly subtracting $(G2331)$ and $(G2441)$ we see that

$$-4\Gamma^4_{24}\left(\left(\alpha^2+2\right)\Gamma^2_{12} + \Gamma^1_{11}\right) = 0.$$

We obtain $\Gamma^1_{11} = -(2+\alpha_1^2)\Gamma^2_{12}$. The equations $(G1442)$ and $(G2332)$ now become

$$\begin{aligned}
&\alpha\left((\alpha^2+1)(\Gamma^2_{12})^2 - 4(\Gamma^4_{24})^2\right) = 0,\\
&4\left(\alpha^2+1\right)(\Gamma^2_{12})^2 - 4(\Gamma^4_{24})^2 = 0,
\end{aligned}$$

which leads to a contradiction.
Now if $\Gamma^4_{23} = \Gamma^4_{24} = 0$, then,
the fact that $\lambda_1 \neq 0$, implies from $(G1441)$ that $\Gamma^4_{14} \neq 0$. So $(G2441)$ immediately implies that $\Gamma^1_{21} = 0$.
Using (3.8), we deduce from $(G1243)$ and $(G1341)$ that $\Gamma^2_{43} = 0$ and $\Gamma^4_{13} = 0$ respectively. The equations $((G2442), (G1441), (G1331), (G2332))$ now reduce to

$$\begin{aligned}
&-(3\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14})(2\Gamma^1_{11} + 3\Gamma^2_{12} + 2\Gamma^4_{14}) = 0,\\
&-(\Gamma^1_{11} - \Gamma^2_{12} - 2\Gamma^4_{14})(\Gamma^2_{12} - 2\Gamma^4_{14}) = 0,\\
&-\Gamma^2_{12}(3\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) = 0,\\
&\Gamma^2_{12}(-\Gamma^1_{11} + \Gamma^2_{12} + 2\Gamma^4_{14}) = 0.
\end{aligned}$$

We immediately see that the above system admits no solution which satisfies the condition (3.8).

Type 2.3.b We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = \Gamma^3_{12} = \Gamma^4_{12} = \Gamma^1_{34} = 0$ and $\Gamma^4_{14} = -\frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12})$. By the choice of E_3 and E_4 we also have that

$$\begin{cases}\Gamma^2_{34} = 0,\\ \Gamma^4_{24} \neq 0 \neq \Gamma^1_{11} + \Gamma^2_{12}.\end{cases} \tag{3.9}$$

In exactly the same way as in the previous case we get $\lambda_2 = \Gamma^4_{33} = \Gamma^4_{43} = \Gamma^2_{31} = \Gamma^2_{41} = 0$ and $\Gamma^2_{11} = \Gamma^1_{21}$. From $(G1244)$ we obtain that $\Gamma^2_{21} = \Gamma^2_{12}$ and from $(G1432)$ and $(G2432)$ we see that $\Gamma^4_{13} = \Gamma^4_{23} = 0$. Subtracting $(G1331)$ and $(G1441)$ it follows that $\Gamma^1_{21} = 0$. Equation $(G2331)$ now implies that $\Gamma^2_{12} = -\frac{1}{2}\Gamma^1_{11}$. From $(G1212)$ we now see that λ_1 is negative and if necessary by replacing E_1 buy $-E_1$ we see that $\Gamma^1_{11} = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}$. Similarly from $(G2332)$ we get that $\Gamma^4_{24} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}$. Therefore we have shown the following lemma.

Lemma 3.2. *Let M be an affine homogeneous hypersurface which is of type 2.3.b. Then λ_1 is negative and $\lambda_2 = 0$. Moreover, we can choose a local frame*

$\{E_1, E_2, E_3, E_4\}$ *such that*

$$\Gamma^1_{11} = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^2_{12} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^3_{13} = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1},$$
$$\Gamma^4_{14} = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^2_{21} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^1_{22} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1},$$
$$\Gamma^3_{23} = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^4_{24} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^1_{33} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1},$$
$$\Gamma^2_{33} = \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^1_{44} = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, \qquad \Gamma^2_{44} = -\frac{2}{\sqrt{3}}\sqrt{-\lambda_1},$$

and all the other connection coefficients vanish.

From this we can now show the following.

Theorem 3.1. *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$. Assume the affine shape operator of M has two distinct real eigenvalues, both multiplicity 2 and that M is of Type 2.3.b. Then M is locally affine congruent to an open part of the hypersurface*

$$\left(x_1 - x_4^2\right)^3 \left(x_2 - x_5^2\right)^3 x_3^2 = 1. \tag{3.10}$$

Proof. Using the local frame of the previous lemma, we have that the Lie brackets are given by

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}E_3, \quad [E_1, E_4] = -\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}E_4,$$
$$[E_2, E_3] = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}E_3, \quad [E_2, E_4] = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}E_4, \quad [E_3, E_4] = 0.$$

So if we now take $E_1^* = E_1, E_2^* = E_2, E_3^* = \rho_1 E_3, E_4^* = \rho_2 E_4$, we see that with respect to the new frame the Lie brackets become

$$\begin{aligned}
&[E_1^*, E_2^*] = 0,\\
&[E_1^*, E_3^*] = (E_1(\rho_1) - \rho_1 \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1})E_3,\\
&[E_1^*, E_4^*] = (E_1(\rho_2) - \rho_2 \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1})E_4,\\
&[E_2^*, E_3^*] = (E_2(\rho_1) + \rho_1 \frac{1}{\sqrt{3}}\sqrt{-\lambda_1})E_3,\\
&[E_2^*, E_4^*] = (E_2(\rho_2) - \rho_1 \frac{1}{\sqrt{3}}\sqrt{-\lambda_1})E_4,\\
&[E_3^*, E_4^*] = 0.
\end{aligned}$$

Now a straightforward computations shows that the one forms ω_1 and ω_2 respectively defined by

$$\begin{aligned}
&\omega_1(E_1) = \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \qquad \omega_1(E_2) = -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && (3.11)\\
&\omega_1(E_3) = 0, \qquad \omega_1(E_4) = 0,\\
&\omega_2(E_1) = \frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}, \qquad \omega_2(E_2) = \frac{1}{\sqrt{3}}\sqrt{-\lambda_1}, && (3.12)\\
&\omega_2(E_3) = 0, \qquad \omega_2(E_4) = 0.
\end{aligned}$$

are closed one forms. Hence there exist local functions such that $\omega_1 = d\ln\rho_1$ and $\omega_2 = d\ln\rho_2$. Using these functions we now see that with respect to the new frame all connection coefficients vanish. Hence there exist local coordinates (u, v, x, y) such that

$$\frac{\partial}{\partial u} = E_1, \quad \frac{\partial}{\partial v} = E_2, \quad \frac{\partial}{\partial x}\rho_1 E_3, \quad \frac{\partial}{\partial y} = \rho_2 E_4. \tag{3.13}$$

By integrating (3.11) and (3.12), and (3.13), we get the functions ρ_1 and ρ_2

$$\rho_1 = \alpha_1\, e^{\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}(u-2v)}, \qquad \rho_2 = \alpha_2\, e^{\frac{1}{2\sqrt{3}}\sqrt{-\lambda_1}(u+2v)}, \tag{3.14}$$

where α_1 and α_2 are positive constants. Of course by translating the coordinates u and v we may assume that $\alpha_1 = \alpha_2 = 1$.

