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## On some curvature conditions of pseudosymmetry type

Ryszard Deszcz · Małgorzata Głogowska ·  
Marian Hotłoś · Georges Zafindratafa

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**Abstract** It is known that the difference tensor  $R \cdot C - C \cdot R$  and the Tachibana tensor  $Q(S, C)$  of any semi-Riemannian Einstein manifold  $(M, g)$  of dimension  $n \geq 4$  are linearly dependent at every point of  $M$ . More precisely  $R \cdot C - C \cdot R = (1/(n-1)) Q(S, C)$  holds on  $M$ . In the paper we show that there are quasi-Einstein, as well as non-quasi-Einstein semi-Riemannian manifolds for which the above mentioned tensors are linearly dependent. For instance, we prove that every non-locally symmetric and non-conformally flat manifold with parallel Weyl tensor (essentially conformally symmetric manifold) satisfies  $R \cdot C = C \cdot R = Q(S, C) = 0$ . Manifolds with parallel Weyl tensor having Ricci tensor of rank two form a subclass of the class of Roter type manifolds. Therefore we also investigate Roter type manifolds for which the tensors  $R \cdot C - C \cdot R$  and  $Q(S, C)$  are linearly dependent. We determine necessary and sufficient conditions for a Roter type manifold to be a manifold having that property.

**Keywords** Einstein manifold · Quasi-Einstein manifold · Manifold with parallel Weyl tensor · Roter type manifold · Pseudosymmetric manifold · Generalized Einstein metric condition · Tachibana tensor

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R. Deszcz (✉) · M. Głogowska  
Department of Mathematics, Wrocław University of Environmental and Life Sciences,  
Grunwaldzka 53, 50-357 Wrocław, Poland  
e-mail: ryszard.deszcz@up.wroc.pl

M. Głogowska  
e-mail: malgorzata.glogowska@up.wroc.pl

M. Hotłoś  
Institute of Mathematics and Computer Science, Wrocław University of Technology,  
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland  
e-mail: marian.hotlos@pwr.edu.pl

G. Zafindratafa  
Laboratoire de Mathématiques et Applications de Valenciennes,  
Université de Valenciennes et du Hainaut-Cambrésis, Le Mont Houy,  
Lamath-ISTV2, 59313 Valenciennes Cedex 9, France  
e-mail: georges.zafindratafa@univ-valenciennes.fr

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### 1 Introduction

Let  $\nabla$ ,  $R$ ,  $S$ ,  $\kappa$  and  $C$  be the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Ricci tensor, the scalar curvature tensor and the Weyl conformal curvature tensor of a semi-Riemannian manifold  $(M, g)$ ,  $n = \dim M \geq 2$ , respectively.

It is well-known that the manifold  $(M, g)$ ,  $n \geq 3$ , is said to be an *Einstein manifold* ([1]) if at every point of  $M$  its Ricci tensor  $S$  is proportional to the metric tensor  $g$ , i.e.,  $S = \frac{\kappa}{n} g$  on  $M$ . In particular, if  $S$  vanishes on  $M$  then it is called *Ricci flat*. We denote by  $\mathcal{U}_S$  the set of all points of  $(M, g)$  at which  $S$  is not proportional to  $g$ , i.e.,  $\mathcal{U}_S := \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ . The manifold  $(M, g)$ ,  $n \geq 3$ , is said to be a *quasi-Einstein manifold* if at every point  $x \in \mathcal{U}_S$  we have  $\text{rank}(S - \alpha g) = 1$ , for some  $\alpha \in \mathbb{R}$ , i.e.,  $S = \alpha g + \varepsilon w \otimes w$ , for some  $\alpha \in \mathbb{R}$ , where  $w$  is a non-zero covector at  $x$  and  $\varepsilon = \pm 1$ . We mention that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, e.g., [2] and references therein.

An extension of the class of semi-Riemannian Einstein manifolds is formed by the manifolds for which we have  $\nabla S = 0$ . Manifolds satisfying the last condition are called *Ricci-symmetric*. *Locally symmetric manifolds*, for which we have  $\nabla R = 0$ , constitute an important subclass of the class of Ricci-symmetric manifolds. The last relation implies the integrability condition

$$\mathcal{R}(X, Y) \cdot R = 0, \tag{1.1}$$

where  $\mathcal{R}(X, Y) \cdot$  denotes the derivation obtained from the curvature endomorphism  $\mathcal{R}(X, Y)$  and  $X, Y$  are vector fields on  $M$ . We refer to Sect. 2 for the precise definitions of the symbols used here. Manifolds satisfying (1.1) are called *semisymmetric manifolds* ([3]). Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , is said to be *pseudosymmetric* ([2, 4, 5]) if the tensors  $\mathcal{R}(X, Y) \cdot R$  and  $(X \wedge_g Y) \cdot R$  are linearly dependent at every point of  $M$ . This is equivalent to

$$\mathcal{R}(X, Y) \cdot R = L_R (X \wedge_g Y) \cdot R \tag{1.2}$$

on  $\mathcal{U}_R := \{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$ , where  $L_R$  is some function on this set and the tensor  $G$  is defined by  $G(X, Y, W, Z) = g(X \wedge_g Y(Z), W)$ . A geometric interpretation of the notion of pseudosymmetry is given in [6]. Further, a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , is said to be a *manifold with pseudosymmetric Weyl tensor* ([2, 4, 7]) if the tensors  $\mathcal{C}(X, Y) \cdot C$  and  $(X \wedge_g Y) \cdot C$  are linearly dependent at every point of  $M$ . This is equivalent to

$$\mathcal{C}(X, Y) \cdot C = L_C (X \wedge_g Y) \cdot C \tag{1.3}$$

on  $\mathcal{U}_C := \{x \in M \mid C \neq 0 \text{ at } x\}$ , where  $L_C$  is some function on this set,  $\mathcal{C}(X, Y) \cdot$  denotes the derivation obtained from the Weyl conformal curvature endomorphism  $\mathcal{C}(X, Y)$ , and the Weyl conformal curvature tensor  $C$  is defined by  $C(X, Y, W, Z) = g(\mathcal{C}(X, Y)(Z), W)$ . It is known that (1.3) is invariant under the conformal deformations of the metric tensor  $g$ . We also note that  $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ .

In what follows, for a  $(0, k)$ -tensor  $T$  and a symmetric  $(0, 2)$ -tensor  $A$  on a manifold  $(M, g)$  we will denote the tensors  $\mathcal{R}(X, Y) \cdot T$ ,  $\mathcal{C}(X, Y) \cdot T$  and  $(X \wedge_A Y) \cdot T$  by  $R \cdot T$ ,

$C \cdot T$  and  $Q(A, T)$ , respectively. The tensor  $Q(A, T)$  is called the *Tachibana tensor* (see, e.g., [8]). In particular, we have the following  $(0, 6)$ -tensors:  $R \cdot R, R \cdot C, C \cdot R, C \cdot C$  and the Tachibana tensors:  $Q(g, R), Q(S, R), Q(g, C), Q(S, C)$ . Then we can write (1.2) and (1.3) in the form

$$R \cdot R = L_R Q(g, R) \tag{1.4}$$

and

$$C \cdot C = L_C Q(g, C), \tag{1.5}$$

respectively. We note that if (1.4) and (1.5) hold on the subset  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C$  of a semi-Riemannian manifold  $(M, g), n \geq 4$ , then

$$Q \left( S - \left( L_C - L_R + \frac{\kappa}{n-1} \right) g, C - \lambda G \right) = 0 \tag{1.6}$$

on this set, where  $\lambda$  is some function ([7] Theorem 3.1). In addition, if  $(M, g)$  is a non-quasi-Einstein manifold then from (1.6) it follows that on some open subset  $\mathcal{U}_1 \subset \mathcal{U}$  its curvature tensor  $R$  is a linear combination of the Kulkarni–Nomizu products  $S \wedge S, g \wedge S$  and  $G = \frac{1}{2} g \wedge g$ , i.e.,

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G \tag{1.7}$$

on  $\mathcal{U}_1$ , where  $\phi, \mu$  and  $\eta$  are some functions on this set ([7] Theorem 3.2 (ii)). A semi-Riemannian manifold  $(M, g), n \geq 4$ , satisfying (1.7) on  $\mathcal{U}_S \cap \mathcal{U}_C \subset M$  is called a *Roter type manifold* ([9]). We refer to [10] for a survey on that class of manifolds.

