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# Stabilization of second–order evolution equations with time delay

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## Abstract

We consider second–order evolution equations in an abstract setting with damping and time delay and give sufficient conditions ensuring exponential stability. Our abstract framework is then applied to the wave equation, the elasticity system and the Petrovsky system.

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**Keywords and Phrases:** second–order evolution equations, delay feedbacks, stabilization

## 1 Introduction

Let  $H$  be a real Hilbert space with norm and inner product denoted respectively by  $\|\cdot\|_H$  and  $\langle\cdot,\cdot\rangle_H$  and let  $A : \mathcal{D}(A) \rightarrow H$  be a positive self–adjoint operator with a compact inverse in  $H$ . Denote by  $V := \mathcal{D}(A^{\frac{1}{2}})$  the domain of  $A^{\frac{1}{2}}$ . Moreover, for  $i = 1, 2$ , let  $U_i$  be real Hilbert spaces with norm and inner product denoted respectively by  $\|\cdot\|_{U_i}$  and  $\langle\cdot,\cdot\rangle_{U_i}$  and let  $B_i : U_i \rightarrow V'$  be linear operators. In this setting we consider the problem

$$u_{tt}(t) + Au(t) + B_1B_1^*u_t(t) + B_2B_2^*u_t(t - \tau) = 0 \quad t > 0, \quad (1.1)$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, \quad (1.2)$$

$$B_2^*u_t(t) = f^0(t) \quad t \in (-\tau, 0), \quad (1.3)$$

where the constant  $\tau > 0$  is the time delay. We assume that the delay feedback operator  $B_2$  is bounded, that is  $B_2 \in \mathcal{L}(U_2, H)$ , while the standard one  $B_1 \in \mathcal{L}(U_1, V')$  may be unbounded.

Time delays are often present in applications and practical problems and it is by now well–known that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly exponentially stable in absence of delay. For some examples in this sense we refer to [8, 9, 21, 27].

We are interested in giving stability results for the above problem under a suitable assumption on the “size” of the feedback operator  $B_2$ , when the feedback  $B_1$  is a stabilizing one. More precisely, we will show that for a system which is exponentially stable in absence of time delay, i.e. for  $B_2 = 0$ , the exponential stability is preserved if  $\|B_2^*\|$  is sufficiently small.

In this sense this paper extends and generalizes the result of [25] for wave equation with local damping and time delay. On the other hand it completes the analysis of [21, 23]. Indeed here we do not assume

$$\exists \alpha < 1 \text{ such that } \|B_2^*u\|_{U_2} \leq \alpha \|B_1^*u\|_{U_1}, \quad \forall u \in V;$$

as in [23] (cfr. assumption (1.8) of [21] for the wave equation).

Assuming that an observability inequality holds for the system (1.1), (1.2) when  $B_2 = 0$ , through the definition of a suitable energy (see (3.2)) and the use of a perturbation argument as in [25], we obtain sufficient conditions ensuring exponential stability. Our abstract framework is then applied to some concrete examples, namely the wave equation, the elasticity system and the Petrovsky system.

The paper is organized as follows. In section 2 a well-posedness result of the abstract system is proved. In section 3 we obtain exponential stability results for the abstract system under suitable conditions. Finally, in sections 4, 5 and 6 we apply our abstract results to the wave equation with local and boundary dampings, the elasticity system and the Petrovsky system respectively. Other examples (like wave or beam equations on networks) could be given, we skip them for shortness.

## 2 Well-posedness

In this section we will give well-posedness results for problem (1.1)–(1.3) using semigroup theory.

As in [21] we introduce the function

$$z(\rho, t) := B_2^*u_t(t - \tau\rho), \quad \rho \in (0, 1), \quad t > 0. \quad (2.1)$$

Then, problem (1.1)–(1.3) can be rewritten as

$$u_{tt}(t) + Au(t) + B_1B_1^*u_t(t) + B_2z(1, t) = 0 \quad t > 0, \quad (2.2)$$

$$\tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0, \quad (2.3)$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, \quad (2.4)$$

$$z(\rho, 0) = f^0(-\tau\rho) \quad \rho \in (0, 1), \quad (2.5)$$

$$z(0, t) = B_2^*u_t(t), \quad t > 0. \quad (2.6)$$

If we denote  $\mathcal{U} := (u, u_t, z)^T$ , then  $\mathcal{U}' = (u_t, u_{tt}, z_t)^T$  and  $\mathcal{U}$  satisfies

$$\begin{cases} \mathcal{U}' = \mathcal{A}\mathcal{U} \\ \mathcal{U}(0) = (u_0, u_1, f^0(-\tau\cdot))^T, \end{cases} \quad (2.7)$$

where the operator  $\mathcal{A}$  is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ -Au - B_1B_1^*v - B_2z(1) \\ -\tau^{-1}z_\rho \end{pmatrix}, \quad (2.8)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \{ (u, v, z)^T \in V \times V \times H^1((0, 1); U_2) : Au + B_1 B_1^* v \in H \text{ and } z(0) = B_2^* v \}. \quad (2.9)$$