We now note the immersion $F : M \hookrightarrow \mathbb{R}^{n+1}$. From the definition of the affine shape operator and Lemma 3.2 we get that

$$\xi_u = -\lambda_1 F_u, \qquad \xi_v = -\lambda_1 F_v, \qquad \xi_x = \xi_y = 0,$$

and

$$\begin{aligned}
F_{uu} &= \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{uv} &= -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_v,\\
F_{ux} &= 0,\\
F_{uy} &= 0,\\
F_{vv} &= -\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{vx} &= 0,\\
F_{vy} &= 0,\\
F_{xx} &= \rho_1^2(-\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u + \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}\partial_v F_v + \xi),\\
F_{xy} &= 0,\\
F_{yy} &= \rho_2^2(-\frac{1}{\sqrt{3}}\sqrt{-\lambda_1}F_u - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_v + \xi).
\end{aligned} \tag{3.15}$$

Taking the derivative of the first equation of (3.15) we get

$$F_{uuu} - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_{uu} + \lambda_1 F_u = 0.$$

Its solutions are given by

$$F(u,v,x,y) = A_1(v,x,y) + A_2(v,x,y)e^{\mu_1 u} + A_3(v,x,y)e^{\mu_2 u}, \tag{3.16}$$

where $\mu_1 = -\frac{\sqrt{-\lambda_1}}{\sqrt{3}}$, $\mu_2 = \frac{3\sqrt{-\lambda_1}}{\sqrt{3}}$ and A_1, A_2, A_3 are functions in M. It immediately follows from (3.15) that A_2 depends only on v, A_1 dpends only on x and y and A_3 is a constant vector. Therefore

$$F(u,v,x,y) = A_1(x,y) + A_2(v)e^{\mu_1 u} + A_3 e^{\mu_2 u}. \tag{3.17}$$

It now follows that

$$\xi = F_{uu} - \frac{2}{\sqrt{3}}\sqrt{-\lambda_1}F_u = -\lambda_1(A_2(v)e^{\mu_1 u} + A_3 e^{\mu_2 u}).$$

It now follows that for A_2 we get the following differential equation:

$$A_2''(v) = -\tfrac{4}{3}A_2\lambda_1.$$

Its solution are

$$A_2(v) = c_1.e^{-2\mu_1 v} + c_2.e^{2\mu_1 v},$$

where c_1, c_2 are constant vectors. The remaining differential equations now become

$$\begin{aligned}
(A_1)_{xx} &= -\frac{8}{3}c_1\lambda_1,\\
(A_1)_{xy} &= 0,\\
(A_1)_{yy} &= -\frac{8}{3}c_2\lambda_1.
\end{aligned}$$

Therefore

$$A_1(x,y) = c_3 x + c_4 y - \frac{4}{3}c_1\lambda_1 x^2 - \frac{4}{3}c_2\lambda_1 y^2 + c_5.$$

Of course by a translation, we may now suppose that c_5 vanishes and by an affine transformation we may assume that c_1, c_2, c_3, c_4, A_3 is the canonical basis of $\mathbb{R}^5$. This completes the proof of the theorem. □

Type 2.3.c We have that $\Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{31} = \Gamma^1_{41} = \Gamma^3_{12} = \Gamma^4_{12} = \Gamma^1_{34} = \Gamma^2_{34} = 0$, $\Gamma^4_{14} = -\frac{1}{2}(\Gamma^1_{11} + \Gamma^2_{12})$ and $\Gamma^4_{24} = 0$. The difference tensor is given by

$$\begin{aligned}
&K(E_1,E_1) = \Gamma^1_{11}E_1 + \Gamma^1_{21}E_2, && K(E_1,E_2) = \Gamma^1_{21}E_1 + \Gamma^2_{12}E_2,\\
&K(E_2,E_2) = \Gamma^1_{21}E_1 + \Gamma^2_{12}E_2, && K(E_1,E_3) = \tfrac{-1}{2}(\Gamma^1_{11} + \Gamma^2_{12})E_3,\\
&K(E_2,E_3) = K(E_4,E_4) = \tfrac{-1}{2}(\Gamma^1_{11} + \Gamma^2_{12})E_1, && K(E_1,E_4) = \tfrac{-1}{2}(\Gamma^1_{11} + \Gamma^2_{12})E_4,\\
&K(E_2,E_3) = K(E_2,E_4) = K(E_3,E_4) = 0.
\end{aligned}$$

For this type of hypersurface we want to prove the following theorem.

Theorem 3.2. *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$ whose affine shape operator has two distinct real eigenvalues with multiplicity two. Assume that M is of Type 2.3.c. Then λ_1 is negative and λ_2 vanishes. Moreover M is affine equivalent with an open part of*

$$x_2^3\left(x_1-(x_3^2+x_4^2)-\frac{x_5^2}{x_2}\right)^5=1. \tag{3.18}$$

Before starting the proof of this theorem we first need the following lemma.

Lemma 3.3. *We can choose the frame $\{E_1,E_2,E_3,E_4\}$ such that*

$$\Gamma^1_{11}=-\frac{2}{\sqrt{15}}\sqrt{-\lambda_1},\qquad \Gamma^2_{12}=-\frac{1}{\sqrt{15}}\sqrt{-\lambda_1},\qquad \Gamma^3_{13}=\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$
$$\Gamma^4_{14}=\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},\qquad \Gamma^2_{21}=-\sqrt{\frac{5}{3}}\sqrt{-\lambda_1},\qquad \Gamma^1_{22}=\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$
$$\Gamma^1_{33}=\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},\qquad \Gamma^1_{44}=\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},$$

and all the other connection coefficients vanish.