We can check that on any Einstein manifold  $(M, g), n \geq 4$ , the tensors  $Q(g, R), Q(S, R), Q(g, C)$  and  $Q(S, C)$  satisfy

$$\frac{\kappa}{n} Q(g, R) = Q(S, R) = Q(S, C) = \frac{\kappa}{n} Q(g, C). \tag{1.8}$$

Further, in [11](Theorem 3.1) it was stated that on every Einstein manifold  $(M, g), n \geq 4$ , the following identity is satisfied:

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g, R). \tag{1.9}$$

The remarks above lead to the problem of investigation of curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds  $(M, g), n \geq 4$ , satisfying at every point of  $M$  the curvature condition, of the following form: the difference tensor  $R \cdot C - C \cdot R$  is proportional to  $Q(g, R), Q(S, R), Q(g, C)$  and  $Q(S, C)$ . Such conditions are strongly related to some pseudosymmetry type curvature conditions, see, e.g., [2] and references therein. In Sect. 2 we present the definitions of the most important conditions of pseudosymmetry type. For instance, (1.2) and (1.3) are conditions of this kind. We also note that there are manifolds for which the difference tensor  $R \cdot C - C \cdot R$  is a linear combination of the Tachibana tensors above, see, e.g., Sect. 5 of the present paper, [12](Theorem 5.1) and [13](Propositions 2.1 and 3.2).

In this paper we will investigate semi-Riemannian manifolds  $(M, g), n \geq 4$ , satisfying at every point of  $M$  the following condition:

$$\text{the tensors } R \cdot C - C \cdot R \text{ and } Q(S, C) \text{ are linearly dependent.} \tag{*}$$

It is obvious that (\*) is satisfied at every point of  $M$  at which  $C$  vanishes. It is also clear that (1.8) and (1.9) imply that

$$R \cdot C - C \cdot R = \frac{1}{n-1} Q(S, C)$$

holds on any Einstein manifold  $(M, g), n \geq 4$ . Therefore we will restrict our considerations to manifolds  $(M, g), n \geq 4$ , satisfying (\*) on the set  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ . We will investigate on  $\mathcal{U}$  the condition

$$R \cdot C - C \cdot R = L Q(S, C), \tag{1.10}$$

where  $L$  is some function on this set. We mention that if the tensor  $R \cdot C - C \cdot R$  vanishes on  $\mathcal{U}$ , then on this set we have ([11] Theorem 4.1)

$$R \cdot C = C \cdot R = 0. \tag{1.11}$$

On the other hand, if  $Q(S, C)$  vanishes on  $\mathcal{U}$  then at every point  $x \in \mathcal{U}$  we have: (i) if  $\text{rank } S = 1$  at  $x$  then  $\kappa = 0$  and (1.11) hold at  $x$  (see Sect. 3) or (ii) if  $\text{rank } S > 1$  at  $x$  then  $C \cdot R = 0$  and  $R \cdot C = \frac{\kappa}{n-1} Q(g, C)$  hold at  $x$  (see Sect. 4). Thus we see that in the case (ii), if a manifold satisfies (\*) then its scalar curvature must vanish on  $\mathcal{U}$ .

The main result of Sect. 3 (Theorem 3.4) states that pseudosymmetric manifolds satisfying some additional curvature conditions are quasi-Einstein manifolds satisfying the conditions:  $C \cdot C = 0, C \cdot R = 0$  and (1.10) with the function  $L = \frac{1}{n-1}$ . In that section an example of warped product manifolds satisfying assumptions of Theorem 3.4 is also given.

The main result of Sect. 4 states that every essentially conformally symmetric manifold (e.c.s. manifold) satisfies  $R \cdot C = C \cdot R = Q(S, C) = 0$ . We refer to [14–17] for recent results on e.c.s. manifolds. We also mention that some e.c.s. metrics are realized on compact manifolds ([15, 17]).

In Sect. 5 Roter type manifolds satisfying (1.10) are investigated. We prove (Theorem 5.2) that if  $(M, g), n \geq 4$ , is a Roter type manifold with vanishing scalar curvature  $\kappa$  on  $\mathcal{U} \subset M$  then (1.10), with  $L = -1$ , holds on this set. In Theorem 5.3 we present some converse statement. We show (Example 5.4) that under some conditions the Cartesian product of two semi-Riemannian spaces of constant curvature satisfies assumptions of Theorem 5.3.

## 2 Preliminaries

Throughout this paper, all manifolds are assumed to be connected paracompact manifolds of class  $C^\infty$ . Let  $(M, g)$  be an  $n$ -dimensional,  $n \geq 3$ , semi-Riemannian manifold, let  $\nabla$  be its Levi-Civita connection and  $\mathfrak{X}(M)$  the Lie algebra of vector fields on  $M$ . We define on  $M$  the endomorphisms  $X \wedge_A Y$  and  $\mathcal{R}(X, Y)$  of  $\mathfrak{X}(M)$  by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$$

and

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

respectively, where  $A$  is a symmetric  $(0, 2)$ -tensor on  $M$  and  $X, Y, Z \in \mathfrak{X}(M)$ . The Ricci tensor  $S$ , the Ricci operator  $\mathcal{S}$  and the scalar curvature  $\kappa$  of  $(M, g)$  are defined by  $S(X, Y) = \text{tr}\{Z \rightarrow \mathcal{R}(Z, X)Y\}$ ,  $g(SX, Y) = S(X, Y)$  and  $\kappa = \text{tr } \mathcal{S}$ , respectively. The endomorphism  $\mathcal{C}(X, Y)$  is given by

$$C(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2} \left( X \wedge_g SY + SX \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right) Z.$$

Finally, the  $(0, 4)$ -tensor  $G$ , the Riemann–Christoffel curvature tensor  $R$  and the Weyl conformal curvature tensor  $C$  of  $(M, g)$  are defined by

$$\begin{aligned} G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \\ R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(\mathcal{C}(X_1, X_2)X_3, X_4), \end{aligned}$$

respectively, where  $X_1, X_2, \dots \in \mathfrak{X}(M)$ .

Let  $\mathcal{B}$  be a tensor field sending any  $X, Y \in \mathfrak{X}(M)$  to a skew-symmetric endomorphism  $\mathcal{B}(X, Y)$ , and let  $B$  be a  $(0, 4)$ -tensor associated with  $\mathcal{B}$  by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{2.1}$$

It is well-known that the tensor  $B$  is said to be a *generalized curvature tensor* if the following conditions are fulfilled:  $B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2)$  and

$$B(X_1, X_2, X_3, X_4) + B(X_3, X_1, X_2, X_4) + B(X_2, X_3, X_1, X_4) = 0.$$

For  $\mathcal{B}$  as above, let  $B$  be again defined by (2.1). We extend the endomorphism  $\mathcal{B}(X, Y)$  to a derivation  $\mathcal{B}(X, Y) \cdot$  of the algebra of tensor fields on  $M$ , requiring that it commutes with contractions and  $\mathcal{B}(X, Y) \cdot f = 0$  for any smooth function  $f$  on  $M$ . Now for a  $(0, k)$ -tensor field  $T, k \geq 1$ , we can define the  $(0, k + 2)$ -tensor  $B \cdot T$  by

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \dots, X_k) \\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k). \end{aligned}$$

If  $A$  is a symmetric  $(0, 2)$ -tensor then we define the  $(0, k + 2)$ -tensor  $Q(A, T)$  by

$$\begin{aligned} Q(A, T)(X_1, \dots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \dots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k). \end{aligned}$$

In this manner we obtain the  $(0, 6)$ -tensors  $B \cdot B$  and  $Q(A, B)$ . Substituting  $\mathcal{B} = \mathcal{R}$  or  $\mathcal{B} = \mathcal{C}$ ,  $T = R$  or  $T = C$  or  $T = S$ ,  $A = g$  or  $A = S$  in the above formulas, we get the tensors  $R \cdot R, R \cdot C, C \cdot R, R \cdot S, Q(g, R), Q(S, R), Q(g, C)$  and  $Q(g, S)$ .