Denote by  $\mathcal{H}$  the Hilbert space

$$\mathcal{H} := V \times H \times L^2((0, 1); U_2), \quad (2.10)$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} := \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \tilde{u} \rangle_H + \langle v, \tilde{v} \rangle_H + \xi \int_0^1 \langle z(\rho), \tilde{z}(\rho) \rangle_{U_2} d\rho, \quad (2.11)$$

where  $\xi$  is any fixed positive number.

The following well-posedness result holds.

**Proposition 2.1** *For any initial datum  $U_0 \in \mathcal{H}$  there exists a unique solution  $U \in C([0, +\infty), \mathcal{H})$  of problem (2.7). Moreover, if  $U_0 \in \mathcal{D}(\mathcal{A})$ , then*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

**Proof.** We will show that the operator  $\mathcal{A}$  defined by (2.8), (2.9) generates a strongly continuous semigroup in the Hilbert  $\mathcal{H}$  defined in (2.10), (2.11).

Denoting by  $I$  the identity operator, we first show that there exists a positive constant  $c$  such that  $\mathcal{A} - cI$  is dissipative (cfr. [1]). Let  $(u, v, z)^T \in \mathcal{D}(\mathcal{A})$ , then

$$\begin{aligned} \left\langle \mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &:= \left\langle \begin{pmatrix} v \\ -Au - B_1 B_1^* v - B_2 z(1) \\ -\tau^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \langle A^{\frac{1}{2}} v, A^{\frac{1}{2}} u \rangle_H - \langle Au + B_1 B_1^* v + B_2 z(1), v \rangle_H - \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho. \end{aligned}$$

Since  $Au + B_1 B_1^* v + B_2 z(1) \in H \subset V'$ , by duality we have

$$\begin{aligned} \left\langle \mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &= \langle A^{\frac{1}{2}} v, A^{\frac{1}{2}} u \rangle_H - \langle Au, v \rangle_{V', V} - \langle B_1 B_1^* v, v \rangle_{V', V} \\ &\quad - \langle B_2 z(1), v \rangle_{V', V} - \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho \\ &= -\|B_1^* v\|_{U_1} - \langle z(1), B_2^* v \rangle_{U_2} - \xi \int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho. \end{aligned}$$

Integrating by parts and using the relation  $z(0) = B_2^* v$ , we get

$$\int_0^1 \langle z_\rho(\rho), z(\rho) \rangle_{U_2} d\rho = \frac{1}{2} (\|z(1)\|_{U_2}^2 - \|B_2^* v\|_{U_2}^2),$$

thus, using also Young's inequality

$$\left\langle \mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \leq \left( \frac{\xi}{2} + \frac{1}{2\xi} \right) \|B_2^* v\|_{U_2}^2 \leq c \|v\|_H^2,$$

for a suitable constant  $c > 0$ . Hence, the operator  $\mathcal{A} - cI$  is dissipative.

Now, we show that  $\lambda I - \mathcal{A}$  is surjective for some  $\lambda > 0$ . Given  $(f, g, h)^T \in \mathcal{H}$  we seek  $(u, v, z)^T \in \mathcal{D}(\mathcal{A})$  such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.$$

This is equivalent to

$$\lambda u - v = f; \quad (2.12)$$

$$\lambda v + Au + B_1 B_1^* v + B_2 z(1) = g; \quad (2.13)$$

$$\lambda z + \tau^{-1} z_\rho = h. \quad (2.14)$$

Analogously to [21], suppose that we have found  $u$  with the appropriate regularity. Then, by (2.12),

$$v = \lambda u - f \in V. \quad (2.15)$$

Moreover, from (2.6), (2.14) and (2.15),  $z$  is given by

$$z(\rho) = \lambda B_2^* u e^{-\lambda \tau \rho} - B_2^* f e^{-\lambda \tau \rho} + \tau e^{-\lambda \tau \rho} \int_0^\rho e^{\lambda \sigma \tau} h(\sigma) d\sigma, \quad \rho \in (0, 1). \quad (2.16)$$

In particular,

$$z(1) = \lambda B_2^* u e^{-\lambda \tau} + z_0, \quad (2.17)$$

where

$$z_0 = -B_2^* f e^{-\lambda \tau} + \tau e^{-\lambda \tau} \int_0^1 e^{\lambda \sigma \tau} h(\sigma) d\sigma,$$

is a fixed element of  $U_2$  depending only on  $f$  and  $h$ .