Proof. We will use some of the Codazzi equations for the difference tensor K. A straightforward computation yield that

$$\begin{aligned}
&Glc1221 && 3(\Gamma^2_{11}-\Gamma^1_{21})\Gamma^1_{21}+(\Gamma^1_{11}-2\Gamma^2_{12})(\Gamma^2_{12}-\Gamma^2_{21})=0,\\
&Glc1331 && \frac{1}{2}(-(\Gamma^1_{11}+\Gamma^2_{12})(2\Gamma^1_{11}+\Gamma^2_{12})-\lambda_1+\lambda_2)=0,\\
&Glc1332 && -\frac{1}{2}\Gamma^2_{11}(\Gamma^1_{11}+\Gamma^2_{12})=0,\\
&Glc2331 && -\frac{1}{2}(\Gamma^1_{11}+\Gamma^2_{12})\Gamma^1_{21}=0,\\
&Glc2442 && \frac{1}{2}(-(\Gamma^1_{11}+\Gamma^2_{12})\Gamma^2_{21}-\lambda_1+\lambda_2)=0,\\
&Glc3432 && \frac{1}{2}(\Gamma^1_{11}+\Gamma^2_{12})\Gamma^2_{41}=0,\\
&Glc3442 && -\frac{1}{2}(\Gamma^1_{11}+\Gamma^2_{12})\Gamma^2_{31}=0,
\end{aligned}$$

where we denote $Glcijk\ell$ the ℓ-th component of the equation (2.11) with $X=E_i,\quad Y=E_j,\quad Z=E_k$. Note that of course there are other non vanishing components. These however will not be used in the proof of the lemma.

From the equations $((Glc2331),(Glc1332),(Glc3432),(Glc3442))$, and the fact that M is of Type 2.3.c, we get that $\Gamma^1_{21}=\Gamma^2_{11}=\Gamma^2_{41}=\Gamma^2_{31}=0$. In view of the Type of M we also have that $\Gamma^1_{11}\neq 0$. Solving now the three remaining equations we get after if necessary changing the sign of e_1 that

$$\Gamma^1_{11}=-\frac{2\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}},\quad \Gamma^2_{12}=-\frac{\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}},\quad \Gamma^2_{21}=-\sqrt{\frac{5}{3}}\sqrt{-\lambda_1+\lambda_2}.$$

The remaining unknown connection coefficients are $\{\Gamma^4_{13}, \Gamma^4_{23}, \Gamma^4_{33}, \Gamma^4_{43}\}$. Note however that these connection coefficients are not necessary constants.

The remaining Gauss equations become

$$\begin{aligned}
&G1221 && \lambda_2 = 0,\\
&G1234 && -\frac{4\Gamma^4_{23}\sqrt{-\lambda_1+\lambda_2}}{\sqrt{15}} - E_2(\Gamma^4_{13}) + E_1(\Gamma^4_{23}) = 0,\\
&G1343 && \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1+\lambda_2}\Gamma^4_{33} + \Gamma^4_{13}\Gamma^4_{43} + E_3(\Gamma^4_{13}) - E_1(\Gamma^4_{33}) = 0,\\
&G2343 && \Gamma^4_{23}\Gamma^4_{43} + E_3(\Gamma^4_{23}) - E_2(\Gamma^4_{33}) = 0,\\
&G2443 && -\Gamma^4_{23}\Gamma^4_{33} + E_4(\Gamma^4_{23}) - E_2(\Gamma^4_{43}) = 0,\\
&G3443 && -(\Gamma^4_{33})^2 - (\Gamma^4_{43})^2 - \lambda_2 + E_4(\Gamma^4_{33}) - E_3(\Gamma^4_{43}) = 0,\\
&G1434 && -\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1+\lambda_2}\Gamma^4_{43} + \Gamma^4_{13}\Gamma^4_{33} - E_4(\Gamma^4_{13}) + E_1(\Gamma^4_{43}) = 0.
\end{aligned}$$

The equation $(G1221)$ now immediately implies that $\lambda_2 = 0$. The other equations are correspond precisely to the conditions that the one form ω defined by

$$\omega(E_1) = -\Gamma^4_{13}, \quad \omega(E_2) = -\Gamma^4_{23}, \quad \omega(E_3) = -\Gamma^4_{33}, \quad \omega(E_4) = -\Gamma^4_{43}, \tag{3.19}$$

is a closed one form. Hence there exists a function φ such that $\omega = d\varphi$. We now consider the following change of frame

$$(E_1^*, E_2^*, E_3^*, E_4^*) = (E_1, E_2, \cos\varphi\, E_3 + \sin\varphi\, E_4, -\sin\varphi\, E_3 + \cos\varphi\, E_4),$$

This frame still satisfies all of the previous conditions. However we moreover have that

$$\begin{aligned}
\hat\nabla_{E_1^*} E_3^* &= \left(E_1(\varphi) + \Gamma^4_{13}\right) E_4^* = 0,\\
\hat\nabla_{E_2^*} E_3^* &= \left(E_2(\varphi) + \Gamma^4_{23}\right) E_4^* = 0,\\
\hat\nabla_{E_3^*} E_3^* &= \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_1^* + \left((E_3(\varphi) + \Gamma^4_{33})\cos\varphi + (E_4(\varphi) + \Gamma^4_{43})\sin\varphi\right) E_4^*\\
&= \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_1^*,\\
\hat\nabla_{E_4^*} E_3^* &= -\left((E_3(\varphi) + \Gamma^4_{33})\sin\varphi - (E_4(\varphi) + \Gamma^4_{43})\cos\varphi\right) E_4^* = 0.
\end{aligned}$$

The conclusion of Lemma 3.3 follows from these and the expression of K. □

Proof of Theorem 3.2. We use the local frame constructed in Lemma 3.3. The Lie brackets of the last frame are given by

$$\begin{aligned}
&[E_1, E_2] = \frac{4}{\sqrt{15}}\sqrt{-\lambda_1}E_2, \quad [E_1, E_3] = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_3,\\
&[E_1, E_4] = \frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}E_4, \quad [E_2, E_3] = [E_2, E_4] = [E_3, E_4] = 0.
\end{aligned}$$

Therefore we immediately see that the one forms ω_1 and ω_2 defined by

$$\begin{aligned}
&\omega_1(E_1) = -\frac{4}{\sqrt{15}}\sqrt{-\lambda_1}, \qquad \omega_2(E_1) = -\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1},\\
&\omega_1(E_2) = 0, \qquad \omega_1(E_3) = 0, \qquad \omega_1(E_4) = 0,\\
&\omega_2(E_2) = 0, \qquad \omega_2(E_3) = 0, \qquad \omega_2(E_4) = 0.
\end{aligned}$$

are closed one forms. Hence there exist positive functions ρ_i such that $\omega_i = d\ln\rho_i$. If we now look at the frame given by $E_1, \rho_1 E_2, \rho_2 E_3, \rho_2 E_4$ we have that all the Lie brackets vanish. So there exist local coordinates (u, v, x, y), such that

$$(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = (E_1, \rho_1 E_2, \rho_2 E_3, \rho_2 E_4).$$