For a symmetric  $(0, 2)$ -tensor  $E$  and a  $(0, k)$ -tensor  $T, k \geq 2$ , we define their Kulkarni–Nomizu product  $E \wedge T$  by

$$\begin{aligned} (E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k) &= E(X_1, X_4)T(X_2, X_3, Y_3, \dots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \dots, Y_k) \\ &\quad - E(X_1, X_3)T(X_2, X_4, Y_3, \dots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \dots, Y_k), \end{aligned}$$

see [18]. The tensor  $E \wedge T$  will be called the *Kulkarni–Nomizu tensor* of  $E$  and  $T$ . The following tensors are generalized curvature tensors:  $R, C$  and  $E \wedge F$ , where  $E$  and  $F$  are symmetric  $(0, 2)$ -tensors. We have  $G = \frac{1}{2} g \wedge g$  and

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \tag{2.2}$$

For symmetric  $(0, 2)$ -tensors  $E$  and  $F$  we have (see, e.g., [19] Sect. 3)

$$Q(E, E \wedge F) = -\frac{1}{2} Q(F, E \wedge E). \tag{2.3}$$

We also have (cf. [18] eq. (3))

$$E \wedge Q(E, F) = -\frac{1}{2} Q(F, E \wedge E).$$

For a symmetric  $(0, 2)$ -tensor  $A$  we denote by  $\mathcal{A}$  the endomorphism related to  $A$  by  $g(AX, Y) = A(X, Y)$ . The tensor  $A^p$ ,  $p \geq 2$ , is defined by  $A^p(X, Y) = A^{p-1}(AX, Y)$ . Further, let  $T$  be a  $(0, k)$ -tensor,  $k \geq 2$ . We will call the tensor  $Q(A, T)$  the *Tachibana tensor* of  $A$  and  $T$ , or the Tachibana tensor for short (see, e.g., [8]). By an application of (2.3) we obtain on  $M$  the identities

$$Q(g, g \wedge S) = -Q(S, G) \quad \text{and} \quad Q(S, g \wedge S) = -\frac{1}{2} Q(g, S \wedge S). \tag{2.4}$$

From the tensors  $g, R$  and  $S$  we define the following  $(0, 6)$ -Tachibana tensors:  $Q(S, R), Q(g, R), Q(g, g \wedge S)$  and  $Q(S, g \wedge S)$ . Using (2.3) we can check that the other  $(0, 6)$ -Tachibana tensors that are constructed from  $g, R$  and  $S$  may be expressed by the four Tachibana tensors above or they vanish identically on  $M$ .

**Proposition 2.1** ([20] Proposition 4.1, [21] Lemma 3.4) *Let  $(M, g)$ ,  $n \geq 3$ , be a semi-Riemannian manifold. Let a non-zero symmetric  $(0, 2)$ -tensor  $A$  and a generalized curvature tensor  $B$ , defined at  $x \in M$ , satisfy at this point  $Q(A, B) = 0$ . In addition, let  $Y$  be a vector at  $x$  such that the scalar  $\rho = w(Y)$  is non-zero, where  $w$  is a covector defined by  $w(X) = A(X, Y)$ ,  $X \in T_x M$ . Then we have*

(i)  $A - \rho w \otimes w \neq 0$  and  $B = \lambda A \wedge A$ ,  $\lambda \in \mathbb{R}$ , or (ii)  $A = \rho w \otimes w$  and

$$w(X) B(Y, Z, X_1, X_2) + w(Y) B(Z, X, X_1, X_2) + w(Z) B(X, Y, X_1, X_2) = 0, \quad X, Y, Z, X_1, X_2 \in T_x M. \tag{2.5}$$

Moreover, in both cases the following condition holds at  $x$ :

$$B \cdot B = Q(\text{Ric}(B), B).$$

### 3 Some special generalized curvature tensors

Let  $e_1, e_2, \dots, e_n$  be an orthonormal basis of  $T_x M$  at a point  $x \in M$  of a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , and let  $g(e_j, e_k) = \varepsilon_j \delta_{jk}$ , where  $\varepsilon_j = \pm 1$  and  $h, i, j, k, l, m, r, s \in \{1, 2, \dots, n\}$ . For a generalized curvature tensor  $B$  on  $M$  we denote by  $\text{Ric}(B), \kappa(B)$  and  $\text{Weyl}(B)$  its Ricci tensor, scalar curvature and Weyl tensor, respectively. We have

$$\begin{aligned} \text{Ric}(B)(X, Y) &= \sum_{j=1}^n \varepsilon_j B(e_j, X, Y, e_j), \\ \kappa(B) &= \sum_{j=1}^n \varepsilon_j \text{Ric}(B)(e_j, e_j), \\ \text{Weyl}(B) &= B - \frac{1}{n-2} g \wedge \text{Ric}(B) + \frac{\kappa(B)}{(n-2)(n-1)} G, \end{aligned} \tag{3.1}$$

and we write  $\mathcal{U}_{\text{Ric}(B)} := \{x \in M \mid \text{Ric}(B) - \frac{\kappa(B)}{n} g \neq 0 \text{ at } x\}$  and  $\mathcal{U}_{\text{Weyl}(B)} := \{x \in M \mid \text{Weyl}(B) \neq 0 \text{ at } x\}$ .

Let  $B_{hijk}$ ,  $T_{hijk}$  and  $A_{ij}$  be the local components of the generalized curvature tensors  $B$  and  $T$  and a symmetric  $(0, 2)$ -tensor  $A$  on  $M$ , respectively. The local components  $(B \cdot T)_{hijklm}$  and  $Q(A, T)_{hijklm}$  of the tensors  $B \cdot T$  and  $Q(A, T)$  are the following:

$$\begin{aligned} (B \cdot T)_{hijklm} &= g^{rs} (T_{rijk} B_{shlm} + T_{hrjk} B_{silm} + T_{hirk} B_{sjlm} + T_{hijr} B_{sklm}), \\ Q(A, T)_{hijklm} &= A_{hl} T_{mijk} + A_{il} T_{hmjk} + A_{jl} T_{himk} + A_{kl} T_{hijm} \\ &\quad - A_{hm} T_{lijk} - A_{im} T_{hljk} - A_{jm} T_{hilk} - A_{km} T_{hijl}. \end{aligned}$$

If we contract the last equation with  $g^{ij}$  and  $g^{hm}$ , then we obtain

$$\begin{aligned} g^{rs} Q(A, T)_{hrsklm} &= A_l^s T_{skhm} - A_l^s T_{shmk} - A_m^s T_{skhl} + A_m^s T_{shlk} \\ &\quad + Q(A, Ric(T))_{hkml}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} g^{rs} Q(A, T)_{rijcls} &= -A_l^s T_{sljk} + A_l^s T_{sijk} + A_j^s T_{sikl} + A_k^s T_{silj} \\ &\quad + A_{lk} Ric(T)_{ij} - A_{jl} Ric(T)_{ik} - g^{rs} A_{rs} T_{lijc}. \end{aligned} \tag{3.3}$$

**Proposition 3.1** ([22] Proposition 2.1) *Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ . If*

$$B \cdot B - Q(Ric(B), B) = L Q(g, Weyl(B))$$

on  $\mathcal{U}_{Weyl(B)} \subset M$ , where  $L$  is a function on  $\mathcal{U}_{Weyl(B)}$ , and  $Ric(B)_{ij}$  and  $B_{hijk}$  are the local components of  $Ric(B)$  and  $B$ , respectively, then at every point of  $\mathcal{U}_{Weyl(B)}$  we have

$$Ric(B)_h^s B_{sklm} + Ric(B)_l^s B_{skmh} + Ric(B)_m^s B_{skhl} = 0. \tag{3.4}$$

Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ . The local components  $(B \cdot Weyl(B))_{hijklm}$  and  $(Weyl(B) \cdot B)_{hijklm}$  of the tensors  $B \cdot Weyl(B)$  and  $Weyl(B) \cdot B$  are the following:

$$\begin{aligned} (B \cdot Weyl(B))_{hijklm} &= g^{rs} (Weyl(B)_{rijk} B_{shlm} + Weyl(B)_{hrjk} B_{silm} \\ &\quad + Weyl(B)_{hirk} B_{sjlm} + Weyl(B)_{hijr} B_{sklm}), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (Weyl(B) \cdot B)_{hijklm} &= g^{rs} (B_{rijk} Weyl(B)_{shlm} + B_{hrjk} Weyl(B)_{silm} \\ &\quad + B_{hirk} Weyl(B)_{sjlm} + B_{hijr} Weyl(B)_{sklm}). \end{aligned} \tag{3.6}$$