It remains only to determine  $u$ . From (2.12) and (2.13)  $u$  satisfies

$$\lambda^2 u + Au + \lambda B_1 B_1^* u + B_2 z(1) = g + \lambda f + B_1 B_1^* f,$$

and then, by (2.17),

$$\lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u = g + \lambda f + B_1 B_1^* f - B_2 z^0. \quad (2.18)$$

We denote the right-hand side of (2.18) by  $w$ , namely

$$w := g + \lambda f + B_1 B_1^* f - B_2 z^0 \in H \subset V'.$$

Then, from (2.18), we have

$$\langle \lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda \tau} B_2 B_2^* u, \varphi \rangle_{V', V} = \langle w, \varphi \rangle_{V', V}.$$

Since  $u \in V \subset H$ , we can rewrite

$$\begin{aligned} & \langle \lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda\tau} B_2 B_2^* u, \varphi \rangle_{V',V} \\ &= \lambda^2 \langle u, \varphi \rangle_{V',V} + \langle Au, \varphi \rangle_{V',V} + \lambda \langle B_1^* u, B_1^* \varphi \rangle_{V',V} + \lambda e^{-\lambda\tau} \langle B_2^* u, B_2^* \varphi \rangle_{V',V} \\ &= \lambda^2 \langle u, \varphi \rangle_H + \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \varphi \rangle_H + \lambda \langle B_1^* u, B_1^* \varphi \rangle_{U_1} + \lambda e^{-\lambda\tau} \langle B_2^* u, B_2^* \varphi \rangle_{U_2}. \end{aligned}$$

Therefore, we obtain

$$\lambda^2 \langle u, \varphi \rangle_H + \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \varphi \rangle_H + \lambda \langle B_1^* u, B_1^* \varphi \rangle_{U_1} + \lambda e^{-\lambda\tau} \langle B_2^* u, B_2^* \varphi \rangle_{U_2} = \langle w, \varphi \rangle_{V',V}. \quad (2.19)$$

The left-hand side of (2.19) is a continuous and coercive bilinear form on  $V$ . Then, Lax–Milgram’s lemma implies the existence of a unique solution  $u \in V$  of (2.19) that satisfies

$$\lambda^2 u + Au + \lambda B_1 B_1^* u + \lambda e^{-\lambda\tau} B_2 B_2^* u = w \text{ in } V'.$$

This implies that  $Au \in H$  and by defining  $v$  by (2.15) and  $z$  by (2.16), we have found  $(u, v, z)^T \in \mathcal{D}(\mathcal{A})$  satisfying (2.12)–(2.14). This implies that  $\lambda I - \mathcal{A}$  is surjective for all  $\lambda > 0$  and the same holds for the operator  $\lambda I - (\mathcal{A} - cI)$ .

Then, the Lumer–Phillips Theorem implies that  $\mathcal{A} - cI$  generates a strongly continuous semigroup of contraction in  $\mathcal{H}$ . Hence, the operator  $\mathcal{A}$  generates a strongly continuous semigroup in  $\mathcal{H}$ .  $\blacksquare$

### 3 Stability result

For a fixed constant  $\xi$  satisfying

$$\xi > 1, \quad (3.1)$$

we define the energy functional for solutions to problem (1.1)–(1.3) as

$$E(t) := E(u, t) = \frac{1}{2} (\|A^{\frac{1}{2}} u(t)\|_H^2 + \|u_t(t)\|_H^2) + \frac{\xi}{2} \int_{t-\tau}^t \|B_2^* u_t(s)\|_{U_2}^2 ds. \quad (3.2)$$

We can obtain a first estimate.

**Proposition 3.1** *For any regular solution of problem (1.1) – (1.3)*

$$E'(t) \leq -\|B_1^* u_t(t)\|_{U_1}^2 + \frac{1+\xi}{2} \|B_2^* u_t(t)\|_{U_2}^2 - \frac{\xi-1}{2} \|B_2^* u_t(t-\tau)\|_{U_2}^2. \quad (3.3)$$

**Proof:** Differentiating  $E(t)$  we get

$$E'(t) = \langle A^{\frac{1}{2}} u(t), A^{\frac{1}{2}} u_t(t) \rangle_H + \langle u_t(t), u_{tt}(t) \rangle_H + \frac{\xi}{2} \|B_2^* u_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* u_t(t-\tau)\|_{U_2}^2.$$