Choosing the initial conditions for the functions ρ_i appropriately we get that

$$\rho_1 = e^{-\frac{4}{\sqrt{15}}\sqrt{-\lambda_1}.u}, \qquad \rho_2 = e^{-\frac{1}{2}\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}.u}.$$

In the same way as in the previous theorem we now determine the system of differential equations for the position vector F and the affine normal of the immersion. We get that

$$\xi_u = -\lambda_1 F_u, \qquad \xi_v = -\lambda_1 F_v, \qquad \xi_x = \xi_y = 0,$$

and

$$\begin{aligned}
F_{uu} &= -\frac{2}{\sqrt{15}}\sqrt{-\lambda_1}F_u + \xi,\\
F_{uv} &= -\frac{5}{\sqrt{15}}\sqrt{-\lambda_1}F_v,\\
F_{ux} &= 0,\\
F_{uy} &= 0,\\
F_{vv} &= \rho_1^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right),\\
F_{vx} &= 0,\\
F_{vy} &= 0,\\
F_{xx} &= \rho_2^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right),\\
F_{xy} &= 0,\\
F_{yy} &= \rho_2^2\left(\sqrt{\frac{3}{5}}\sqrt{-\lambda_1}F_u + \xi\right).
\end{aligned} \tag{3.20}$$

Deriving the first equation with respect to u, we get

$$F_{uuu} + \frac{2}{\sqrt{15}}\sqrt{-\lambda_1}F_{uu} + \lambda_1 F_u = 0.$$

Its solutions are given by

$$F(u, v, x, y) = A_1(v, x, y) + A_2(v, x, y)e^{\mu_1 u} + A_3(v, x, y)e^{\mu_2 u}, \tag{3.21}$$

where $\mu_1 = -\frac{5}{\sqrt{15}}\sqrt{-\lambda_1}$ and $\mu_2 = \frac{3}{\sqrt{15}}\sqrt{-\lambda_1}$, and A_1, A_2, A_3 are functions in M.
The other equations now immediately imply that A_3 is a constant vector, A_2 depends only on v and A_1 is the sum of a function depending only on x and a function depending only on y. The first equations also implies that

$$\xi = -\lambda_1(A_2(v,x,y)e^{\mu_1 u} + A_3(v,x,y)e^{\mu_2 u}).$$

The remaining differential equations now reduce to

$$\begin{aligned}(A_1)_{xx} &= -\tfrac{8}{5}A_3\lambda_1,\\ (A_1)_{yy} &= -\tfrac{8}{5}A_3\lambda_1,\\ (A_2)_{vv} &= -\tfrac{8}{5}A_3\lambda_1.\end{aligned}$$

So there exist constant vectors A_3, c_1, c_2, c_3, c_4, c_5 such that

$$\begin{aligned}F(u,v,x,y) = -\frac{4}{5}A_3\lambda_1(x^2+y^2) + c_1x + c_2y + c_5\\ + (-\frac{4}{5}\lambda_1 A_3 v^2 + c_3 v + c_4)e^{\mu_1 u} + A_3 e^{\mu_2 u}.\end{aligned}$$

As λ_1 is negative, by a homothetic transformation we may of course assume that $-\frac{4}{5}\lambda_1 = 1$. Next by applying an affine transformation we may that c_5 vanishes and c_1, c_2, c_3, c_4, A_5 are mapped to the standard basis. So we can write

$$F(u,v,x,y) = ((x^2+y^2) + v^2e^{\mu_1 u} + e^{\mu_2 u},\ e^{\mu_1 u},\ x,\ y,\ ve^{\mu_1 u}).$$

which is contained in

$$x_2^3\left(x_1 - (x_3^2 + x_4^2) - \frac{x_5^2}{x_2}\right)^5 = 1.$$

□

3.3. Hypersurfaces of type 3

In this case we have that both $V_1 = 0$ and $W_2 = 0$. We again will use properties of the difference tensor K in order to determine a unique frame. Note that by the classical Berwald theorem, we know that if K vanishes identically, M is congruent with a quadric (and therefore an affine sphere). So we may assume that K does not vanish identically. This leaves the following possibilities.

1. Suppose that the restriction of K to either $\mathcal{E}_1$ or $\mathcal{E}_2$ does not vanish identically. If necessary by interchanging both distributions we may assume that $h(K(\mathcal{E}_1), \mathcal{E}_1), \mathcal{E}_1)$ does not vanish identically. In that case we can pick e_1 such that $h(K(E_1, E_1), E_2) = 0$ and $h(K(E_1, E_1), E_1) \neq 0)$.
   (a) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can pick E_3 and E_4 in the direction of the corresponding eigenspaces,

(b) the symmetric operator K_{E_1} restricted to $\mathcal{E}_2$ is a multiple of the identity. Note that because $V_1 = 0$ this implies that K_{E_1} restricted to $\mathcal{E}_2$ vanishes. However the symmetric operator K_{E_2} restricted to $\mathcal{E}_2$ has two distinct eigenvalues. In this case we can again pick E_3 and E_4 in the direction of the corresponding eigenspaces,

(c) both the symmetric operators K_{E_1} and K_{E_2} restricted to $\mathcal{E}_2$ vanish. However $h(K(\mathcal{E}_2, \mathcal{E}_2), \mathcal{E}_2)$ does not vanish. In that case we can pick E_3 such that $h(K(E_3, E_3), E_4) = 0$ and $h(K(E_3, E_3), E_3) \neq 0)$

(d) We have $K(E_1, E_1) = \nu_1 E_1 = -K(E_2, E_2)$, $K(E_1, E_2) = -\nu_1 E_2$ and all other components of the difference tensor vanish. In this case we still have a rotation freedom in $\mathcal{E}_2$ and the frame is not determined uniquely.

2. Suppose that the restriction of K to both $\mathcal{E}_1$ and $\mathcal{E}_2$ vanishes identically. We can now look at the restriction of K to $\mathcal{E}_1$ and $\mathcal{E}_2$. This gives us two maps

$$K^1 = K|_{\mathcal{E}_1} : \mathcal{E}_1 \times \mathcal{E}_1 \to \mathcal{E}_2, \qquad K^2 = K|_{\mathcal{E}_2} : \mathcal{E}_2 \times \mathcal{E}_2 \to \mathcal{E}_1.$$

By the apolarity and the symmetry of K we now that $K(E_2, E_2) = -K(E_1, E_1)$ and $K(E_4, E_4) = -K(E_3, E_3)$. Hence the maps map the unit circle in respectively $\mathcal{E}_1$ and $\mathcal{E}_2$ to an ellipse (or a line), a circle or a point. Using the fact that K can not vanish identically this leaves us with the following possibilities:

(a) The image of K^1 is an ellipse (or line). We can choose E_3 and E_4 as the axes of the ellipse and we can choose E_1 such that $K(E_1, E_1)$ is in the direction of E_3.