Using (3.1), (3.5) and (3.6) we can check that the local components  $(B \cdot Weyl(B) - Weyl(B) \cdot B)_{hijklm}$  of the difference tensor  $B \cdot Weyl(B) - Weyl(B) \cdot B$  can be expressed as follows:

$$\begin{aligned} &(n - 2)(B \cdot Weyl(B) - Weyl(B) \cdot B)_{hijklm} \\ &= Q(Ric(B), B)_{hijklm} \\ &\quad - \frac{\kappa}{n - 1} Q(g, B)_{hijklm} + g_{hl} V_{mijk} - g_{hm} V_{lijc} - g_{il} V_{mhjk} + g_{im} V_{lhjk} \\ &\quad + g_{jl} V_{mkhi} - g_{jm} V_{lkhi} - g_{kl} V_{mjhi} + g_{km} V_{ljhi} - g_{ij} (B \cdot Ric(B))_{hkml} \\ &\quad - g_{hk} (B \cdot Ric(B))_{ijlm} + g_{ik} (B \cdot Ric(B))_{hjlm} + g_{hj} (B \cdot Ric(B))_{iklm}, \end{aligned}$$

where  $V_{mijk} = g^{rs} Ric(B)_{mr} B_{sijk}$  (see [2, 11, 12]).

According to [9], a generalized curvature tensor  $B$  on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , is called a Roter type tensor if

$$B = \frac{\phi}{2} Ric(B) \wedge Ric(B) + \mu g \wedge Ric(B) + \eta G \tag{3.7}$$

on  $\mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)}$ , where  $\phi$ ,  $\mu$  and  $\eta$  are some functions on this set. Manifolds admitting Roter type tensors were investigated (e.g.) in [23]. We have



**Proposition 3.2** *Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , satisfying (3.7) on  $\mathcal{U} = \mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$ . Then the following relations hold on  $\mathcal{U}$ :*

(i)

$$(Ric(B))^2 = \alpha_1 Ric(B) + \alpha_2 g, \tag{3.8}$$

$$\alpha_1 = \kappa(B) + \phi^{-1}((n-2)\mu - 1), \quad \alpha_2 = \phi^{-1}(\mu\kappa(B) + (n-1)\eta),$$

(a)  $B \cdot B = L_B Q(g, B), \quad L_B = \phi^{-1}((n-2)(\mu^2 - \phi\eta) - \mu),$

(b)  $B \cdot Weyl(B) = L_B Q(g, Weyl(B)),$

(c)  $B \cdot B = Q(Ric(B), B) + (L_B + \phi^{-1}\mu) Q(g, Weyl(B)), \tag{3.9}$

$$Weyl(B) \cdot B = L_{Weyl(B)} Q(g, B), \quad L_{Weyl(B)} = L_B + \frac{1}{n-2} \left( \frac{\kappa(B)}{n-1} - \alpha_1 \right), \tag{3.10}$$

$$Weyl(B) \cdot Weyl(B) = L_{Weyl(B)} Q(g, Weyl(B)), \tag{3.11}$$

$$Weyl(B) \cdot B = Q(Ric(B), Weyl(B)) + \left( L_B - \frac{\kappa(B)}{n-1} \right) Q(g, Weyl(B)). \tag{3.12}$$

We also have

$$B \cdot Weyl(B) - Weyl(B) \cdot B = \left( \frac{(n-1)\mu - 1}{(n-2)\phi} + \frac{\kappa(B)}{n-1} \right) Q(g, B) + \frac{1}{n-2} Q(Ric(B), B) + \frac{\mu((n-1)\mu - 1) - (n-1)\phi\eta}{(n-2)\phi} Q(Ric(B), G),$$

and, equivalently,

$$B \cdot Weyl(B) - Weyl(B) \cdot B = \left( \phi^{-1} \left( \mu - \frac{1}{n-2} \right) + \frac{\kappa(B)}{n-1} \right) Q(g, B) + \left( \phi^{-1}\mu \left( \mu - \frac{1}{n-2} \right) - \eta \right) Q(Ric(B), G) \tag{3.13}$$

([23] Sects. 1 and 4).

(ii)

$$Q(Ric(B), Weyl(B)) = \phi^{-1} \left( \frac{1}{n-2} - \mu \right) Q(g, B) + \frac{1}{n-2} \left( L_B - \frac{\kappa(B)}{n-1} \right) Q(g, g \wedge Ric(B)). \tag{3.14}$$

Moreover, if  $L_B = \frac{\kappa(B)}{n-1}$ , resp.,  $\kappa(B) = 0$ , then we have

$$Q(Ric(B), Weyl(B)) = L_{Weyl(B)} Q(g, B), \tag{3.15}$$

and

$$B \cdot Weyl(B) - Weyl(B) \cdot B = -Q(Ric(B), Weyl(B)), \tag{3.16}$$

respectively.

(iii) If  $L_{Weyl(B)} = 0$  on  $\mathcal{U}_1 \subset \mathcal{U}$  then on this set we have

$$\begin{aligned} &\phi^{-1} \left( \frac{1}{n-2} - \mu \right) (B \cdot Weyl(B) - Weyl(B) \cdot B) \\ &= (n-2) \left( \phi^{-1} \mu \left( \mu - \frac{1}{n-2} \right) - \eta \right) Q(Ric(B), Weyl(B)). \end{aligned} \tag{3.17}$$

*Proof* (ii) Using (3.1), (2.4), (3.7) and suitable formulas of (i) we get

$$\begin{aligned} &Q(Ric(B), Weyl(B)) \\ &= Q(Ric(B), B) - \frac{1}{n-2} Q(Ric(B), g \wedge Ric(B)) \\ &\quad + \frac{\kappa(B)}{(n-2)(n-1)} Q(Ric(B), G) \\ &= \phi^{-1} \left( \frac{1}{n-2} - \mu \right) Q(g, \frac{\phi}{2} Ric(B) \wedge Ric(B)) \\ &\quad - \left( \eta + \frac{\kappa(B)}{(n-2)(n-1)} \right) Q(g, g \wedge Ric(B)) \\ &= \phi^{-1} \left( \frac{1}{n-2} - \mu \right) Q(g, B - \mu g \wedge Ric(B)) \\ &\quad - \left( \eta + \frac{\kappa(B)}{(n-2)(n-1)} \right) Q(g, g \wedge Ric(B)), \\ &= \phi^{-1} \left( \frac{1}{n-2} - \mu \right) Q(g, B) \\ &\quad - \left( \phi^{-1} \mu \left( \frac{1}{n-2} - \mu \right) + \eta + \frac{\kappa(B)}{(n-2)(n-1)} \right) Q(g, g \wedge Ric(B)), \end{aligned} \tag{3.18}$$

which leads to (3.14).

If  $L_B = \frac{\kappa(B)}{n-1}$  then, (3.12) reduces to (3.15). It is clear that (3.15) also follows from (3.18).

If  $\kappa(B) = 0$ , then (3.9)(b) and (3.12) yield (3.16). We can also obtain (3.16) from (3.13), by making use of (2.3), (3.9)(a) and (3.14).