Hence using the definition of  $A$  and (1.1) we get successively

$$\begin{aligned} E'(t) &= \langle Au(t), u_t(t) \rangle_{V',V} - \langle u_t(t), Au(t) \rangle_{V,V'} - \langle u_t(t), B_1 B_1^* u_t(t) \rangle_{V,V'} \\ &\quad - \langle u_t(t), B_2 B_2^* u_t(t-\tau) \rangle_{V,V'} + \frac{\xi}{2} \|B_2^* u_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* u_t(t-\tau)\|_{U_2}^2. \end{aligned}$$

Then,

$$E'(t) = -\|B_1^* u_t(t)\|_{U_1}^2 - \langle B_2^* u_t(t), B_2^* u_t(t - \tau) \rangle_{U_2} + \frac{\xi}{2} \|B_2^* u_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* u_t(t - \tau)\|_{U_2}^2,$$

and (3.3) follows from Cauchy-Schwarz's inequality.  $\blacksquare$

Note that, from (3.3), the energy of solutions to problem (1.1)–(1.3) is not decreasing in general. Indeed the second term in the right-hand side of (3.3), coming from the delay term in (1.1), is non negative. We now consider, as in [25], the next auxiliary problem which is close to the first one but whose energy is decreasing.

$$\varphi_{tt}(t) + A\varphi(t) + B_1 B_1^* \varphi_t(t) + B_2 B_2^* \varphi_t(t - \tau) + \xi B_2 B_2^* \varphi_t(t) = 0, \quad t > 0, \quad (3.4)$$

$$\varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \quad (3.5)$$

$$B_2^* \varphi_t(t) = g^0(t), \quad t \in (-\tau, 0), \quad (3.6)$$

where  $\xi$  is the same constant as in (3.2).

The well-posedness of system (3.4)–(3.6) can be proved using standard semigroup theory as in Proposition 2.1. Analogously to above we introduce the function

$$\eta(\rho, t) = B_2^* \varphi_t(t - \tau\rho), \quad \rho \in (0, 1), \quad t > 0;$$

and we rewrite the problem in the abstract form

$$\begin{cases} \Phi' = \mathcal{A}^0 \Phi, \\ \Phi(0) = (\varphi_0, \varphi_1, g^0(-\tau \cdot))^T, \end{cases} \quad (3.7)$$

where the operator  $\mathcal{A}^0$  is defined by

$$\mathcal{A}^0 \begin{pmatrix} \varphi \\ \psi \\ \eta \end{pmatrix} := \begin{pmatrix} \psi \\ -A\varphi - B_1 B_1^* \psi - B_2 \eta(1) - \xi B_2 B_2^* \psi \\ -\tau^{-1} \eta_\rho \end{pmatrix},$$

with domain  $\mathcal{D}(\mathcal{A}^0) = \mathcal{D}(\mathcal{A})$  (see (2.9)) in the Hilbert space  $\mathcal{H}$  defined by (2.10) and (2.11).

**Proposition 3.2** *For any initial datum  $\Phi_0 \in \mathcal{H}$  there exists a unique solution  $\Phi \in C([0, +\infty), \mathcal{H})$  of problem (3.7). Moreover, if  $\Phi_0 \in \mathcal{D}(\mathcal{A}^0)$ , then*

$$\Phi \in C([0, +\infty), \mathcal{D}(\mathcal{A}^0)) \cap C^1([0, +\infty), \mathcal{H}).$$

For solutions of problem (3.4)–(3.6) the energy  $F(\cdot)$ ,

$$F(t) := F(\varphi, t) = \frac{1}{2} (\|A^{\frac{1}{2}} \varphi(t)\|_H^2 + \|\varphi_t(t)\|_H^2) + \frac{\xi}{2} \int_{t-\tau}^t \|B_2^* \varphi_t(s)\|_{U_2}^2 ds, \quad (3.8)$$

with  $\xi$  satisfying (3.1), is decreasing in time.

More precisely, we have the following result.

**Proposition 3.3** *For any regular solution of problem (3.4) – (3.6), we have*

$$F'(t) \leq -\|B_1^* \varphi_t(t)\|_{U_1}^2 - \frac{\xi - 1}{2} \|B_2^* \varphi_t(t)\|_{U_2}^2 - \frac{\xi - 1}{2} \|B_2^* \varphi_t(t - \tau)\|_{U_2}^2. \quad (3.9)$$

Then, if  $\xi$  satisfies (3.1), the energy  $F(\cdot)$  is decreasing.