(b) The image of both K^1 and K^2 are circles. We can pick E_3 and E_4 such that

$$\begin{aligned}
&K(E_1, E_1) = \nu_1 E_3 = -K(E_2, E_2), \quad K(E_1, E_2) = \nu_1 E_4,\\
&K(E_1, E_3) = \nu_1 E_1 + \nu_2 E_3 - \epsilon\nu_3 E_4,\\
&K(E_1, E_4) = \nu_1 E_2 - \epsilon\nu_3 E_3 - \nu_2 E_4,\\
&K(E_2, E_3) = -\nu_1 E_2 + \nu_3 E_3 + \epsilon\nu_2 E_4,\\
&K(E_2, E_4) = \nu_1 E_1 + \epsilon\nu_2 E_3 - \nu_3 E_4,\\
&K(E_3, E_3) = \nu_2 E_1 + \nu_3 E_2 = -K(E_4, E_4),\\
&K(E_3, E_4) = \epsilon(-\nu_3 E_1 + \nu_2 E_2),
\end{aligned}$$

where $\epsilon = \pm 1$. Making now the following change of basis $E_1^* = \cos\varphi.E_1 + \sin\varphi.E_2$, $E_2^* = -\sin\varphi.E_1 + \cos\varphi.E_2$, $E_3^* = \cos 2\varphi.E_3 + \sin 2\varphi.E_4$ and $E_4^* = -\sin 2\varphi.E_3 + \cos 2\varphi.E_4$, we see that $\nu_1^* = \nu_1$ and

$$\begin{aligned}
\nu_2^* = \tfrac{1}{2}(&\nu_2((1+\epsilon)\cos(3\varphi) + (\epsilon - 1)\cos(5\varphi))\\
&+ \nu_3(-(1+\epsilon)\sin(3\varphi) + (1-\epsilon)\sin(5\varphi)),\\
\nu_3^* = \tfrac{1}{2}(&\nu_2((1+\epsilon)\sin(3\varphi) + (1-\epsilon)\sin(5\varphi))\\
&+ \nu_3((1+\epsilon)\cos(3\varphi) - (1-\epsilon)\cos(5\varphi)).
\end{aligned}$$

Hence by assuming that ν_3^* has to vanish we can fix the frame completely.

(c) The image of K_1 is a circle and $h(K(\mathcal{E}_2, \mathcal{E}_2), \mathcal{E}_1) = 0$. Given E_1 and E_2, we can choose E_3 and E_4 such that

$$\begin{aligned} &K(E_1, E_1) = \nu_1 E_3 = -K(E_2, E_2), && K(E_1, E_2) = \nu_1 E_4, \\ &K(E_1, E_3) = \nu_1 E_1, && K(E_1, E_4) = \nu_1 E_2, \\ &K(E_2, E_3) = -\nu_1 E_2, && K(E_2, E_4) = \nu_1 E_1. \end{aligned}$$

Note that in this case making the following change of basis preserves the difference tensor
$E_1^* = \cos\varphi.E_1 + \sin\varphi.E_2$, $E_2^* = -\sin\varphi.E_1 + \cos\varphi.E_2$,
$E_3^* = \cos 2\varphi.E_3 + \sin 2\varphi.E_4$ and $E_4^* = -\sin 2\varphi.E_3 + \cos 2\varphi.E_4$.

So we see that except in case 3.1.d and case 3.2.c, the frame is uniquely determined and we can apply the Lemma 3.1.

Type 3: Case 1.a, Case 1.b and Case 1.c These cases can be treated simultaneously. We have that $\Gamma_{11}^1 = -\Gamma_{12}^2$, $\Gamma_{22}^2 = -\Gamma_{21}^1$, $\Gamma_{44}^4 = -\Gamma_{43}^3$ and $\Gamma_{34}^4 = -\Gamma_{33}^3$. Note that we have also that $\Gamma_{12}^2 \neq 0$, $\Gamma_{21}^1 = 0$ and $\Gamma_{34}^1 = 0$.

From $(G1211)$ we first get that $\Gamma_{11}^2 = 0$. The equation $(G1341)$ now implies that $\Gamma_{13}^4\Gamma_{14}^4 = 0$. Let us first assume that $\Gamma_{14}^4 = 0$. Then $(G1441)$ implies that $\lambda_1 = 0$. Then $(G1212)$ and $(G1221)$ respectively imply that

$$\Gamma_{21}^2(\Gamma_{12}^2 + \Gamma_{21}^2) = 0 = -(2\Gamma_{12}^2 - \Gamma_{21}^2)(3\Gamma_{12}^2 - \Gamma_{21}^2).$$

This leads to $\Gamma_{12}^2 = 0$ which is a contradiction.

Therefore we must have that $\Gamma_{14}^4 \neq 0$ and $\Gamma_{13}^4 = 0$. Adding $(G1221)$ and $(G1212)$ we get that $\Gamma_{21}^2 = \Gamma_{12}^2$. So that $(G1212)$ implies that $\lambda_1 = -2(\Gamma_{12}^2)^2$. From $(G1234)$, $(G1243)$, we then deduce that $\Gamma_{23}^4 = \Gamma_{34}^2 = 0$. Whereas $(G1434)$ and $(G1343)$ imply that $\Gamma_{43}^4 = 0$ and $\Gamma_{33}^4 = 2\Gamma_{43}^3$. The equations $(G1331)$ and $(G1441)$ now reduce to

$$2(\Gamma_{12}^2 - \Gamma_{14}^4)(\Gamma_{12}^2 + 2\Gamma_{14}^4) = 0 = 2(\Gamma_{12}^2 - 2\Gamma_{14}^4)(\Gamma_{12}^2 + \Gamma_{14}^4),$$

which again lead to the contradiction that $\Gamma_{12}^2 = 0$. Hence these cases are not possible.

Type 3.1.d Due to the form of the difference tensor we have that $\Gamma_{21}^1 = \Gamma_{34}^1 = \Gamma_{31}^1 = \Gamma_{41}^1 = \Gamma_{12}^3 = \Gamma_{12}^4 = \Gamma_{14}^4 = \Gamma_{24}^4 = \Gamma_{34}^2 = \Gamma_{33}^3 = \Gamma_{43}^3 = 0$. This implies that

$$h(\nabla_{\mathcal{E}_2}\mathcal{E}_2, \mathcal{E}_1) = 0,$$

and in view of the form of K also

$$h(\hat{\nabla}_{\mathcal{E}_2}\mathcal{E}_2, \mathcal{E}_1) = 0.$$

So by a direct computation we get that

$$(\hat{\nabla}K)(E_1, E_3, E_3) - (\hat{\nabla}K)(E_3, E_1, E_3) = h(E_1\hat{\nabla}_{E_3}E_3) = 0.$$

On the other hand, from the Codazzi equation for K, (2.11) we deduce that

$$(\hat{\nabla}K)(E_1, E_3, E_3) - (\hat{\nabla}K)(E_3, E_1, E_3) = \tfrac{1}{2}(\lambda_1 - \lambda_2)E_1.$$

Hence this case also leads to a contradiction.

