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Abstract

We consider a control problem where the system is driven by a decoupled as well as a coupled forward-backward stochastic differential equation. We prove the existence of an optimal control in the class of relaxed controls, which are measure-valued processes, generalizing the usual strict controls. The proof is based on some tightness properties and weak convergence on the space $\mathcal{D}$ of càdlàg functions, endowed with the Jakubowsky S-topology. Moreover, under some convexity assumptions, we show that the relaxed optimal control is realized by a strict control.

Keywords: Forward-backward stochastic differential equation; stochastic control; relaxed control; tightness; Meyer-Zheng topology; Jakubowsky S-topology.

MSC 2000 subject classifications: 93E20, 60H10, 60H30.

1 Introduction

In this paper, we investigate the existence of optimal controls, for systems driven by forward-backward stochastic differential equations (FBSDEs), of the form:

$$\begin{cases} dX_t & = & b\left(t, X_t, Y_t, U_t\right) dt + \sigma\left(t, X_t, Y_t, U_t\right) dW_t, \\ -dY_t & = & h\left(t, X_t, Y_t, U_t\right) dt - Z_t dW_t - dM_t, \\ X_0 = x, & & Y_T = \varphi\left(X_T\right), \end{cases} \tag{1.1}$$

where (M_t) is a square integrable martingale, which is orthogonal to the Brownian motion (W_t). The expected cost over the time interval $[0, T]$ is given by

$$J\left(U_{\cdot}\right) = E\left[\psi\left(X_T\right) + g\left(Y_0\right) + \int_0^T l\left(t, X_t, Y_t, U_t\right) dt\right]. \tag{1.2}$$

Backward stochastic differential equations (BSDEs) have been first introduced by Pardoux and Peng, in the seminal paper [20]. Since then, the BSDE theory became a powerful tool in many fields, such as mathematical finance, optimal control, semi-linear and quasi-linear partial differential equations. When the BSDE is associated to some forward stochastic differential equation, the system is called a forward-backward stochastic differential equation (FBSDE). The earliest version of such an equation appeared in Bismut [7], in the stochastic version of Pontriagin maximum principle. See [11, 16] for a complete account on the subject, and the references therein.

Control problems for systems governed by BSDEs and FBSDEs model many problems arising in financial mathematics, especially the minimization of risk measures (El Karoui and Barrieu [6], Oksendal and Sulem [18]), the recursive utility problems and the portfolio optimization problems. Therefore it becomes quite

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natural to investigate this kind of problems, for themselves as a class of interesting dynamical systems and for their connections to real life problems. Many papers have been devoted to this subject, see e.g. [3, 9, 12, 21, 22, 24] and the references therein. These papers have been concerned by various forms of the stochastic maximum principle. Existence of optimal relaxed controls for systems driven by BSDEs has been studied for the first time in Buckdahn and al. [8], by combining probabilistic arguments as well as PDEs techniques. Then, Bahlali and al. [1] investigate a control problem, with a general cost functional by using probabilistic tools. The authors suppose that the generator is linear and assume convexity of the cost function, as well as the action space. They showed existence of an optimal strong control, that is an optimal control adapted to the original filtration of the Brownian motion. In a second paper [2], they proved existence of a relaxed as well as of a strict control for a system of controlled decoupled non linear FBSDE, where the diffusion coefficient is not controlled and the generator does not depend on the second variable Z.

Our aim is to prove existence of optimal controls for systems driven by FBSDEs. In the first part, we suppose that our FBSDE is decoupled, that is the forward part of the equation does not contain the backward parts Y and Z and the diffusion coefficient depends explicitly on the control variable. We use the formulation by martingale problems for the forward SDE and the Meyer-Zheng compactness criteria, to prove the existence result. The proof is inspired from a technique used by Pardoux [19]. Note that, our result improves [2] to the case where the diffusion coefficient is controlled, [10] to FBSDEs and [8] to continuous connficients in the case where the generator does not depend upon Z. Moreover, note that under our assumptions (the continuity of the coefficients b and σ) there are difficulties to apply directly HJB techniques as in [8], to obtain the necessary estimates of the solution of the HJB PDE, as well as its gradient. It should be mentioned that in [8], the coefficients of the forward equation are Lipschitz continuous. Moreover, our approach based on probabilistic techniques could be used for more general FBSDEs and BSDEs, namely non Markov BSDEs, for which PDE (HJB) techniques do not work. Of course, it should be mentioned that in [8], the authors treat the case of a generator depending explicitely upon Z.

In the second part, we deal with a coupled FBSDEs, where the coefficients depend on X and Y, but not on the second variable Z, with an uncontrolled diffusion coefficient. We use Jakubowsky's S-topology and a suitable version of the Skorokhod theorem to prove the main result. Under some additional convexity assumption, we show that the relaxed optimal control, which is a measure-valued process, is in fact realized as a strict control.

2 Formulation of the problem

We study the existence of optimal controls for systems driven by FBSDEs of the form (1.1) where the cost functional over the time interval $[0, T]$ is given by (1.2).

We assume that b, σ, l, h, g and ψ are given mappings, $(W_t, t \geq 0)$ is a standard Brownian motion, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, satisfying the usual conditions, where $(\mathcal{F}_t)$ is not necessarily the Brownian filtration. (M_t) is a square integrable martingale which is orthogonal to the Brownian motion (W_t) and X, Y, Z are square integrable adapted processes. The control variable U_t, called strict control, is a measurable, $\mathcal{F}_t-$ adapted process with values in some compact metric space K.

The objective of the controller is to minimize this cost functional, over the class $\mathbb{U}$ of admissible controls, that is, adapted processes with values in the set K, called the action space. A control $\widehat{U}$ satisfying $J\left(\widehat{U}\right) = \inf\left\{J\left(U\right), U \in \mathbb{U}\right\}$ is called optimal.

Without additional convexity conditions, an optimal control may fail to exist in the set $\mathbb{U}$ of strict controls even in deterministic control. It should be noted that the set $\mathbb{U}$ is not equipped with a compact topology. The idea is then to introduce a new class of admissible controls, in which the controller chooses at time t, a probability measure $q_t(du)$ on the control set K, rather than an element $U_t \in \mathbb{U}$. These are called relaxed controls. It turns out that this class of controls enjoys good topological properties. If $q_t(du) = \delta_{U_t}(du)$ is a Dirac measure charging U_t for each t, then we get a strict control as a special case. Thus the set of strict controls may be identified as a subset of relaxed controls.

To be convinced on the fact that strict controls may not exist even in the simplest cases, let us consider a deterministic example.

The problem is to minimize the following cost function: $J(U) = \int_0^T \left(X^U(t)\right)^2 dt$ over the set $\mathbb{U}$ of measurable functions $U : [0, T] \to \{-1, 1\}$, where $X^U(t)$ denotes the solution of $dX^{U.}(t) = U(t)dt$, $X(0) = 0$. We have $\inf_{U. \in \mathbb{U}} J(U.) = 0$.