(iii) Conditions (3.8), (3.9)(a) and (3.10) yield

$$\phi^{-1} \left( \mu - \frac{1}{n-2} \right) + \frac{\kappa(B)}{n-1} = (n-2) \left( \phi^{-1} \mu \left( \mu - \frac{1}{n-2} \right) - \eta \right).$$

Now (3.13) turns into

$$\begin{aligned} &B \cdot Weyl(B) - Weyl(B) \cdot B \\ &= (n-2) \left( \phi^{-1} \mu \left( \mu - \frac{1}{n-2} \right) - \eta \right) Q(g, Weyl(B)). \end{aligned} \tag{3.19}$$

Further, (3.10) yields  $L_B - \frac{\kappa(B)}{n-1} = \phi^{-1} \left( \mu - \frac{1}{n-2} \right)$ , hence (3.14) takes the form

$$Q(Ric(B), Weyl(B)) = \phi^{-1} \left( \frac{1}{n-2} - \mu \right) Q(g, Weyl(B)),$$

which together with (3.19) gives (3.17). Our proposition is thus proved. □

**Proposition 3.3** *Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ . If the conditions*

$$B \cdot B = \frac{\kappa(B)}{(n-1)n} Q(g, B), \tag{3.20}$$

$$B \cdot B - Q(Ric(B), B) = -\frac{(n-2)\kappa(B)}{(n-1)n} Q(g, Weyl(B)), \tag{3.21}$$

$$B \cdot Weyl(B) = \frac{1}{n-1} Q(Ric(B), Weyl(B)) \tag{3.22}$$

are satisfied on  $\mathcal{U} = \mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$ , then on this set we have

$$Weyl(B) \cdot Weyl(B) = 0, \tag{3.23}$$

$$rank(Ric(B) - \frac{\kappa(B)}{n} g) = 1, \tag{3.24}$$

$$Weyl(B) \cdot B = 0, \tag{3.25}$$

$$(n-1)(B \cdot Weyl(B) - Weyl(B) \cdot B) = Q(Ric(B), Weyl(B)), \tag{3.26}$$

and the tensor  $Weyl(B)$  satisfies (2.5) for some non-zero covector  $w$ .

*Proof* First of all we note that (3.20) yields

$$B \cdot Weyl(B) = \frac{\kappa(B)}{(n-1)n} Q(g, Weyl(B)). \tag{3.27}$$

Next, comparing the right-hand sides of (3.22) and (3.27) we obtain

$$Q\left(Ric(B) - \frac{\kappa(B)}{n} g, Weyl(B)\right) = 0. \tag{3.28}$$

From this, in view of Proposition 4.1 of [20] (see also [21] Lemma 3.4), we get

$$Weyl(B) \cdot Weyl(B) = Q(Ric(Weyl(B)), Weyl(B)). \tag{3.29}$$

Since  $Ric(Weyl(B)) = 0$ , (3.29) reduces to (3.23).

Suppose that  $rank(Ric(B) - \frac{\kappa(B)}{n} g) > 1$  at  $x \in \mathcal{U}$ . From (3.28), in view of Proposition 4.1 of [20] (see also [21] Lemma 3.4 or [24] Lemma 3.1), it follows that

$$Weyl(B) = \frac{\phi}{2} \left( Ric(B) - \frac{\kappa(B)}{n} g \right) \wedge \left( Ric(B) - \frac{\kappa(B)}{n} g \right) \tag{3.30}$$

at  $x$ , where  $\phi \in \mathbb{R} \setminus \{0\}$ . Applying (3.1) in (3.30) we immediately get

$$\begin{aligned} B &= \frac{\phi}{2} Ric(B) \wedge Ric(B) + \left( \frac{1}{n-2} - \frac{\phi\kappa(B)}{n} \right) g \wedge Ric(B) \\ &\quad + \kappa(B) \left( \frac{\phi\kappa(B)}{n^2} - \frac{1}{(n-2)(n-1)} \right) G. \end{aligned}$$

This, together with (3.9) and (3.20), yields

$$B \cdot B - Q(Ric(B), B) = \left( \frac{1}{(n-2)\phi} - \frac{(n-2)\kappa(B)}{(n-1)n} \right) Q(g, Weyl(B)). \tag{3.31}$$

Comparing the right-hand sides of (3.21) and (3.31) we get  $Q(g, Weyl(B)) = 0$ , and, as a consequence,  $Weyl(B) = 0$ , which is a contradiction. Therefore (3.24) holds on  $\mathcal{U}$ . Thus

from (3.28), in view of Proposition 2.1 (ii), it follows that the tensor  $Weyl(B)$  satisfies (2.5) for some non-zero covector  $w$ . Further, we note that (3.24) is equivalent to

$$\frac{1}{2} Ric(B) \wedge Ric(B) = \frac{\kappa(B)}{n} g \wedge S - \frac{(\kappa(B))^2}{n^2} G \tag{3.32}$$

(cf. [25] eq. (21)). Now from (3.28), by making use of (2.3), (3.1) and (3.32), we obtain

$$Q(Ric(B), B) - \frac{\kappa(B)}{n} Q(g, B) = -\frac{\kappa(B)}{(n-1)n} Q(g, g \wedge Ric(B)). \tag{3.33}$$

In terms of tensor components,

$$Q(Ric(B), B)_{rijkl} - \frac{\kappa(B)}{n} Q(g, B)_{rijkl} = -\frac{\kappa(B)}{(n-1)n} Q(g, g \wedge Ric(B))_{rijkl}.$$

Contracting this with  $g^{rs}$  and using (3.3), (3.4) and (3.32) in the form

$$g^{rs} Ric(B)_{ir} B_{sljk} = \frac{\kappa(B)}{(n-1)n} (g_{jl} Ric(B)_{ik} - g_{kl} Ric(B)_{ij}) - \frac{\kappa(B)}{n} \left( B_{lijk} - \frac{\kappa(B)}{(n-1)n} G_{lijk} \right)$$

Applying this, (3.1), (3.33) and (3.20) in the form

$$(B \cdot B)_{hijklm} = \frac{\kappa(B)}{(n-1)n} Q(g, B)_{hijklm}$$

to the identity (3.6), we obtain (3.25). Finally, (3.22) and (3.25) give (3.26), which completes the proof.  $\square$

As an immediate consequence of Proposition 3.3 we have

**Theorem 3.4** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold. If the following conditions*

$$R \cdot R = \frac{\kappa}{(n-1)n} Q(g, R), \tag{3.34}$$

$$R \cdot R - Q(S, R) = -\frac{(n-2)\kappa}{(n-1)n} Q(g, C), \tag{3.35}$$

$$R \cdot C = \frac{1}{n-1} Q(S, C) \tag{3.36}$$

are satisfied on  $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ , then on this set we have

$$\begin{aligned} C \cdot C &= 0, \\ rank(S - \frac{\kappa}{n} g) &= 1, \\ C \cdot R &= 0, \end{aligned} \tag{3.37}$$

$$(n-1)(R \cdot C - C \cdot R) = Q(S, C). \tag{3.38}$$

**Proposition 3.5** *Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , satisfying at every point  $x \in \mathcal{U} = \mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$  the conditions*

$$Ric(B) = \alpha g + \beta w \otimes w, \quad w \in T_x^* M, \tag{3.39}$$

$$\begin{aligned} w(X) Weyl(B)(Y, Z, X_1, X_2) + w(Y) Weyl(B)(Z, X, X_1, X_2) \\ + w(Z) Weyl(B)(X, Y, X_1, X_2) = 0, \quad X, Y, Z, X_1, X_2 \in T_x M. \end{aligned} \tag{3.40}$$

Then (3.20), (3.21), (3.22) and (3.24) hold on  $\mathcal{U}$ . Moreover, the scalar curvature  $\kappa(B)$  is non-zero at every point of  $\mathcal{U}$ .

*Proof* We can easily adopt the proof of Theorem 4.2 of [26] to verify our proposition.  $\square$

As an immediate consequence of Propositions 3.3 and 3.6 we get

**Proposition 3.6** *If  $B$  is a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , satisfying (3.39) and (3.40) on  $\mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$ , then (3.25) and (3.26) hold on this set.*

This proposition implies

**Theorem 3.7** *If the Ricci tensor  $S$  and the Weyl tensor  $C$  of a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , satisfy (3.39) and (3.40) on  $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ , then (3.36), (3.37), and, as a consequence, (3.38) hold on this set.*

- Example 3.8* (i) Let  $\overline{M} \times_F \widetilde{N}$  be the warped product of an 1-dimensional manifold  $(\overline{M}, \overline{g})$ , with  $\overline{g} = \varepsilon = \pm 1$ , a warping function  $F$  given by  $F(x^1) = a \exp(bx^1)$ ,  $a = const. > 0$ ,  $b = const. \neq 0$ , and an  $(n - 1)$ -dimensional fibre  $(\widetilde{N}, \widetilde{g})$ ,  $n \geq 4$ .  
 (ii) Proposition 4.2 of [27] states that the above defined warped product  $\overline{M} \times_F \widetilde{N}$ : such that the fibre  $(\widetilde{N}, \widetilde{g})$  is a conformally flat semi-Riemannian manifold with rank  $\widetilde{S} = 1$  and the vanishing scalar curvature  $\widetilde{\kappa}$  satisfies (3.34)–(3.36). We note that the scalar curvature  $\kappa$  of  $\overline{M} \times_F \widetilde{N}$  is a non-zero constant. Precisely, we have ([27] Eq. (28))

$$\kappa = -\frac{(n - 1)n}{4} \varepsilon a^2. \tag{3.41}$$

An example of a family of such warped products is given in Example 4.1 of [27].