Type 3.2 Case a or Case b These cases can be again treated simultaneously as we have again the possibility to define an affine invariant frame. From the choice of the frame we have that

$$\Gamma^1_{21} = \Gamma^2_{12} = \Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{41} = \Gamma^3_{12} = 0$$

Note also that in both Case a and Case b, we have that $\Gamma^1_{31} \neq 0$. Case a can be recognised by $\Gamma^4_{12} \neq \pm 2\Gamma^1_{31}$, whereas Case b can be recognised by $\Gamma^4_{14} \neq 0$, $\Gamma^4_{24} = 0$ and $\Gamma^4_{12} = \pm 2\Gamma^1_{31}$. From $(G1313)$ and $(G1311)$ it immediately follows that $\Gamma^4_{14} = 0$ (and therefore Case b can not occur) as well as that $\lambda_2 = -4(\Gamma^1_{31})^2$. Similarly from $(G2322)$ we deduce that $\Gamma^4_{24} = 0$. From $(G1212)$ we get that

$$\lambda_1 = -(\Gamma^2_{11})^2 - (\Gamma^2_{21})^2.$$

The equation $(G3443)$

$$4(\Gamma^1_{31})^2 = (\Gamma^4_{33})^2 + (\Gamma^4_{43})^2,$$

implies that Γ^4_{33} and Γ^4_{43} can not cancel simultaneously. Therefore from $(G1333)$ and $(G1444)$ it follows that $\Gamma^1_{34} = 0$. Similarly from $(G2333)$ and $(G2444)$ we get that $\Gamma^2_{34} = 0$. The equations $(G1213)$ and $(G1224)$ become

$$-\Gamma^4_{12}\Gamma^4_{13} + 4\Gamma^2_{11}\Gamma^1_{31} = 0 = 2\Gamma^2_{11}\Gamma^4_{12} - 2\Gamma^4_{13}\Gamma^1_{31}.$$

As we are necessary in Case (a), we have that $(\Gamma^4_{12})^2 - 4(\Gamma^1_{31})^2 \neq 0$ and therefore the previous system of equations implies that $\Gamma^4_{13} = \Gamma^2_{11} = 0$. Similarly from $(G1214)$ and $(G1223)$ we get that $\Gamma^2_{21} = \Gamma^4_{23} = 0$. Repeating the same argument respectively for $(G1314)$ and $(G1323)$ and $(G1423)$ and $(G1414)$ we have that $\Gamma^2_{31} = \Gamma^4_{33} = 0$ and $\Gamma^2_{41} = 0$ and $\Gamma^4_{43} = -2\Gamma^1_{31}$. A contradiction now follows from $(G2424)$ which becomes

$$2((\Gamma^4_{12})^2 - 4(\Gamma^1_{31})^2) = 0.$$

Type 3.2.c For this type of hypersurfaces, we want to show the following theorem

Theorem 3.3. *Let M^4 be a locally strongly convex, locally homogeneous, affine hypersurface in $\mathbb{R}^5$ whose affine shape operator has two distinct real eigenvalues both of multiplicity 2. Assume that M is of Type 3.2.c. Then $\lambda_1 = 0$ and λ_2 is negative. Moreover M is affine congruent with an open part of the affine hypersurface given by*

$$2x_2x_3x_4 - x_4^2 - x_1\left(x_3^2 - 2x_5\right) - 2x_2^2x_5 = 1.$$

Before starting the proof of the theorem, we need the following lemma.

Lemma 3.4. *We can choose the frame $\{E_1, E_2, E_3, E_4\}$ such that*

$$\Gamma^3_{11} = \sqrt{-\lambda_2}, \qquad \Gamma^4_{12} = \sqrt{-\lambda_2}, \qquad \Gamma^4_{21} = \sqrt{-\lambda_2},$$
$$\Gamma^3_{22} = -\sqrt{-\lambda_2}, \qquad \Gamma^1_{31} = \frac{\sqrt{-\lambda_2}}{2}, \qquad \Gamma^2_{32} = -\frac{\sqrt{-\lambda_2}}{2},$$
$$\Gamma^2_{41} = \sqrt{-\lambda_2}, \qquad \Gamma^4_{43} = \sqrt{-\lambda_2}, \qquad \Gamma^3_{44} = -\sqrt{-\lambda_2},$$

and all the other gammas are vanished.

Proof. From the choice of the frame we have that

$$\Gamma^1_{21} = \Gamma^2_{12} = \Gamma^3_{43} = \Gamma^3_{33} = \Gamma^1_{41} = \Gamma^3_{12} = 0.$$

We also have that, if necessary after changing the sign of e_2 that $\Gamma^4_{12} = 2\Gamma^1_{31} \neq 0$. Moreover, we have also $\Gamma^4_{14} = \Gamma^4_{24} = \Gamma^1_{34} = \Gamma^2_{34} = 0$.

Using the Codazzi equation for K we now find by a straightforward computation that

$$\begin{array}{ll}
Glc1231 & (2\Gamma^2_{11} - \Gamma^4_{13})\Gamma^1_{31} = 0, \\
Glc1241 & (2\Gamma^2_{21} - \Gamma^4_{23})\Gamma^1_{31} = 0, \\
Glc1331 & \frac{1}{2}\left(4(\Gamma^1_{31})^2 - \lambda_1 + \lambda_2\right) = 0, \\
Glc1341 & \Gamma^1_{31}(2\Gamma^2_{31} - \Gamma^4_{33}) = 0, \\
Glc2431 & \Gamma^1_{31}(2\Gamma^1_{31} - 2\Gamma^2_{41} + \Gamma^4_{43}) = 0.
\end{array}$$

Therefore we get that

$$\Gamma^4_{13} = 2\Gamma^2_{11}, \quad \Gamma^4_{33} = 2\Gamma^2_{31}, \quad \Gamma^4_{23} = 2\Gamma^2_{21}, \quad \Gamma^4_{43} = -2\Gamma^1_{31} + 2\Gamma^2_{41}.$$

Moreover, we have that

$$\Gamma^1_{31} = \frac{\sqrt{\lambda_1 - \lambda_2}}{2}.$$

The remaining unknown coefficients coefficients are $\Gamma^2_{11}, \Gamma^2_{21}, \Gamma^2_{31}, \Gamma^2_{41}$. As the frame is not unique these are not necessarily constants.