Indeed, consider the following sequence of controls:

$$U_n(t) = (-1)^k \text{ if } \frac{kT}{n} \leq t \leq \frac{(k+1)T}{n},\ 0 \leq k \leq n-1.$$

Then clearly $|X^{U_n}(t)| \leq 1/n$ and $|J(U_n)| \leq T/n^2$ which implies that $\inf_{u\in\mathbb{U}} J(u) = 0$. There is however no control U such that $J(U_.) = 0$. If this would have been the case, then for every t, $X^{U_\cdot}(t) = 0$. This in turn would imply that $U_t = 0$, which is impossible. The problem is that the sequence (U_n) has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality. If we identify $U_n(t)$ with the Dirac measure $\delta_{U_n(t)}(du)$ and set $q^n(dt, du) = \delta_{U_n(t)}(du).dt$, we get a measure on $[0, T] \times K$. Then $(q^n(dt, du))_n$ converges weakly to $(T/2)dt\cdot[\delta_{-1} + \delta_1](du)$. This suggests that the set of strict controls is too narrow and should be embedded into a wider class enjoying compactness properties. The idea of a relaxed control is to replace the K-valued process (U_t) with a $\mathcal{P}(K)$-valued process (q_t), where $\mathcal{P}(K)$ is the space of probability measures equipped with the topology of weak convergence.

Let $\mathbb{V}$ be the set of Radon measures on $[0, T] \times K$, whose projections on $[0, T]$ coincide with the Lebesgue measure dt. Equipped with the topology of stable convergence of measures, $\mathbb{V}$ is a compact metric space (see [15]). Stable convergence is required for bounded measurable functions $\phi(t, u)$, such that for each fixed $t \in [0, T]$, $\phi(t,\cdot)$ is continuous. That is, a sequence (μ^n) in $\mathbb{V}$ converges in the stable topology to μ, if for every bounded measurable function $\phi : [0, T] \times K \to \mathbb{R}$ such that for each fixed $t \in [0, T]$, $\phi(t,\cdot)$ is continuous,

$$\int_0^T \int_K \phi(t, u).\mu^n(dt, du) \text{ converges to } \int_0^T \int_K \phi(t, u).\mu(dt, du).$$

Definition 2.1. A measure-valued control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable q with values in $\mathbb{V}$, such that $q(\omega, dt, du) = dt.q(\omega, t, du)$ and where $q(\omega, t, du)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.q$ is $\mathcal{F}_t-$measurable. We denote by $\mathcal{R}$ the set of such processes q.

Definition 2.2. A strict control is a term $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, U_t, W_t, X_t, Y_t, Z_t)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) U_t is a K-valued process, progressively measurable with respect to $(\mathcal{F}_t)$.

(3) W_t is a $(\mathcal{F}_t, P)$- Brownian motion and $(W_t, X_t, Y_t, Z_t, M_t)$ satisfies FBSDE (1.1), where (M_t) is a square integrable martingale, orthogonal to (W_t).

The controls, as defined in the last definition, are called weak controls, because of the possible change of the probability space and the Brownian motion with U_t.

Definition 2.3. A relaxed control is a term $\alpha = (\Omega, \mathcal{F}, \mathcal{F}_t, P, q_t, W_t, X_t, Y_t, Z_t, M_t)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) q is a measure-valued control on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

(3) W_t is a $(\mathcal{F}_t, P)$- Brownian motion and $(W_t, X_t, Y_t, Z_t, M_t)$ satisfies the following FBSDE

$$\begin{cases} X_t = x + \int_0^t \int_K b\,(s, X_s, Y_s, u)\, q(s, du).ds + \int_0^t \sigma\,(s, X_s, Y_s, Z_s)\, dW_s, \\ Y_t = \varphi\,(X_T) + \int_t^T \int_K h\,(s, X_s, Y_s, u)\, q(s, du).ds - \int_t^T Z_s dW_s - (M_T - M_t). \end{cases} \tag{2.1}$$

where (M_t) is a square integrable martingale, orthogonal to (W_t).

Accordingly, the relaxed cost functional is defined by

$$J\,(q) = E\left[\psi(X_T) + g\,(Y_0) + \int_0^T \int_K l\,(s, X_s, Y_s, , Z_s, u)\, q(s, du) ds\right]. \tag{2.2}$$

Remark 2.4. The appearance of the orthogonal martingale (M_t) in 1.1 is due to the fact that the filtration associated to the optimal control is usually larger than the Brownian filtration. By the Kunita-Watanabe representation theorem, the conditional expectation with respect to this filtration is a sum of a Brownian stochastic integral and an orthogonal martingale.

Remark 2.5. As in classical control problems driven by Itô SDEs, the cost functional 1.2 may be defined as $J\,(U) = E\left[\overline{g}\left(\overline{Y}_0\right)\right]$ where $\overline{Y}_t = \left(Y_t, Y_t^{k+1}\right)$, with Y_t^{k+1} the solution of the one-dimensional BSDE

$$\begin{cases} -dY_t^{k+1} = l\,(t, X_t, Y_t, U_t)\, dt - \overline{Z}_t dW_t - d\overline{M}_t, \\ Y_T^{k+1} = \psi(X_T), \end{cases}$$

and $\overline{g}\left(\overline{Y}_0\right) = g\,(Y_0) + Y_0^{k+1}$.

This property is known in optimal control as the equivalence between the Bolza and Mayer problems.

Notations
In the sequel we denote by:

- $\mathcal{C}([0,T];\mathbb{R}^d)$: the space of continuous functions from $[0,T]$ into $\mathbb{R}^d$, equipped with the topology of uniform convergence,
- $\mathcal{D}([0,T];\mathbb{R}^m)$: the Skorohod space of càdlàg functions from $[0,T]$ into $\mathbb{R}^m$, that is functions which are continuous from the right with left hand limits endowed with the Meyer-Zheng topology of convergence in dt-measure,
- $\mathcal{S}^2([0,T];\mathbb{R}^n) = \{X : [0,T]\times\Omega \to \mathbb{R}^n; X$ is progressively measurable and $E(\sup_{0\le t\le T}|X_t|^2) < +\infty\}$,
- $\mathcal{M}^2([0,T];\mathbb{R}^{k\times m}) = \{Z : [0,T]\times\Omega \to \mathbb{R}^{k\times m};$ Z is progressively measurable and $E\int_0^T |Z_t|^2 dt < +\infty\}$,
- $L^2_{\mathcal{F}}([0,T];\mathbb{R}^k) = \{f(t,\omega) : [0,T]\times\Omega \to \mathbb{R}^k;$ $\mathcal{F}_t$ - adapted such that $E\left[\int_0^T |f(t,\omega)|^2 dt\right] < +\infty\}$.

3 Control of decoupled FBSDEs

In this section we deal with decoupled FBSDEs, that is the forward part does not contain (Y,Z) the solution of the backward part. We suppose that the diffusion coefficient σ depends explicitly on the control variable and the driver h as well as the instantaneous cost l do not depend on Z. More precisely, our system is governed by the following equation

$$\begin{cases} dX_t &= b(t,X_t,U_t)\,dt + \sigma(t,X_t,U_t)\,dW_t, \\ X(0) &= x, \\ -dY_t &= h(t,X_t,Y_t,U_t)\,dt - Z_t dW_t - dM_t, \\ Y_T &= \varphi(X_T), \end{cases} \tag{3.1}$$

defined on some filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, (W_t) is an m-dimensional Brownian motion and (M_t) is a square integrable martingale which is orthogonal to (W_t). The coefficients of our FBSDE are defined as follows

$$\begin{aligned} &b : [0,T]\times\mathbb{R}^d\times K \to \mathbb{R}^d, \\ &\sigma : [0,T]\times\mathbb{R}^d\times K \to \mathbb{R}^{d\times m}, \\ &h : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times K \to \mathbb{R}^k \\ &\varphi : \mathbb{R}^d \to \mathbb{R}^k. \end{aligned}$$

Let us define the cost functional over $[0,T]$ by

$$J(U) = E\left[\psi(X_T) + g(Y_0) + \int_0^T l(t,X_t,Y_t,U_t)\,dt\right], \tag{3.2}$$

where

$$\begin{aligned} &l : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times K \to \mathbb{R}, \\ &\psi : \mathbb{R}^d \to \mathbb{R}, \\ &g : \mathbb{R}^k \to \mathbb{R}. \end{aligned}$$

$(\mathbf{H_1})$ Assume that the functions b, σ, h, φ are continuous and bounded.