- (iii) Let now the fibre  $(\widetilde{N}, \widetilde{g})$  be the semisymmetric manifold defined in Example 4.1 of [19]. This manifold satisfies rank  $\widetilde{S} = 1$  and  $\widetilde{\kappa} = 0$ . The scalar curvature  $\kappa$  of  $\overline{M} \times_F \widetilde{N}$  satisfies (3.41) ([24] Eq. (54)). Moreover, the warped product  $\overline{M} \times_F \widetilde{N}$  satisfies (3.34)–(3.36) ([24] Example 5.1, [2] Sect. 6 pp. 15–16). As it was stated in [24] (Example 5.4), the manifold  $\overline{M} \times_F \widetilde{N}$  can be locally realized as a hypersurface in a semi-Riemannian space of non-zero constant curvature.

**Proposition 3.9** *Let  $B$  be a generalized curvature tensor on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ .*

- (i) *If the conditions*

$$Q(Ric(B), B) = 0, \tag{3.42}$$

$$rank(Ric(B)) = 1 \tag{3.43}$$

*are satisfied on  $\mathcal{U} = \mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$ , then on this set we have*

$$\kappa(B) = 0, \tag{3.44}$$

$$B \cdot Weyl(B) = Weyl(B) \cdot B = Q(Ric(B), Weyl(B)) = 0. \tag{3.45}$$

- (ii) *If the conditions (3.43) and*

$$Q(Ric(B), Weyl(B)) = 0 \tag{3.46}$$

*are satisfied on  $\mathcal{U} = \mathcal{U}_{Ric(B)} \cap \mathcal{U}_{Weyl(B)} \subset M$ , then on this set we have (3.44) and*

$$B \cdot Weyl(B) = Weyl(B) \cdot B = 0.$$

*Proof* Let  $B_{hijk}$ ,  $Ric(B)_{ij}$  and  $Weyl(B)_{hijk}$  be the local components of the tensors  $B$ ,  $Ric(B)$  and  $Weyl(B)$  at  $x \in \mathcal{U}$ , respectively.

- (i) By (3.43), at  $x$  we have  $Ric(B) = \rho w \otimes w$  and  $Ric(B)_{ij} = \rho w_i w_j$ , where  $w_i$  are the components of the covector  $w$ . Since (3.42) and (3.43) hold at  $x$ , (2.5) yields

$$w_h B_{ijkl} + w_i B_{jhkl} + w_j B_{hikl} = 0. \tag{3.47}$$

Contracting this with  $g^{hl}$  and using (3.43) we get

$$w^r B_{ijk r} = 0, \quad w^r = g^{rs} w_s. \tag{3.48}$$

Transvecting now (3.47) with  $w^h$  and using (3.48) we get (3.44). Now (3.42)-(3.44), together with (2.3) and (3.1), yield

$$\begin{aligned} Q(Ric(B), Weyl(B)) &= Q(Ric(B), B) - \frac{1}{n-2} Q(Ric(B), g \wedge Ric(B)) \\ &= Q(Ric(B), B) + \frac{1}{2(n-2)} Q(g, Ric(B) \wedge Ric(B)) = Q(Ric(B), B) = 0. \end{aligned}$$

Further, in view of Proposition 2.1, (3.42) yields  $B \cdot B = Q(Ric(B), B)$ , and, consequently,  $B \cdot B = 0$ , which implies  $B \cdot Weyl(B) = 0$ . Now, Proposition 3.3 completes the proof of (i).

- (ii) Since (3.46) and (3.43) hold at  $x$ , (2.5) yields

$$w_h Weyl(B)_{ijkl} + w_i Weyl(B)_{jhkl} + w_j Weyl(B)_{hikl} = 0. \tag{3.49}$$

Contracting this with  $g^{hl}$  we get

$$w^r Weyl(B)_{ijk r} = 0, \quad w^r = g^{rs} w_s. \tag{3.50}$$

Transvecting (3.49) with  $w^h$  and using (3.50) we get (3.44). Next, by making use of (2.3), (3.1), (3.43), (3.44) and (3.46), we get

$$\begin{aligned} 0 &= Q(Ric(B), Weyl(B)) = Q(Ric(B), B) - \frac{1}{n-2} Q(Ric(B), g \wedge Ric(B)) \\ &= Q(Ric(B), B) + \frac{1}{2(n-2)} Q(g, Ric(B) \wedge Ric(B)) = Q(Ric(B), B). \end{aligned}$$

Now (i) completes the proof of (ii). □

*Example 3.10* (i) Let  $\overline{M} \times_F \widetilde{N}$  be the warped product of a 3-dimensional manifold  $(\overline{M}, \overline{g})$ , with a warping function  $F$  and a 1-dimensional fibre  $(\widetilde{N}, \widetilde{g})$ . In Proposition 3.3(i) of [28] it was proved that if the conditions

$$\kappa = 0, \quad S^2 = 0, \tag{3.51}$$

are satisfied on  $\overline{M} \times_F \widetilde{N}$ , then  $\text{rank } S \leq 1$ . In addition, if

$$S \cdot C = 0, \quad R \cdot R = Q(S, R) \tag{3.52}$$

on  $\overline{M} \times_F \widetilde{N}$ , then it is a semisymmetric manifold ([28], Proposition 3.3(ii)). Thus, by (3.52), on  $\overline{M} \times_F \widetilde{N}$  we have  $Q(S, R) = 0$ . Now, in view of Proposition 3.6(i),

$$R \cdot C = C \cdot R = Q(S, C) = 0 \tag{3.53}$$

on  $\mathcal{U}_S \cap \mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ .

- (ii) In Sect. 4 of [28] a class of warped products  $\overline{M} \times_F \widetilde{N}$  of dimension  $\geq 4$  were investigated. Among others, in this class of warped products there are manifolds satisfying (3.51)-(3.53).

#### 4 Manifolds with parallel Weyl conformal curvature tensor

Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold whose Weyl conformal curvature tensor is parallel, i.e.,  $\nabla C = 0$  on  $M$ . It is obvious that the last condition implies  $R \cdot C = 0$ . Suppose, moreover, that the manifold  $(M, g)$  is neither conformally flat nor locally symmetric. Such manifolds are called *essentially conformally symmetric manifolds*, e.c.s. manifolds, in short (see, e.g., [29] and [30]). E.c.s. manifolds are semisymmetric manifolds ( $R \cdot R = 0$ , [29] Theorem 9) satisfying  $\kappa = 0$  and  $Q(S, C) = 0$  ([29] Theorems 7 and 8). In addition,

$$FC = \frac{1}{2} S \wedge S \tag{4.1}$$

holds on  $M$ , where  $F$  is some function on  $M$ , called the *fundamental function* ([30]). At every point of  $M$  we also have  $\text{rank } S \leq 2$  ([30] Theorem 5). We mention that the local structure of e.c.s. manifolds has already been described. We refer to [14] and [16] for the final results related to this subject. We also mention that certain e.c.s. metrics are realized on compact manifolds ([15, 17]).

Suppose that  $F = 0$  at  $x \in M$ . Now (4.1) implies  $\text{rank } S \leq 1$  at  $x$ . It is clear that if  $S$  vanishes, then (3.53) holds at  $x$ . If  $\text{rank } S = 1$ , then in view Proposition 3.9(ii) we also have (3.53) at  $x$ . Next, we assume that  $F$  is non-zero at  $x \in M$ . Then  $\text{rank } S = 2$  at  $x$ . In this case (4.1) turns into (3.7) with  $B = R$ ,  $\text{Ric}(B) = S$ ,  $\phi = F^{-1}$ ,  $\mu = \frac{1}{n-2}$  and  $\eta = 0$ . Therefore (3.10) and (3.11) reduce to  $C \cdot R = 0$  and  $C \cdot C = 0$ , respectively. Consequently, (3.53) holds at  $x$ . Thus we have proved the following

**Theorem 4.1** *Condition (3.53) is satisfied on every essentially conformally symmetric manifold  $(M, g)$ .*

*Remark 4.2* (i) E.c.s. warped product manifolds were investigated in [31], where examples of such manifolds are given.