Now the Gauss equation ($G1331$) is given by

$$\lambda_1 = 0.$$

The final Gauss equations now reduce to

$$\begin{array}{ll}
G1212 & (\Gamma^2_{11})^2 + (\Gamma^2_{21})^2 - E_2(\Gamma^2_{11}) + E_1(\Gamma^2_{21}), \\
G1312 & \Gamma^2_{21}\Gamma^2_{31} - 2\Gamma^2_{11}\Gamma^2_{41} + \frac{3}{2}\Gamma^2_{11}\sqrt{-\lambda_2} - E_3(\Gamma^2_{11}) + E_1(\Gamma^2_{31}) = 0, \\
G1334 & 2\Gamma^2_{21}\Gamma^2_{31} - 4\Gamma^2_{11}\Gamma^2_{41} + 3\Gamma^2_{11}\sqrt{-\lambda_2} - 2E_3(\Gamma^2_{11}) + 2E_1(\Gamma^2_{31}) = 0, \\
G1412 & 2\Gamma^2_{11}\Gamma^2_{31} + \Gamma^2_{21}\Gamma^2_{41} - E_4(\Gamma^2_{11}) + E_1(\Gamma^2_{41}) = 0, \\
G2312 & -\Gamma^2_{11}\Gamma^2_{31} - 2\Gamma^2_{21}\Gamma^2_{41} + +\frac{1}{2}\Gamma^2_{21}\sqrt{-\lambda_2} - E_3(\Gamma^2_{21}) + E_2(\Gamma^2_{31}) = 0, \\
G2412 & 2\Gamma^2_{21}\Gamma^2_{31} - \Gamma^2_{11}\Gamma^2_{41} + \Gamma^2_{11}\sqrt{-\lambda_2} - E_4(\Gamma^2_{21}) + E_2(\Gamma^2_{41}) = 0, \\
G3412 & 2\left((\Gamma^2_{31})^2 + (\Gamma^2_{41})^2 - \Gamma^2_{41}\sqrt{-\lambda_2}\right) - E_4(\Gamma^2_{31}) + E_3(\Gamma^2_{41}) = 0.
\end{array}$$

These equations are precisely the existence conditions, see Appendix 2 of [7]

$$[E_i, E_j]\varphi = \nabla_{E_i}E_j(\varphi) - \nabla_{E_j}E_i(\varphi) = E_i(E_j(\varphi)) - E_j(E_i(\varphi)), \text{ for } 1 \leq i, j \leq 4.$$

for the following system of differential equations

$$\begin{cases} E_1(\varphi) = -\Gamma^2_{11}, \\ E_2(\varphi) = -\Gamma^2_{21}, \\ E_3(\varphi) = -\Gamma^2_{31} - \frac{\sqrt{-\lambda_2}}{2}\sin 2\varphi, \\ E_4(\varphi) = -\Gamma^2_{41} + \frac{\sqrt{-\lambda_2}}{2}\cos 2\varphi + \frac{\sqrt{-\lambda_2}}{2}, \end{cases} \tag{3.22}$$

Taking therefore a solution of this system and using the freedom in the frame to define,

$$\begin{cases} E_1^* = \cos\varphi\,.E_1 + \sin\varphi\,.E_2, \\ E_2^* = -\sin\varphi\,.E_1 + \cos\varphi\,.E_2, \end{cases} \qquad \begin{cases} E_3^* = \cos 2\varphi\,.E_3 + \sin 2\varphi\,.E_4, \\ E_4^* = -\sin 2\varphi\,.E_3 + \cos 2\varphi\,.E_4, \end{cases}$$

we see that the new frame satisfies all previous conditions and that moreover:

$$\begin{aligned} \hat{\nabla}_{E_1^*}E_1^* &= \left((E_1(\varphi)+\Gamma^2_{11})\cos\varphi + (E_2(\varphi)+\Gamma^2_{21})\sin\varphi\right)E_2^* + \frac{\sqrt{-\lambda_2}}{2}E_3^* \\ &= \frac{\sqrt{-\lambda_2}}{2}E_3^*, \\ \hat{\nabla}_{E_2^*}E_1^* &= \left(-(E_1(\varphi)+\Gamma^2_{11})\sin\varphi + (E_2(\varphi)+\Gamma^2_{21})\cos\varphi\right)E_2^* + \frac{\sqrt{-\lambda_2}}{2}E_4^* \\ &= \frac{\sqrt{-\lambda_2}}{2}E_4^*, \\ \hat{\nabla}_{E_3^*}E_3^* &= 2\left((E_3(\varphi)+\Gamma^2_{31})\cos 2\varphi + \left(E_4(\varphi)+\Gamma^2_{41} - \frac{\sqrt{-\lambda_2}}{2}\right)\sin 2\varphi\right)E_4^* = 0, \\ \hat{\nabla}_{E_4^*}E_3^* &= 2\left(-(E_3(\varphi)+\Gamma^2_{31})\sin 2\varphi + \left(E_4(\varphi)+\Gamma^2_{41} - \frac{\sqrt{-\lambda_2}}{2}\right)\cos 2\varphi\right)E_4^* \\ &= \frac{\sqrt{-\lambda_2}}{2}E_4^*, \end{aligned}$$

Therefore we may assume that

$$\Gamma^2_{11} = 0, \quad \Gamma^2_{21} = 0, \quad \Gamma^2_{31} = 0, \quad \Gamma^2_{41} = \sqrt{-\lambda_2}.$$

This completes the proof of the lemma. □

Proof of Theorem 3.3. We use the local frame constructed in the previous lemma. By applying a homothety we may assume that $\lambda_2 = -1$. The Lie brackets of the last frame are given by

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -\frac{\sqrt{-\lambda_2}}{2}E_1, \quad [E_1, E_4] = -\sqrt{-\lambda_2}E_2,$$
$$[E_2, E_3] = \frac{\sqrt{-\lambda_2}}{2}E_2, \quad [E_2, E_4] = 0, \quad [E_3, E_4] = -\sqrt{-\lambda_2}E_2.$$