$(\mathbf{H_2})$ h is Lipschitz in the variable y uniformly in (t,x,u), i.e: there exists a constant $C>0$ such that for every $t\in[0,T]$, $u\in K$, y , $y'\in\mathbb{R}^d$,

$$|h(t,x,y,u) - h(t,x,y',u)| \le C\,|y-y'|\,.$$

$(\mathbf{H_3})$ Assume that l, ψ and g are continuous and bounded functions.

The infinitesimal generator L, associated with the forward part of our equation, acting on functions f in $C_b^2(R^d;R)$, is defined by

$$Lf(t,x,u)=\left(\frac{1}{2}\sum_{i,j}a_{ij}\frac{\partial^2 f}{\partial x_i x_j}+\sum_i b_i\frac{\partial f}{\partial x_i}\right)(t,x,u),$$

where $a_{ij}(t,x,u)$ denotes the generic term of the symmetric matrix $\sigma\sigma^*(t,x,u)$.

As it is well known, weak solutions for Itô SDEs are equivalent to the existence of solutions for the corresponding martingale problem. Then one can rewrite definition 2.1 and definition 2.2, by using the formulation of martingale problems for the forward part. This simplifies taking limits and does not pose the problem of the relaxation of the stochastic integral part see [5, 4, 10]. We can define a strict control using martingale problems as follows.

Definition 3.1. A strict control is a term $\alpha=(\Omega,\mathcal{F},\mathcal{F}_t,P,U_t,X_t,Y_t,M_t)$ such that

(1) $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) U_t is an K-valued process, progressively measurable with respect to $(\mathcal{F}_t)$.

(3i) (X_t) is $\mathbb{R}^d-$valued $\mathcal{F}_t-$ adapted, with continuous paths, such that

$$f(X_t)-f(x)-\int_0^t Lf(s,X_s,U_s)ds \text{ is a P}-\text{martingale.} \tag{3.3}$$

(3ii) (Y_t,M_t) is the solution of the following backward SDE

$$Y_t=\varphi\left(X_T\right)+\int_t^T h\left(s,X_s,Y_s,U_s\right)ds-\left(M_T-M_t\right), \tag{3.4}$$

where (M_t) is a square integrable $(\mathcal{F}_t)-$ martingale.

Definition 3.2. A relaxed control is a term $\alpha=(\Omega,\mathcal{F},\mathcal{F}_t,P,q_t,X_t,Y_t,M_t)$ such that

(1) $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) q is a measure-valued control on $(\Omega,\mathcal{F},\mathcal{F}_t,P)$.

(3i) (X_t) is $\mathbb{R}^d-$valued $\mathcal{F}_t-$adapted, with continuous paths, such that

$$f(X_t)-f(x)-\int_0^t\int_K Lf(s,X_s,u).q(s,du)ds \text{ is a P}-\text{martingale.} \tag{3.5}$$

(3ii) (Y_t,M_t) is the solution of the following backward SDE

$$Y_t\;=\;\varphi\left(X_T\right)+\textstyle\int_t^T\int_K h\left(s,X_s,Y_s,u\right).q(s,du).ds-\left(M_T-M_t\right), \tag{3.6}$$

where (M_t) is a square integrable $(\mathcal{F}_t)-$martingale.

By a slight abuse of notation, we will often denote a relaxed control by q instead of specifying all the components.

The cost functional associated to a relaxed control is now defined by

$$J\left(q\right)=E\left[\psi(X_T)+g\left(Y_0\right)+\int_0^T\int_K l\left(s,X_s,Y_s,u\right)q(s,du)ds\right]. \tag{3.7}$$

The main result of this section is given by the following Theorem.

Theorem 3.3. *Under assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$, *the relaxed control problem has an optimal solution.*

The proof is based on some auxiliary results on the tightness of the processes under consideration and the identification of the limits.

Let $(q^n)_{n\geq 0}$ be a minimizing sequence, that is $\lim\limits_{n\to\infty}J\left(q^n\right)=\inf\limits_{q\in\mathcal{R}}J\left(q\right)$ and let (X^n,Y^n,M^n) be a solution of our FBSDE, where:

(i) (X_t^n) is $\mathbb{R}^d-$valued $\mathcal{F}_t-$adapted, with continuous paths, such that

$$f(X_t^n) - f(x) - \int_0^t \int_K Lf(s, X_s^n, u).q^n(s, du).ds \text{ is a } P\text{-martingale}$$

(ii) (Y_t^n, M_t^n) is the solution of the following backward SDE

$$Y_t^n = \varphi(X_T^n) + \int_t^T \int_K h(s, X_s^n, Y_s^n, u) .q^n(s, du).ds - (M_T^n - M_t^n),$$

and (M_t^n) is a square integrable continuous $(\mathcal{F}_t)-$martingale.

The proof of the main result consists in proving that the sequence of distributions of the processes (q^n, X^n, Y^n, M^n) is tight for a certain topology, on the state space and then show that, we can extract a subsequence, which converges in law to a process $(\widehat{q}, \widehat{X}, \widehat{Y}, \widehat{M})$, satisfying the same FBSDE. To complete the proof, we show that under some regularity conditions the sequence of cost functionals $(J(q^n))_n$ converge to $J(\widehat{q})$ which is equal to $\inf_{q \in \mathcal{R}} J(q)$ and then $(\widehat{q}, \widehat{X}, \widehat{Y}, \widehat{M})$ is optimal.

Lemma 3.4. *The family of relaxed controls $(q^n)_n$ is tight in $\mathbb{V}$.*

Proof. $[0, T] \times K$ being compact, then by Prokhorov's theorem, the space $\mathbb{V}$ of probability measures on $[0, T] \times K$ is also compact for the topology of weak convergence. The fact that q^n, $n \geq 0$ are random variables with values in the compact set $\mathbb{V}$ yields that the family of distributions associated to $(q^n)_{n \geq 0}$ is tight. ■

Lemma 3.5. *i) The family $(q^n, X^n)_n$ of solutions of the martingale problem is tight on the space $\mathbb{V} \times \mathcal{C}(([0, T]; R^d)$.*

ii) There exists a subsequence which converges in law to $(\widehat{q}, \widehat{X})$, whose law is a solution of the martingale problem, that is for each $f \in \mathcal{C}_b^2$, $f(\widehat{X}_t) - f(x) - \int_0^t \int_K Lf(s, \widehat{X}_s, u).\widehat{q}(s, du).ds$ is a P-martingale

Proof. Let us give the outlines of the proof which is inspired from [10], Theorem 3.4.
i) Following [23], Theorem 1.4.6, it is sufficient to show that for each positive f in $\mathcal{C}_b^2$, there exists a constant A_f such that: $f(X_t) + A_f.t$ is a supermartingale under the distribution $P_n = P^{(q^n, X^n)}$ on the canonical space $\mathbb{V} \times \mathcal{C}([0, T]; R^d)$ of the couple (q^n, X^n). Let

$$A_f = \sup \left\{ |Lf(t, x, u)|; (t, x, u) \in [0, T] \times \mathbb{R}^d \times K \right\}.$$