- (ii) The manifolds studied in this section satisfy (1.10). They can be quasi-Einstein or not. Moreover, the tensor  $C \cdot C$  of such manifolds is the zero tensor.

#### 5 Roter type manifolds satisfying (1.10)

We recall that if the curvature tensor  $R$  of a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 4$ , is a linear combination of the Kulkarni–Nomizu products  $S \wedge S$ ,  $g \wedge S$  and  $G = \frac{1}{2} g \wedge g$  on  $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ , i.e., (1.7) holds on this set, then  $(M, g)$  is called a *Roter type manifold*. Such manifolds were investigated among others in [32] and [33]. We also refer to [10] for a survey on Roter type manifolds, as well as on Roter type hypersurfaces. Curvature properties of manifolds satisfying (1.7) are presented in Proposition 3.2 (for  $B = R$ ).

*Remark 5.1* ([11] Theorem 4.1 and Corollary 4.1) Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold and let  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ . If  $R \cdot C - C \cdot R = L Q(g, C)$  holds on  $\mathcal{U}$  for some function  $L$ , then  $R \cdot R = L Q(g, R)$  and  $C \cdot R = 0$  on this set. In particular, if  $R \cdot C = C \cdot R$  holds on  $\mathcal{U}$ , then  $R \cdot R = R \cdot C = C \cdot R = 0$  on this set. Therefore we consider manifolds satisfying (1.10) and (1.7) on  $\mathcal{U}$  with non-zero function  $L$ .

As an immediate consequence of Proposition 3.2(ii) we get

**Theorem 5.2** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold satisfying (1.7) on  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ . If  $\kappa = 0$  on  $\mathcal{U}$  then (1.10), with  $L = -1$ , holds on this set.*

We also have a converse to this result.

**Theorem 5.3** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold satisfying (1.10) and (1.7) on  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ , and let  $\mathcal{U}_1 \subset \mathcal{U}$  be the set of all points at which the functions  $L$  and  $L_C$ , defined by (1.10), (3.10) and (3.11) (for  $B = R$ ), respectively, are nowhere zero on this set. Then on  $\mathcal{U}_1$  we have*

$$L = -1, \tag{5.1}$$

$$\kappa = 0. \tag{5.2}$$

*Proof* From (3.9)(b) and (3.10) we obtain on  $\mathcal{U}$

$$R \cdot C - C \cdot R = L_R Q(g, C) - L_C Q(g, R).$$

This, together with (1.10), yields

$$L Q(S, C) = L_R Q(g, C) - L_C Q(g, R),$$

which, via (2.3), turns into

$$L Q(S, C) - \frac{L_C}{n-2} Q(S, G) = L_R Q(g, C) - L_C Q(g, R) + \frac{L_C}{n-2} Q(g, g \wedge S).$$

We restrict our considerations to the set  $\mathcal{U}_1$ . Now the last equation turns into

$$\begin{aligned} L Q\left(S, C - \frac{L_C L^{-1}}{n-2} G\right) &= (L_R - L_C) Q(g, C), \\ Q\left(S, C - \frac{L_C L^{-1}}{n-2} G\right) &= -(L_C - L_R) L^{-1} Q\left(g, C - \frac{L_C L^{-1}}{n-2} G\right), \end{aligned}$$

and

$$Q(S + (L_C - L_R) L^{-1} g, C - \frac{L_C L^{-1}}{n-2} G) = 0. \tag{5.3}$$

We note that if we had  $\text{rank}(S + (L_C - L_R) L^{-1} g) = 1$  at a point of  $\mathcal{U}_1$  then - in a standard way - we would obtain  $C = 0$  from (1.7), which is a contradiction. Therefore (5.3) implies ([20] Proposition 4.1, [21] Lemma 3.4):  $\text{rank}(S + (L_C - L_R) L^{-1} g) \geq 2$  and

$$\begin{aligned} C - \frac{L_C L^{-1}}{n-2} G &= \frac{\lambda}{2} (S + (L_C - L_R) L^{-1} g) \\ &\wedge (S + (L_C - L_R) L^{-1} g), \end{aligned} \tag{5.4}$$

$$\left(C - \frac{L_C L^{-1}}{n-2} G\right) \cdot \left(C - \frac{L_C L^{-1}}{n-2} G\right) = -\frac{n-1}{n-2} L_C L^{-1} Q(g, C) \tag{5.5}$$

on  $\mathcal{U}_1$ , where  $\lambda$  is a function on this set. Now (5.4), via (2.2), turns into

$$\begin{aligned} R &= \frac{\lambda}{2} S \wedge S + \left(\frac{1}{n-2} + \lambda(L_C - L_R) L^{-1}\right) g \wedge S \\ &+ \left(\lambda(L_C - L_R)^2 L^{-2} + \frac{L_C L^{-1}}{n-2} - \frac{\kappa}{(n-2)(n-1)}\right) G. \end{aligned} \tag{5.6}$$



The decomposition of  $R$  is unique on  $\mathcal{U}_1$  (see, e.g., [24] Lemma 3.1). Therefore (1.7) and (5.6) yield  $\phi = \lambda$  and

$$\begin{aligned} \text{(a)} \quad \mu &= \frac{1}{n-2} + \phi(L_C - L_R)L^{-1}, \\ \text{(b)} \quad \eta &= \phi(L_C - L_R)^2L^{-2} + \frac{L_C L^{-1}}{n-2} - \frac{\kappa}{(n-2)(n-1)}. \end{aligned} \tag{5.7}$$

Since  $L_C - L_R = \frac{1}{n-2}(\phi^{-1}(1 - (n-2)\mu) - \frac{n-2}{n-1}\kappa)$ , (5.7)(a) can be written as

$$(L + 1)(1 - (n-2)\mu) = \frac{n-2}{n-1}\kappa\phi. \tag{5.8}$$

On the other hand, we have

$$\begin{aligned} \left(C - \frac{L_C L^{-1}}{n-2} G\right) \cdot \left(C - \frac{L_C L^{-1}}{n-2} G\right) &= \left(C - \frac{L_C L^{-1}}{n-2} G\right) \cdot C \\ &= C \cdot C - \frac{L_C L^{-1}}{n-2} G \cdot C = C \cdot C - \frac{L_C L^{-1}}{n-2} Q(g, C), \end{aligned}$$

which, together with (5.5), gives  $C \cdot C = -L_C L^{-1} Q(g, C)$ . From the last relation, via (3.11), it follows that

$$(L + 1)L_C Q(g, C) = 0 \tag{5.9}$$

on  $\mathcal{U}_1$ . Now from (5.8) and (5.9) we conclude that (5.1) and (5.2) are satisfied on this set, which completes the proof.  $\square$

*Example 5.4* (i) Let  $(M, g) = N_{s_1}^p(c_1) \times N_{s_2}^{n-p}(c_2)$  be a Cartesian product of semi-Riemannian spaces of constant curvature, where  $2 \leq p \leq n - 2$ ,  $c_1 = \frac{\kappa_1}{(p-1)p}$ ,  $c_2 = \frac{\kappa_2}{(n-p-1)(n-p)}$ ,  $\kappa_1$  and  $\kappa_2$  are the scalar curvatures of  $N_{s_1}^p(c_1)$  and  $N_{s_2}^{n-p}(c_2)$ , respectively. For the scalar curvature  $\kappa$  of  $(M, g)$  we have

$$\kappa = \kappa_1 + \kappa_2 = p(p-1)c_1 + (n-p)(n-p-1)c_2. \tag{5.10}$$

It is well-known that (3.9), with  $L_R = 0$ , holds on  $M$  ([3] Theorem 4.5). Moreover, (3.11) is satisfied on  $M$ , with

$$L_C = -\frac{(p-1)(n-p-1)}{(n-2)(n-1)}(c_1 + c_2) \tag{5.11}$$

([34] Sect. 4).