Therefore a straightforward computation shows that the integrabilty conditions, see Appendix 2 of [7], of the following system of differential equations

for the functions ρ_1, ρ_2 and ρ_3

$$\begin{aligned}
&E_1(\rho_1) = 0, \quad E_2(\rho_1) = 0, \quad E_3(\rho_1) = \rho_1 \frac{\sqrt{-\lambda_2}}{2}, \quad E_4(\rho_1) = 0,\\
&E_1(\rho_2) = 0, \quad E_2(\rho_2) = 0, \quad E_3(\rho_2) = 0, \quad E_4(\rho_2) = \frac{\sqrt{-\lambda_2}}{\rho_1^2},\\
&E_1(\rho_3) = \rho_1\sqrt{-\lambda_2}, \quad E_2(\rho_3) = 0, \quad E_3(\rho_3) = 0, \quad E_4(\rho_3) = 0,
\end{aligned}$$

are satisfied. We now define

$$\begin{aligned}
E_1^* &= \tfrac{1}{\rho_1} E_1 - \tfrac{1}{2}\rho_1\rho_2 E_2,\\
E_2^* &= \rho_1 E_2,\\
E_3^* &= E_3,\\
E_4^* &= \tfrac{1}{2}\rho_1\rho_3 E_2 + \rho_1^2 E_4.
\end{aligned}$$

We then get that $[E_i^*, E_j^*] = 0$. So there exist local coordinates (x, z, u, y), such that

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial u}, \frac{\partial}{\partial y}) = (\tfrac{1}{\rho_1} E_1 - \tfrac{1}{2}\rho_1\rho_2 E_2, \rho_1 E_2, E_3, \tfrac{1}{2}\rho_1\rho_3 E_2 + \rho_1^2 E_4).$$

Choosing the initial conditions for the functions ρ_i appropriately we get that

$$\rho_1 = e^{\frac{\sqrt{-\lambda_2}}{2}u}, \qquad \rho_2 = y, \qquad \rho_3 = x.$$

Note that

$$\begin{cases}
E_1 = \rho_1 \left(\partial_x + \dfrac{1}{2} y \partial_z\right),\\
E_2 = \dfrac{1}{\rho_1}\partial_z,\\
E_3 = \partial_u,\\
E_4 = \dfrac{1}{\rho_1^2}\left(\partial_y - \dfrac{1}{2} x \partial_z\right).
\end{cases}$$

In the same way as in the previous theorems we now determine the system of differential equations for the position vector F and the affine normal of the immersion. We get that

$$\xi_x = \xi_z = 0, \qquad \xi_u = F_u, \qquad \xi_y = F_y - \tfrac{1}{2} x F_z,$$

and

$$\begin{aligned}
F_{uu} &= \xi,\\
F_{ux} &= 0,\\
F_{uy} &= F_y - \tfrac{1}{2}xF_z,\\
F_{uz} &= 0,\\
F_{zz} &= e^u\left(-F_u + \xi\right),\\
F_{zx} &= \tfrac{1}{2}e^{-u}\left(2F_y - xF_z + ye^{2u}F_u - ye^{2u}\xi\right),\\
F_{zy} &= \tfrac{1}{2}xe^u\left(-F_u + \xi\right),\\
F_{xx} &= \frac{1}{4}e^{-u}\left(\left(4 - e^{2u}y^2\right)F_u - 4F_y + 2xyF_z + \left(-e^{2u}y^2 + 4\right)\xi\right),\\
F_{xy} &= \tfrac{1}{4}e^{-u}\left(x\left(e^{2u}yF_u + 2F_y - e^{2u}\xi y\right) + \left(2e^u - x^2\right)F_z\right),\\
F_{yy} &= \tfrac{1}{4}e^u(4e^u + x^2)\left(-F_u + \xi\right).
\end{aligned} \tag{3.23}$$

Deriving the first equation with respect to u, we get

$$F_{uuu} - F_u = 0.$$

Its solutions are given by

$$F(u,x,y,z) = A_1(x,y,z) + A_2(x,y,z)e^u + A_3(x,y,z)e^{-u}, \tag{3.24}$$

The other differential equations involving the variable u then imply that A_3 is a constant, A_2 depends only on the variable y and

$$(A_1)_y = \tfrac{1}{2}x(A_1)_z. \tag{3.25}$$

From the differential equations F_{zz} we obtain that

$$\xi = e^{-u}\left((A_1)_{zz} - A_3 + A_2e^{2u}\right).$$

Substituting this in the differential equation for F_{yy} we see that

$$(A_1)_{zz} = (A_2)_{yy}$$

The differential equations for ξ imply also that

$$(A_1)_{zz} = 2A_3.$$

Therefore it follows that we can write

$$A_2(y) = A_3y^2 + C_1y + C_2,$$

where C_1 and C_2 are constant vectors. Moreove, using also the differential equation for F_{xz} we see that we can write

$$A_1(x,y,z) = A_3z^2 + (xyA_3 + C_1x + B_1(y))z + B_2(x,y).$$

Expressing now (3.25) yields

$$xzA_3 + (B_1)_yz + (B_2)_y = xzA_3 + \tfrac{1}{2}(x^2yA_3 + C_1x^2 + B_1(y)x)).$$

Therefore $B_1(y) = C_3$ a constant vector and

$$B_2(x,y) = \tfrac{1}{4}x^2y^2A_3 + \tfrac{1}{2}x^2yC_1 + \tfrac{1}{2}xyC_3 + B_3(x).$$

The only remaining equation of our system of differential equations now becomes

$$(B_3)_{xx} = 2C_2.$$

So after applying a translation in $\mathbb{R}^5$ we see that we can write

$$B_3(x) = C_2x^2 + C_4x.$$

Mapping now $C_2, C_4, C_3, C_1, 2A_3$ to the standard basis completes the proof of the theorem □

Note that all of the examples appearing in the theorem are indeed affine homogeneous. As the first two already appear as generalised Calabi products in [3] we only have to consider the final example.

We look at the set of equiaffine matrices given by

$$b(x,y,z,u) = \begin{pmatrix} e^u & 2e^{u/2}x & 0 & 0 & 0 & x^2 \\ 0 & e^{u/2} & 0 & 0 & 0 & x \\ 0 & e^{u/2}y & e^{-u/2} & 0 & 0 & \frac{1}{2}(xy+2z) \\ e^u y & e^{u/2}\left(\frac{3xy}{2}+z\right) & e^{-u/2}x & 1 & 0 & \frac{1}{2}x(xy+2z) \\ \frac{e^u y^2}{2} & \frac{1}{2}e^{u/2}y(xy+2z) & e^{-u/2}\left(\frac{xy}{2}+z\right) & y & e^{-u} & \frac{1}{8}(xy+2z)^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As

$$\begin{aligned} &b(x,y,z,u)b(x',y',z',u') \\ &\quad = b(x+e^{\frac{u}{2}}x', y+e^{-u}y', -\tfrac{1}{2}e^{-u}xy' + \tfrac{1}{2}e^{u/2}x'y + e^{-u/2}z' + z, u+u'), \end{aligned}$$

the set of above matrices forms a group. It is clear that the image of our surface is the orbit of the point $(1,0,0,0,1,1)$.

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