A_f is finite since the coefficients b and σ defining the operator L are bounded.
Since for each n, $f(X_t) - f(x) - \int_0^t \int_K Lf(s, X_s, u).q(s, du).ds := C_t f(x, q)$ is a P_n-martingale, then $f(X_t) + A_f.t$ is a positive supermartingale. Then (X^n) is tight in $\mathcal{C}$ endowed with the topology of uniform convergence.
ii) The sequence (q^n, X_n) being tight, then we can extract a subsequence still denoted by (q^n, X^n) which converges weakly to $(\widehat{q}, \widehat{X})$. In particular, for every bounded (x, q)-continuous, $\mathcal{C}_s \otimes \mathbb{V}_s$-measurable functions ($\mathcal{C}_s$ and $\mathbb{V}_s$ are the $\sigma-$fields generated by the coordinates until for $t \leq s$), we have

$$P_n [\phi(q, x) (C_t f(x, q) - C_s f(x, q))] \text{ converges to } \widehat{P} [\phi(q, x) (C_t f(x, q) - C_s f(x, q))]$$

where $\widehat{P}$ denotes the law of the couple $(\widehat{q}, \widehat{X})$ in the space $\mathbb{V} \times \mathcal{C}([0, T]; R^d)$.
$C_t f(x, q)$ being a P_n-martingale, then $P_n [\phi(x, q) (C_t f(x, q) - C_s f(x, q))] = 0$.
Hence the limit $\widehat{P} [\phi(q, x) (C_t f(x, q) - C_s f(x, q))] = 0$ and thus the law $\widehat{P}$ of the couple $(\widehat{q}, \widehat{X})$ is a solution of the martingale problem. ■

Lemma 3.6. *i) The sequence (Y^n, M^n) is tight on the space $\mathcal{D}^2$ equipped with the Meyer-Zheng topology.*

ii) There exists a subsequence still denoted by (q^n, X^n, Y^n, M^n) which converges weakly to $(\widehat{q}, \widehat{X}, \widehat{Y}, \widehat{M})$, in the space $\mathbb{V} \times \mathcal{C} \times \mathcal{D}^2$. Moreover $(\widehat{q}, \widehat{X}, \widehat{Y}, \widehat{M})$ satisfies:

For every $f \in \mathcal{C}_b^2$,

$$f(\widehat{X}_t) - f(x) - \int_0^t \int_K Lf(s, \widehat{X}_s, u).\widehat{q}(s, du).ds \textit{ is a } \mathcal{F}_t^{\widehat{q}, \widehat{X}, \widehat{Y}} - \textit{martingale}$$

$$\widehat{Y}_t = \varphi\left(\widehat{X}_T\right) + \int_t^T \int_K h\left(s, \widehat{X}_s, \widehat{Y}_s, u\right) \widehat{q}(s, du).ds - \left(\widehat{M}_T - \widehat{M}_t\right) \tag{3.8}$$

where $\left(\widehat{M}_t\right)$ is a square integrable $\mathcal{F}_t^{\widehat{q}, \widehat{X}, \widehat{Y}}-$martingale.

Proof. i) Using standard techniques from BSDEs theory it is not difficult to prove that $E\left(\sup_{0\leq t\leq T}|Y_t^n|^2\right)\leq CE\left(|\varphi(X_T^n)|^2+\int_0^T\sup_{u\in K}\left|h\left(s,\widehat{X}_s,0,u\right)\right|^2ds\right)$. Then using assumption $\mathbf{(H_1)}$ it follows that:

$$\sup_n E\left(\sup_{0\leq t\leq T}|Y_t^n|^2+\langle M^n\rangle_T\right)<+\infty.$$

Let us denote the conditional variation

$$V_t(Y^n)=\sup E\left(\sum\left|E\left[\left(Y_{t_{i+1}}^n-Y_{t_i}^n\right)/F_{t_i}^{X^n}\right]\right|\right),$$

where the supremum is taken over all the partitions of the interval $[0,t]$. One can easily prove that

$$V_t(Y^n)\leq E\left(\int_0^T\sup_{u\in K}|h(s,X_s^n,Y_s^n,u)|\,ds\right).$$

The assumptions made on the coefficients ensure that

$$\sup_n\left[V_t(Y^n)+\sup_{0\leq t\leq T}E|Y_t^n|+\sup_{0\leq t\leq T}E|M_t^n|\right]<+\infty.$$

Then following [19, 17], the sequence (Y^n,M^n) satisfies the Meyer-Zheng criterion for tightness of families of quasi-martingales.

ii) Since the sequence $(q^n,X^n,Y^n,M^n)_n$ is tight, then there exist a subsequence still denoted by $(q^n,X^n,Y^n,M^n)_n$ which converges weakly to $(\widehat{q},\widehat{X},\widehat{Y},\widehat{M})$ on the space $\mathbb{V}\times\mathcal{C}\times\mathcal{D}^2$, where $\mathcal{C}$ is equipped with the topology of uniform convergence and $\mathcal{D}$ is equipped with the Meyer-Zheng topology. By using the fact that for each $t\leq T$, the mapping $(q,x,y)\longrightarrow\int_t^T\int_K h(t,x_s,y_s,u).q(s,du)ds$ is continuous from $\mathbb{V}\times\mathcal{C}\times\mathcal{D}$ into $\mathbb{R}$, one can pass to the limit in the BSDE and get

$$\widehat{Y}_t=\varphi\left(\widehat{X}_T\right)+\int_t^T\int_K h\left(s,\widehat{X}_s,\widehat{Y}_s,u\right).\widehat{q}(s,du).ds-\left(\widehat{M}_T-\widehat{M}_t\right).$$

Let us show that $\widehat{M}_t$ and $\phi(\widehat{X}_t)-\phi(\widehat{X}_s)-\int_0^t\int_K L\phi(s,\widehat{X}_s,u).\widehat{q}(s,du).ds$ are both martingales with respect to the natural filtration $\mathcal{F}_t=\mathcal{F}_t^{\widehat{q},\widehat{X},\widehat{Y}}$. For any s,t such that $0\leq s\leq t\leq T$ and Φ_s a bounded continuous mapping from $\mathbb{V}_s\times\mathcal{C}([0,s],\mathbb{R})\times\mathcal{D}([0,s],\mathbb{R})$ and for each $\phi\in C_b^2$:

$$E\left[\Phi_s(q^n,X^n,Y^n)\left(\phi(X_t^n)-\phi(X_s^n)-\int_0^t\int_K L\phi(s,X_s^n,u).q^n(s,du).ds\right)\right]\longrightarrow 0\text{ as }n\longrightarrow\infty$$

and for each n

$$E\left[\Phi_s(q^n,X^n,Y^n)\left(\int_0^\varepsilon\left(M_{t+r}^n-M_{s+r}^n\right).dr\right)\right]=0,$$

where $\mathbb{V}_s$ denotes the restriction of measures to the interval $[0,s]$. The fact that (q^n,X^n,Y^n) is weakly convergent and $E(\sup_{0\leq t\leq T}|M_t^n|^2)$ is finite yield

$$E\left[\Phi_s(\widehat{q},\widehat{X},\widehat{Y})\left(\phi(\widehat{X}_t)-\phi(\widehat{X}_s)-\int_0^t\int_K L\phi(s,\widehat{X}_s,u).\widehat{q}(s,du).ds\right)\right]=0$$
$$E\left[\Phi_s(\widehat{q},\widehat{X},\widehat{Y})\left(\int_0^\varepsilon\left(\widehat{M}_{t+r}-\widehat{M}_{s+r}\right).dr\right)\right]=0.$$