(ii) We assume that  $c_1$  and  $c_2$  satisfy

$$\text{(a)} \quad c_1 + c_2 \neq 0 \quad \text{and} \quad \text{(b)} \quad (p-1)c_1 - (n-p-1)c_2 \neq 0. \tag{5.12}$$

Then  $(M, g)$  is a non-conformally flat and non-Einstein manifold. More precisely, we have  $\mathcal{U}_S \cap \mathcal{U}_C = M$  (cf. [35] Sect. 3). Moreover, as it was stated in Sect. 3 of [35], (3.7) holds on  $M$  with

$$\begin{aligned} \phi &= \tau(c_1 + c_2), \quad \eta = \tau c_1 c_2 ((p-1)^2 c_1 + (n-p-1)^2 c_2), \\ \mu &= -(n-2)\tau c_1 c_2, \quad \tau = ((p-1)c_1 - (n-p-1)c_2)^{-2}. \end{aligned} \tag{5.13}$$

(iii) In addition, we assume that  $n \neq 2p$ ,  $\kappa = 0$  and  $c_1 \neq 0$  hold on  $M$ . Now (5.10) reduces to  $c_2 = -\frac{p(p-1)}{(n-p)(n-p-1)} c_1$ . Applying this in (5.11), (5.12) and (5.13) we obtain

$$\begin{aligned} c_1 + c_2 &= \frac{(n-1)(n-2p)}{(n-p)(n-p-1)} c_1, & (p-1)c_1 - (n-p-1)c_2 &= \frac{n(p-1)}{n-p} c_1, \\ \phi &= \frac{(n-1)(n-p)(n-2p)}{n^2(n-p-1)(p-1)^2} c_1^{-1}, & \mu &= \frac{(n-2)p(n-p)}{n^2(n-p-1)(p-1)}, \\ \eta &= \frac{(n-2p)p}{n^2(n-p-1)} c_1, & L_C &= -\frac{(p-1)(n-2p)}{(n-2)(n-p)} c_1. \end{aligned}$$

Thus we see that  $(M, g)$  is a Roter type manifold satisfying

$$R \cdot C - C \cdot R = -Q(S, C),$$

with the non-zero function  $L_C$  and zero scalar curvature  $\kappa$ .

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## References

1. A.L. Besse, *Einstein Manifolds*, *Ergeb. Math. Grenzgeb.*, 3. Folge, Bd. 10 (Springer-Verlag, Berlin, Heidelberg, New York, 1987)
2. R. Deszcz, M. Głogowska, M. Hotłoś, K. Sawicz, in *A survey on generalized einstein metric conditions*, ed. by S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner. *Advances in Lorentzian geometry: Proceedings of the Lorentzian Geometry Conference in Berlin*, AMS/IP Studies in Advanced Mathematics 49, 27–46, 2011
3. Z.I. Szabó, Structure theorems on Riemannian spaces satisfying  $R(X, Y) \cdot R = 0$ . I. The local version. *J. Differ. Geom.* **17**, 531–582 (1982)
4. R. Deszcz, On pseudosymmetric spaces. *Bull. Belg. Math. Soc. Ser. A* **44**, 1–34 (1992)
5. R. Deszcz, W. Grycak, On some class of warped product manifolds. *Bull. Inst. Math. Acad. Sin.* **15**, 311–322 (1987)
6. S. Haesen, L. Verstraelen, Properties of a scalar curvature invariant depending on two planes. *Manuscr. Math.* **122**, 59–72 (2007)
7. R. Deszcz, Ş. Yaprak, Curvature properties of certain pseudosymmetric manifolds. *Publ. Math. Debr.* **45**, 334–345 (1994)
8. R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz, M. Scherfner, On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type. *Kragujev. J. Math.* **35**, 223–247 (2011)
9. R. Deszcz, On some Akivis–Goldberg type metrics. *Publ. Inst. Math. (Beograd) (N.S.)* **74**(88), 71–83 (2003)
10. M. Głogowska, On Roter-type identities, in *Pure and Applied Differential Geometry-PADGE 2007, Berichte aus der Mathematik* (Shaker Verlag, Aachen, 2007), pp. 114–122
11. R. Deszcz, M. Hotłoś, Z. Şentürk, On some family of generalized Einstein metric conditions. *Demonstr. Math.* **34**, 943–954 (2001)
12. R. Deszcz, M. Głogowska, M. Hotłoś, L. Verstraelen, On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms. *Colloq. Math.* **96**, 149–166 (2003)
13. R. Deszcz, M. Hotłoś, Z. Şentürk, On curvature properties of certain quasi-Einstein hypersurfaces. *Int. J. Math.* **23**, 1250073 (2012). (17 pages)
14. A. Derdzinski, W. Roter, Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds. *Tohoku Math. J.* **59**, 565–602 (2007)

15. A. Derdzinski, W. Roter, On compact manifolds admitting indefinite metrics with parallel Weyl tensor. *J. Geom. Phys.* **58**, 1137–1147 (2008)
16. A. Derdzinski, W. Roter, The local structure of conformally symmetric manifolds. *Bull. Belg. Math. Soc. Simon Stevin* **16**, 117–128 (2009)
17. A. Derdzinski, W. Roter, Compact pseudo-Riemannian manifolds with parallel Weyl tensor. *Ann. Glob. Anal. Geom.* **37**, 73–90 (2010)
18. R. Deszcz, M. Głogowska, Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces. *Publ. Inst. Math. (Beograd) (N.S.)* **72**(86), 81–93 (2002)
19. R. Deszcz, M. Głogowska, M. Hotłoś, Z. Şentürk, On certain quasi-Einstein semisymmetric hypersurfaces. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **41**, 151–164 (1998)
20. F. Defever, R. Deszcz, On semi-Riemannian manifolds satisfying the condition  $R \cdot R = Q(S, R)$ . In: *Geometry and Topology of Submanifolds*, vol. III (World Scientific, River Edge 1991), pp. 108–130
21. R. Deszcz, M. Hotłoś, On certain subclass of pseudosymmetric manifolds. *Publ. Math. Debr.* **53**, 29–48 (1998)
22. R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotłoś, M. Yawata, On semi-Riemannian manifolds satisfying some conformally invariant curvature condition. *Colloq. Math.* **131**, 149–170 (2013)
23. D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes. *Tsukuba J. Math.* **30**, 363–381 (2006)
24. R. Deszcz, M. Hotłoś, On hypersurfaces with type number two in spaces of constant curvature. *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.* **46**, 19–34 (2003)
25. M. Głogowska, On quasi-Einstein Cartan type hypersurfaces. *J. Geom. Phys.* **58**, 599–614 (2008)
26. R. Deszcz, M. Hotłoś, Z. Şentürk, On the equivalence of the Ricci-pseudosymmetry and pseudosymmetry. *Colloq. Math.* **79**, 211–227 (1999)
27. F. Defever, R. Deszcz, M. Hotłoś, M. Kucharski, Z. Şentürk, Generalisations of Robertson–Walker spaces. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **43**, 13–24 (2000)
28. F. Defever, R. Deszcz, M. Głogowska, V.V. Goldberg, L. Verstraelen, A class of four-dimensional warped products. *Demonstr. Math.* **35**, 853–864 (2002)
29. A. Derdziński, W. Roter, Some theorems on conformally symmetric manifolds. *Tensor (N.S.)* **32**(1978), 11–23 (1978)
30. A. Derdziński, W. Roter, Some properties of conformally symmetric manifolds which are not Ricci-recurrent. *Tensor (N.S.)* **34**, 11–20 (1980)
31. M. Hotłoś, On conformally symmetric warped products. *Ann. Acad. Paedagog. Crac.* **23**, 75–85 (2004)
32. R. Deszcz, M. Plaue, M. Scherfner, On Roter type warped products with 1-dimensional fibres. *J. Geom. Phys.* **69**, 1–11 (2013)
33. R. Deszcz, M. Scherfner, On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces. *Colloq. Math.* **109**, 13–29 (2007)
34. F. Defever, R. Deszcz, On Riemannian manifolds satisfying a certain curvature condition imposed on the Weyl curvature tensor. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **110**(32), 27–34 (1993)
35. M. Głogowska, Curvature conditions on hypersurfaces with two distinct principal curvatures. in *Banach Center Publications*. 69, Institute of Mathematics Polish Academy of Sciences, 133–143, 2005