In the last equality, dividing by ε and then sending it to 0 and by the right continuity of the martingale $\left(\widehat{M}_t\right)$, we get

$$E\left[\Phi_s(\widehat{q},\widehat{X},\widehat{Y})\left(\widehat{M}_t-\widehat{M}_s\right)\right]=0.$$

These identities are valid for all functions Φ_s described above and for all $s\leq t$. Then $\widehat{M}_t$ and $\phi(\widehat{X}_t)-\phi(\widehat{X}_s)-\int_0^t\int_K L\phi(s,\widehat{X}_s,u).\widehat{q}(s,du).ds$ are both $\mathcal{F}_t^{\widehat{q},\widehat{X},\widehat{Y}}-$martingales. ■

Proof of Theorem 3.5. According to Lemma 3.6 and the assumptions **(H₁)-(H₃)** we have

$$\begin{aligned}
\inf_{q\in\mathcal{R}} J(q) &= \lim_{n\to\infty} J(q^n) \\
&= \lim_{n\to\infty} E\left[\psi(X_T^n) + g(Y_0^n) + \int_0^T \int_K l(t, X_t^n, Y_t^n, u)\, q_t^n(du)\, dt\right] \\
&= E\left[\psi(\hat{X}_T) + g\left(\hat{Y}_0\right) + \int_0^T \int_K l\left(t, \hat{X}_t, \hat{Y}_t, u\right) \hat{q}_t(du)\, dt\right],
\end{aligned}$$

which means that $\hat{q}$ is an optimal control. ■

Corollary 3.7. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Moreover assume that for every* $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k$, *the set*

$$(b,\sigma,h,l)(t,x,y,K) := \left\{ b_i(t,x,u), (\sigma\sigma^*)_{ij}(t,x,u), h_j(t,x,y,u), l(t,x,y,u) / u \in K, i=1,...,d, \quad j=1,...,k \right\} \tag{3.9}$$

is convex in $\mathbb{R}^{d+d\times d+k+1}$. *Then, the relaxed optimal control* $\hat{q}_t$ *has the form of a Dirac measure charging a strict control* $\hat{U}_t$, *that is* $\hat{q}_t(du) = \delta_{\hat{U}_t}(du)$.

Proof. We put

$$\int_K h\left(t,\hat{X}_t,\hat{Y}_t,u\right)\hat{q}_t(du) := \hat{h}(t,w) \in h(t,x,y,\mathbb{U}),\ \int_K l\left(t,\hat{X}_t,\hat{Y}_t,u\right)\hat{q}_t(du) := \hat{l}(t,w) \in l(t,x,y,\mathbb{U}),$$
$$\int_K b\left(t,\hat{X}_t^n,u\right)\hat{q}_t(du) := \widehat{b}(t,w) \in b(t,x,\mathbb{U}),\ \int_K a\left(t,\hat{X}_t^n,u\right)\hat{q}_t(du) := \widehat{a}(t,w) \in a(t,x,\mathbb{U}), \text{ where } a = \sigma\sigma^*.$$

From $(\mathbf{H}_1) - (\mathbf{H}_3)$ and the measurable selection theorem (see [25] p. 74 or [10]), there is a K-valued, $\mathcal{F}^{\hat{X},\hat{Y},\hat{q}}$−adapted process $\hat{U}$, such that for every $s \in [0,\ T]$,

$$\begin{aligned}
\left(\hat{h},\hat{l}\right)(s,w) &= (h,l)\left(s,\hat{X}(s,w),\hat{Y}(s,w),\hat{U}(s,w)\right), \\
\left(\hat{b},\hat{a}\right)(s,w) &= (b,a)\left(s,\hat{X}(s,w),\hat{U}(s,w)\right).
\end{aligned}$$

Hence, for every $t \in [0,\ T]$ and $w \in \hat{\Omega}$, we have

$$\int_K h\left(t,\hat{X}_t,\hat{Y}_t,u\right)\hat{q}_t(du) = h\left(t,\hat{X}_t,\hat{Y}_t,\hat{U}_t\right),$$

$$\int_K l\left(t,\hat{X}_t,\hat{Y}_t,u\right)\hat{q}_t(du) = l\left(t,\hat{X}_t,\hat{Y}_t,\hat{U}_t\right),$$

and

$$\int_K b\left(t,\hat{X}_t,u\right)\hat{q}_t(du) = b\left(t,\hat{X}_t,\hat{U}_t\right),$$

$$\int_K a\left(t,\hat{X}_t,u\right)\hat{q}_t(du) = a\left(t,\hat{X}_t,\hat{U}_t\right).$$

Then the process $(\hat{X}_t, \hat{Y}_t, \widehat{M}_t)$ satisfies, for each $t \in [0,\ T]$:
1) $(\widehat{X}_t)$ is a $\mathbb{R}^d$−valued $\mathcal{F}_t$−adapted, with continuous paths, such that

$$f(\widehat{X}_t) - f(x) - \int_0^t Lf(s,\widehat{X}_s,\hat{U}_s)ds \text{ is a } P\text{-martingale}$$

2) $(\widehat{Y}_t, \widehat{M}_t)$ solves the following BSDE

$$\widehat{Y}_t = \varphi\left(\widehat{X}_T\right) + \int_t^T h\left(s,\widehat{X}_s,\widehat{Y}_s,\hat{U}_s\right)ds - \left(\widehat{M}_T - \widehat{M}_t\right)$$

It follows that $J(\widehat{q}) = J(\hat{U})$, which achieves the proof. ■

4 Control of coupled FBSDEs

In this section, we consider coupled FBSDEs, where all the coefficients depend only on (X, Y) but not on the second backward component Z. Moreover the diffusion coefficient σ does not depend on the control variable. More precisely, the controlled FBSDE takes the form

$$\begin{cases} dX_t & = & b\left(t, X_t, Y_t, U_t\right) dt + \sigma\left(t, X_t, Y_t\right) dW_t, \\ -dY_t & = & h\left(t, X_t, Y_t, U_t\right) dt - Z_t dW_t - dM_t, \\ X_0 = x, & & Y_T = \varphi\left(X_T\right), \end{cases} \tag{4.1}$$

and the cost functional is given by

$$J\left(U_{\cdot}\right) = E\left[\psi(X_T) + g\left(Y_0\right) + \int_0^T l\left(t, X_t, Y_t, U_t\right) dt\right], \tag{4.2}$$

where ψ, g, l satisfy $(\mathbf{H_3})$.
Assume that the coefficients

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times K \to \mathbb{R}^d, \\ \sigma &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^{d\times m}, \\ h &: [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times K \to \mathbb{R}^k, \\ \varphi &: \mathbb{R}^d \to \mathbb{R}^k, \end{aligned}$$

of the FBSDE 4.1 satisfy the following conditions:

$(\mathbf{H}_4)$ b, σ, h are bounded measurable, Lipshitz in (x, y) uniformly in (t, u) and continuous in u.

$(\mathbf{H_5})$ Assume that for each admissible control U, the coefficients b, σ, h, φ of the FBSDE 4.1 satisfy the monotonicity conditions as in [22] Theorem 2.2, page 828.

Remark 4.1. Under assumption $(\mathbf{H_5})$ and for each admissible control U, the FBSDE 4.1 admits a unique strong solution.

The main result of this section is given by the following theorem.

Theorem 4.2. *The relaxed control problem defined by 4.1 and 4.2 has an optimal solution.*

The proof is based on tightness properties of the underlying processes.
As in the last section, let $(q^n)_{n\geq 0}$ be a minimizing sequence for the relaxed control problem, that is

$$\lim_{n\to\infty} J\left(q^n\right) = \inf_{q\in\mathcal{R}} J\left(q\right),$$

Let (X^n, Y^n, Z^n) be the unique strong solution of our FBSDE

$$\begin{cases} X_t^n = x + \int_0^t \int_K b\left(s, X_s^n, Y_s^n, u\right) q^n(s, du).ds + \int_0^t \sigma\left(s, X_s^n, Y_s^n\right) dW_s \\ Y_t^n = \varphi\left(X_T^n\right) + \int_t^T \int_K h\left(s, X_s^n, Y_s^n, u\right) q^n(s, du).ds - \int_t^T Z_s^n dW_s, \end{cases} \tag{4.3}$$

defined on the natural filtration of the Brownian motion (W_t). In this case the orthogonal martingales (M_t^n) disappears, due to the uniqueness of solutions.

The proof of Theorem 4.2 consists in showing that the sequence $(q^n, X^n, Y^n, \int_0^\cdot Z_s^n dW_s)$ is tight and there exists a subsequence converging weakly to $(\widehat{q}, \widehat{X}, \widehat{Y}, \widehat{N})$. Furthermore these processes satisfy the FBSDE

$$\begin{cases} \widehat{X}_t = x + \int_0^t \int_K b\left(s, \widehat{X}_s, \widehat{Y}_s, u\right) \widehat{q}(s, du).ds + \int_0^t \sigma\left(s, \widehat{X}_s, \widehat{Y}_s\right) d\widehat{W}_s, \\ \widehat{Y}_t = \varphi\left(\widehat{X}_T\right) + \int_t^T \int_K h\left(s, \widehat{X}_s, \widehat{Y}_s, u\right) \widehat{q}(s, du).ds - \int_t^T \widehat{Z}_s d\widehat{W}_s - (\widehat{M}_T - \widehat{M}_t), \end{cases} \tag{4.4}$$

where $\widehat{M}_t$ is a square integrable martingale which is orthogonal to the Brownian motion $\widehat{W}_t$.

Lemma 4.3. *Let (X^n, Y^n, Z^n) be the unique solution of equation (4.3). There exists a positive constant C such that*

$$\sup_n E\left(\sup_{0\leq t\leq T} |X_t^n|^2 + \sup_{0\leq t\leq T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds\right) \leq C. \tag{4.5}$$

Proof. Using assumption $(\mathbf{H_5})$, it is easy to check that

$$\sup_n\{E(\sup_{0\leq t\leq T}|X_t^n|^2)\}<\infty \tag{4.6}$$

Using the Burkholder-Davis-Gundy and Schwarz inequalities, it follows that the local martingale $\int_t^T Y_s^n Z_s^n dW_s$ is uniformly integrable. Then by Itô's formula and assumption $(\mathbf{H_2})$, it holds that

$$E(|Y_t^n|^2+\int_t^T|Z_s^n|^2ds)=E\left(|\varphi(X_T^n)|^2+2\int_t^T\int_K\langle Y_s^n,h(s,X_s^n,Y_s^n,u)\rangle q_s^n(du)ds\right).$$

Hence,

$$\begin{aligned}E(|Y_t^n|^2+\int_t^T|Z_s^n|^2ds)&\leq E(|\varphi(X_T^n)|^2+\int_t^T|Y_s^n|^2ds)\\&\quad+E(\int_t^T\int_K|h(s,X_s^n,Y_s^n,u)|^2q_s^n(du)ds).\end{aligned}$$

The result follows from Gronwall's Lemma and BDG inequality

$$\sup_n E\left(\sup_{0\leq t\leq T}|Y_t^n|^2+\int_0^T|Z_s^n|^2ds\right)<\infty.$$

■

Lemma 4.4. *Let (X^n,Y^n,Z^n) be the unique solution of equation (4.3). Then the sequence $(Y^n,\int_0^\cdot Z_s^n dW_s)$ is tight on the space $\mathbb{D}\left([0,T];\mathbb{R}^k\right)\times\mathbb{D}\left([0,T];\mathbb{R}^k\right)$ endowed with the Jakubowski S-topology.*

Proof. Let $0=t_0<t_1<...<t_n=T$, and define the conditional variation by

$$CV\left(Y^n\right):=\sup E\left[\sum_i\left|E\left(Y_{t_{i+1}}^n-Y_{t_i}^n\right)\diagup\mathcal{F}_{t_i}^W\right|\right],$$

where the supremum is taken over all partitions of the interval $[0,T]$. It is proved in [19] that

$$CV\left(Y^n\right)\leq\mathbb{E}\left[\int_0^T\int_K|h\left(s,X_s^n,Y_s^n,u\right)|q_s^n\left(du\right)ds\right].$$

It follows from (4.5) that

$$\sup_n\left[CV\left(Y^n\right)+\sup_{0\leq t\leq T}E\left|Y_t^n\right|+\sup_{0\leq t\leq T}E\left|\int_0^t Z_s^n dW_s\right|\right]<\infty$$

Therefore, the sequence $\left(Y^n,\int_0^\cdot Z_s^n dW_s\right)$ satisfies the Meyer-Zheng criterion [17]. Therefore $(Y^n,\int_0^\cdot Z_s^n dW_s)$ is tight in the Jakubowski S-topology (see the appendix). ■

The next lemma may be proved by standard arguments.

Lemma 4.5. *Let X_t^n be the forward component of equation (4.3).Then the sequence of processes (X^n,W) is tight on the space $\mathcal{C}\left([0,T],\mathbb{R}^d\right)\times\mathcal{C}\left([0,T],\mathbb{R}^m\right)$, endowed with the topology of uniform convergence.*

Proof. of Theorem 4.2 From Lemmas 4.3, 4.4 and 4.5, the sequence $\gamma^n=(q^n,X^n,W,Y^n,N^n)$ where $N^n=\int_0^\cdot Z_s^n dW_s$, is tight on the space $\Gamma=\mathbb{V}\times\mathcal{C}\left([0,T],\mathbb{R}^d\right)\times\mathcal{C}\left([0,T],\mathbb{R}^m\right)\times\left[\mathbb{D}\left([0,T];\mathbb{R}^k\right)\right]^2$.
Using the Skorokhod representation theorem on the space $\mathcal{D}$ endowed with S-topology [13] (see the appendix), there exists a probability space $\left(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}}\right)$, a sequence $\hat{\gamma}^n=\left(\hat{q}^n,\hat{X}^n,\hat{W}^n,\hat{Y}^n,\hat{N}^n\right)$ and $\hat{\gamma}=\left(\hat{q},\hat{X},\hat{W},\hat{Y},\hat{N}\right)$ defined on this space such that:

(i) for each $n\in\mathbb{N}$, law$(\gamma^n)=$ law$(\hat{\gamma}^n)$,

(ii) there exists a subsequence $(\hat{\gamma}^{n_k})$ of $(\hat{\gamma}^n)$, still denoted $(\hat{\gamma}^n)$, which converges to $\hat{\gamma}$, $\hat{\mathbb{P}}$-a.s. on the space Γ,

(iii) the subsequence $(\hat{Y}^n,\hat{N}^n)$ converges to $(\hat{Y},\hat{N})$, $dt\times\hat{\mathbb{P}}-a.s.$, and $(\hat{Y}^n_T,\hat{N}^n_T)$ converges to $(\hat{Y}_T,\hat{N}_T)$ as $n\to\infty$, $\hat{\mathbb{P}}-a.s.$

(iv) $\sup\limits_{0\leq t\leq T}\left|\hat{X}^n_t-\hat{X}_t\right|\to 0,\quad \hat{\mathbb{P}}-a.s.$

According to property (i), it follows that

$$\begin{cases} \hat{X}^n_t &= x+\int_0^t\int_K b\left(s,\hat{X}^n_s,\widehat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds+\int_0^t\sigma\left(s,\hat{X}^n_s,\widehat{Y}^n_s\right)d\hat{W}^n_s \\ \hat{Y}^n_t &= \varphi\left(\hat{X}^n_T\right)+\int_t^T\int_K h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds-\left(\hat{N}^n_T-\hat{N}^n_t\right), \end{cases}\tag{4.7}$$

where $\hat{N}^n_t:=\int_t^T\hat{Z}^n_s d\hat{W}^n_s$.

Combining properties (ii)-(iv), assumptions $(\mathbf{H_2})-(\mathbf{H_5})$ and passing to the limit in the FBSDE (4.7), there exists a countable set $D\subset[0,\ T)$ such that

$$\begin{cases} \hat{X}_t &= x+\int_0^t\int_K b\left(s,\hat{X}_s,\widehat{Y}_s,u\right)\hat{q}_s(du)\,ds+\int_0^t\sigma\left(s,\hat{X}_s,\widehat{Y}_s\right)d\hat{W}_s, \qquad t>0, \\ \hat{Y}_t &= \varphi\left(\hat{X}_T\right)+\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds-\left(\hat{N}_T-\hat{N}_t\right), \qquad t\in[0,\ T]\setminus D. \end{cases}\tag{4.8}$$

Since $\hat{Y}$ and $\hat{N}$ are càdlàg, it follows that for every $t\in[0,T]$,

$$\hat{Y}_t=\varphi\left(\hat{X}_T\right)+\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds+\hat{N}_t-\hat{N}_T.$$

Since all the previous identifications of the limits can be proved similarly, let us prove that:

$$\lim_{n\to\infty}\int_t^T\int_K h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds=\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds.\tag{4.9}$$

We use properties (i), (ii), (iv), Fatou's lemma and Lemma 4.3, to show that there exists a constant C such that:

$$\hat{E}(\int_0^T(|\hat{X}_s|^2+|\hat{Y}_s|^2)ds)\leq C.\tag{4.10}$$

On the other hand, we have

$$\left|\int_t^T\int_K h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds\right|\leq I(n)+J(n),$$

where

$$I(n):=\left|\int_t^T\int_K h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}^n_s(du)\,ds\right|,$$

$$J(n):=\left|\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}^n_s(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds\right|.$$

Let us show that $I(n)$ converges to 0 in probability. Let $\varepsilon>0$ and use the fact that h is Lipschitz in (x,y) to obtain,

$$\begin{aligned}&\hat{\mathbb{P}}\left\{\left|\int_t^T\int_K h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)\hat{q}^n_s(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}^n_s(du)\,ds\right|>\varepsilon\right\}\\ &\leq\frac{1}{\varepsilon}\hat{E}\int_t^T\int_K\left|h\left(s,\hat{X}^n_s,\hat{Y}^n_s,u\right)-h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\right|\hat{q}^n_s(du)\,ds\\ &\leq\frac{C}{\varepsilon}\left[\hat{E}\int_t^T\left|\hat{X}^n_s-\hat{X}_s\right|ds+\hat{E}\int_t^T\left|\hat{Y}^n_s-\hat{Y}_s\right|ds\right].\end{aligned}$$

Now, properties (i)-(iv) and Lemma 4.3 allow us to show that $\hat{E}\int_t^T|\hat{X}^n_s-\hat{X}_s|ds+\hat{E}\int_t^T|\hat{Y}^n_s-\hat{Y}_s|ds$ tends to 0 as n tends to infinity, which yields that $I(n)$ converges to 0 in probability.

Now let us show that $J(n)$ converges to 0 in probability. Let $R>0$ and, put $B:=\{|\hat{X}_s|+|\hat{Y}_s|\leq R\}$ and $\bar{B}:=\Omega-B$. We have,

$$\left|\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s^n(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds\right|\leq I_1(n)+J_1(n),$$

where

$$I_1(n)=:\left|\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)1_B\,\hat{q}_s^n(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)1_B\,\hat{q}_s(du)\,ds\right|,$$

$$J_1(n):=\left|\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)1_{\bar{B}}\,\hat{q}_s^n(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)1_{\bar{B}}\,\hat{q}_s(du)\,ds\right|.$$

Since the function $(s,u)\longmapsto h\left(s,\hat{X}_s,\hat{Y}_s,u\right)1_B$ is bounded, measurable in (s,u) and continuous in u, we deduce by using property (ii) that $I_1(n)$ tends to 0 in probability as n tends to ∞. It remains to prove that $J_1(n)$ tends to 0 in probability as n tends to ∞. We have,

$$\begin{aligned}\hat{E}[J_1(n)]&=\hat{E}(|\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\mathbb{1}_{\bar{B}}\,\hat{q}_s^n(du)\,ds-\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\mathbb{1}_{\bar{B}}\,\hat{q}_s(du)\,ds|)\\&\leq\hat{E}\int_t^T\int_K\left|h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\right|\mathbb{1}_{\bar{B}}\hat{q}_s^n(du)\,ds+\hat{E}\int_t^T\int_K\left|h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\right|\mathbb{1}_{\bar{B}}\hat{q}_s(du)\,ds\\&\leq\frac{C'}{R^2}\hat{E}\int_t^T(\left|\hat{X}_s\right|^2+\left|\hat{Y}_s\right|^2)ds.\end{aligned}$$

We successively pass to the limit in n and R, to show that $\lim_{n\to\infty}J(n)=0$ in probability.

Now, let $\hat{\mathcal{F}}_s:=\mathcal{F}_s^{\hat{X},\hat{Y},\hat{q}}$, be the filtration generated by $(\hat{X}_r,\hat{Y}_r,\hat{q}_r,\ r\leq s)$ completed by $\hat{P}-$nul sets. Combining the estimates (4.5) and Lemma 6.3 in Appendix, one can show that $\left(\hat{N}_t\right)$ is a $\hat{\mathcal{F}}_t$-martingale. Since $\left(\hat{W}\right)$ is a $\left(\hat{\mathcal{F}}_t,\hat{P}\right)-$Brownian motion, then the martingale decomposition theorem yields the existence of a process $\hat{Z}\in\mathcal{M}^2([t,T];\mathbb{R}^{n\times d})$ such that

$$\hat{N}_t=\int_0^t\hat{Z}_s d\hat{W}_s+\hat{M}_t,\quad\text{with}\quad\left\langle\hat{M}_t,\hat{W}\right\rangle_t=0,$$

which implies that

$$\hat{Y}_t=\varphi\left(\hat{X}_T\right)+\int_t^T\int_K h\left(s,\hat{X}_s,\hat{Y}_s,u\right)\hat{q}_s(du)\,ds-\int_t^T\hat{Z}_s d\hat{W}_s-\left(\hat{N}_T-\hat{N}_t\right).$$

To finish the proof of Theorem 4.2, it remains to check that $\hat{q}$ is an optimal control.

According to above properties (i)-(iv) and assumption $(\mathbf{H}_3)$, we have

$$\begin{aligned}\inf_{q\in\mathcal{R}}J(q)&=\lim_{n\to\infty}J(q^n),\\&=\lim_{n\to\infty}E\left[\psi(X_T^n)+g(Y_0^n)+\int_0^T\int_K l(t,X_t^n,Y_t^n,u)\,q_t^n(du)\,dt\right]\\&=\lim_{n\to\infty}\hat{E}\left[\psi(\hat{X}_T^n)+g\left(\hat{Y}_0^n\right)+\int_0^T\int_K l\left(t,\hat{X}_t^n,\hat{Y}_t^n,u\right)\hat{q}_t^n(du)\,dt\right]\\&=\hat{E}\left[\psi(\hat{X}_T)+g\left(\hat{Y}_0\right)+\int_0^T\int_K l\left(t,\hat{X}_t,\hat{Y}_t,u\right)\hat{q}_t(du)\,dt\right].\end{aligned}$$

■

By using the same arguments as in Corollary 3.7, one can prove the following result on the existence of strict controls under convexity assumptions.

Corollary 4.6. *Assume* $(\mathbf{H_2})$- $(\mathbf{H_5})$ *and that for every* $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^k$, *the set*

$$(b,h,l)(t,x,y,K):=\{b_i(t,x,y,u),h_j(t,x,y,u),l(t,x,y,u)/u\in K,i=1,...,d,j=1,...,k\},\tag{4.11}$$

is convex and closed in $\mathbb{R}^{d+k+1}$. *Then, the relaxed optimal control* $\hat{q}_t$ *has the form of a Dirac measure charging a strict control* $\hat{U}_t$, *that is* $\hat{q}_t(du)=\delta_{\hat{U}_t}(du)$.

5 Conclusion

We have proved two results on the existence of an optimal control for systems governed by decoupled as well as coupled FBSDEs, by using probabilistic tools. The ingredients used in the proofs of the main results are based essentially on tightness techniques on the space $\mathcal{C}$ of continuous functions as well as on the space $\mathcal{D}$ of càdlàg functions equipped with Meyer-Zheng topology or Jakubowsky S-topology. The assumptions made on the coefficients are made to ensure weak convergence of the processes under consideration and the corresponding cost functionals. However, a serious difficulty remains in the case where the generator depends on the second backward variable Z. This difficulty consists in finding a natural assumption ensuring the tightness of the second backward variable Z. This is exactly the kind of problems encountered when one deals with weak solutions of BSDEs and coupled FBSDEs with generators depending on Z.

6 Appendix

The S-topology defined by Jakubowski on the space $\mathbb{D}\left([0,T];\mathbb{R}^k\right)$ of càdlàg functions is weaker than the Skorokhod topology and the tightness criteria is the same as for the Meyer-Zheng topology [17]. The topology S arises naturally in limit theorems for stochastic integrals. Let us give some of its properties:

1) If $x_n\rightarrow_S x_0$, then $x_n(t)\rightarrow_S x_0(t)$ for each t except for a countable set.

2) If $x_n(t)\rightarrow_S x_0(t)$ for each t in a dense set containing 0 and T and $\{x_n\}$ is S-relatively compact, then $x_n\rightarrow_S x_0$ (not true for the convergence in measure).

3) We recall (see Meyer and Zheng [17] and Jakubowski [13, 14]) that for a familly $(X^n)_n$ of quasi-martingales on the probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, the following condition ensures the tightness of the familly $(X^n)_n$ on the space $\mathbb{D}\left([0,T];\mathbb{R}^k\right)$ endowed with the S-topology

$$\sup_n\left(\sup_{0\leq t\leq T}E\left|X_t^n\right|+CV\left(X^n\right)\right)<\infty,$$

where, for a quasi-martingale X on $\left(\Omega,\{\mathcal{F}_t\}_{0\leq t\leq T},P\right)$, $CV(X)$ stands for the conditional variation of X on $[0,T]$, and is defined by

$$CV(X)=\sup E\left(\sum_i\left|E\left(X_{t_i+1}-X_{t_i}\right)\big/\mathcal{F}_{t_i}^n\right|\right),$$

where the supremum is taken over all partitions of $[0,T]$.

Let $N^{a,b}(Y)$ denotes the number of up-crossing of the function $Y\in\mathbb{D}\left([0,T];\mathbb{R}^k\right)$ in given levels $a<b$ (recall that $N^{a,b}(Y)\geq k$ if one can find numbers $0\leq t_1<t_2<\cdots<t_{2k-1}<t_{2k}\leq T$ such that $Y(t_{2i-1})<a$ and $Y(t_{2i})>b, i=1,2,...,k$).

Lemma 6.1. *(A criteria for* S*-tightness). A sequence* $(Y^n)_{n\in\mathbb{N}}$ *is* S*-tight if and only if it is relatively compact on the* S*-topology. Let* $(Y^n)_{n\in\mathbb{N}}$ *be a family of stochastic processes in* $\mathbb{D}\left([0,T];\mathbb{R}^k\right)$. *Then this family is tight for the* S *-topology if and only if* $\left(\|Y^n\|_\infty\right)_n$ *and* $N^{a,b}(Y^n)$ *are tight for each* $a<b$.

Lemma 6.2. *(The a.s. Skorokhod representation). Let* $(\mathbb{D},S)$ *be a topological space on which there exists a countable family of* S*-continuous functions separating points in* X. *Let* $\{X_n\}_{n\in\mathbb{N}}$ *be a uniformly tight sequence of laws on* $\mathbb{D}$. *In every subsequence* $\{X_{n_k}\}$ *one can find a further subsequence* $\left\{X_{n_{k_l}}\right\}$ *and stochastic processes* $\{Y_l\}$ *defined on* $\left([0,T],\mathcal{B}_{[0,T]},l\right)$ *such that*

$$Y_l \sim X_{n_{k_l}},\ l = 1, 2, ... \tag{1}$$

for each $w \in [0, T]$

$$Y_l\,(w) \underset{S}{\rightarrow} Y_0\,(w)\,,\ \text{as } l \rightarrow \infty, \tag{2}$$

and for each $\varepsilon > 0$*, there exists an* S*-compact subset* $K_\varepsilon \subset \mathbb{D}$ *such that*

$$P\left(\{w \in [0, T] : Y_l\,(w) \in K_\varepsilon, l = 1, 2, ...\}\right) > 1 - \varepsilon. \tag{3}$$

One can say that (2) and (3) describe "the almost sure convergence in compacts" and that (1), (2) and (3) define the strong a.s. Skorokhod representation for subsequences ("strong" because of condition (3)).

Remark 6.3. The projection $\pi_T : y \in (\mathbb{D}([0, T]; \mathbb{R}), S) \longmapsto y(T)$ is continuous (see Remark 2.4, p.8 in Jakubowski [13]), but $y \longmapsto y(t)$ is not continuous for each $0 \leq t \leq T$.

Lemma 6.4. *Let* (X^n, M^n) *be a multidimensional process in* $\mathbb{D}([0, T]; \mathbb{R}^p)\,(p \in \mathbb{R}^*)$ *converging to* (X, M) *in the S-topology. Let* $(\mathcal{F}_t^{X^n})_{t\geq 0}$ *(resp.* $(\mathcal{F}_t^{X})_{t\geq 0}$*) be the minimal complete admissible filtration for* X^n *(resp.*X*). We assume that* $\sup_n E\left[\sup_{0\leq t\leq T} |M_t^n|^2\right] < C_T\ \forall T > 0$, M^n *is a* $\mathcal{F}^{X^n}$*-martingale and* M *is a* $\mathcal{F}^X$*-adapted. Then* M *is a* $\mathcal{F}^X$*-martingale.*

Lemma 6.5. *Let* $(Y^n)_{n>0}$ *be a sequence of processes converging weakly in* $\mathbb{D}([0, T]; \mathbb{R}^p)$ *to* Y*. We assume that* $\sup_n E\left[\sup_{0\leq t\leq T} |Y_t^n|^2\right] < +\infty$*. Then, for any* $t \geq 0$, $E\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty$.

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