**Proof.** In order to have (3.9) we differentiate (3.8). Hence, using the definition of  $A$  and (3.4), we obtain

$$\begin{aligned} F'(t) &= \langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\varphi_t \rangle_H + \langle \varphi_t, \varphi_{tt} \rangle_H + \frac{\xi}{2} \|B_2^* \varphi_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* \varphi_t(t - \tau)\|_{U_2}^2 \\ &= \langle A\varphi(t), \varphi_t(t) \rangle_{V', V} - \langle \varphi_t(t), A\varphi(t) \rangle_{V, V'} - \langle \varphi_t(t), B_1 B_1^* \varphi_t(t) \rangle_{V, V'} - \xi \langle \varphi_t(t), B_2 B_2^* \varphi_t(t) \rangle_{V, V'} \\ &\quad - \langle \varphi_t(t), B_2 B_2^* \varphi_t(t - \tau) \rangle_{V, V'} + \frac{\xi}{2} \|B_2^* \varphi_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* \varphi_t(t - \tau)\|_{U_2}^2. \end{aligned}$$

Then,

$$\begin{aligned} F'(t) &= -\|B_1^* \varphi_t(t)\|_{U_1}^2 - \xi \|B_2^* \varphi_t(t)\|_{U_2}^2 - \langle B_2^* \varphi_t(t), B_2^* \varphi_t(t - \tau) \rangle_{U_2} \\ &\quad + \frac{\xi}{2} \|B_2^* \varphi_t(t)\|_{U_2}^2 - \frac{\xi}{2} \|B_2^* \varphi_t(t - \tau)\|_{U_2}^2, \end{aligned}$$

and therefore (3.8) follows from Cauchy-Schwarz's inequality.  $\blacksquare$

Consider now the following damped system associated with (1.1) and (1.2),

$$w_{tt}(t) + Aw(t) + B_1 B_1^* w_t = 0 \quad t > 0 \quad (3.10)$$

$$w(0) = w_0 \quad \text{and} \quad w_t(0) = w_1 \quad (3.11)$$

with  $(w_0, w_1) \in V \times H$ . For our stability result we need that this system is exponentially stable or equivalently that the next observability inequality holds (see Lemma 3.2 of [20]). Namely we assume that there exists a time  $\bar{T} > 0$  such that for every time  $T > \bar{T}$  there is a constant  $c$ , depending on  $T$  but independent of the initial data, such that

$$E_S(0) \leq c \int_0^T \|B_1^* w_t(t)\|_{U_1}^2 dt, \quad (3.12)$$

for every weak solution of problem (3.10), (3.11) with initial data  $(w_0, w_1) \in V \times H$ .

Here  $E_S(\cdot)$  denotes the standard energy for wave type equations, that is

$$E_S(t) = E_S(w, t) := \frac{1}{2} (\|A^{\frac{1}{2}} w(t)\|_H^2 + \|w_t(t)\|_H^2).$$

For shortness let us denote by  $C_2$  the norm of  $B_2$

$$\|B_2\| = \|B_2^*\| = C_2. \quad (3.13)$$

We can prove an exponential stability result for the perturbed problem (3.4)–(3.6).

**Theorem 3.4** *Assume that (3.1) holds and that the observability estimate (3.12) holds for problem (3.10) – (3.11). Then, there are two positive constants  $K, \tilde{\mu}$  such that*

$$F(t) \leq K e^{-\tilde{\mu}t} F(0), \quad t > 0, \quad (3.14)$$

for any solution of problem (3.4) – (3.6). In particular,

$$K = \frac{C_0 + 1}{C_0}, \quad (3.15)$$



$$\tilde{\mu} = \frac{1}{2T} \ln \frac{C_0 + 1}{C_0}, \quad (3.16)$$

with  $T$  any fixed time satisfying  $T > \max\{\bar{T}, \tau\}$ ,  $\bar{T}$  being an observability time for (3.12), and

$$C_0 = \max \left\{ 2c, \frac{32cTC_2^2 + \xi}{\xi - 1}, \frac{32cC_2^2T\xi^2}{\xi - 1} \right\}, \quad (3.17)$$

where  $C_2$  is as in (3.13) and  $c := c(T)$  is the observability constant in (3.12).

**Proof.** Following a classical argument (see [28]) we can decompose the solution  $\varphi$  of (3.4) – (3.6) as

$$\varphi = w + \tilde{w}$$

where  $w$  is the solution of system (3.10), (3.11) with  $w_0 = \varphi_0$ ,  $w_1 = \varphi_1$ ; while  $\tilde{w}$  solves

$$\tilde{w}_{tt}(t) + A\tilde{w}(t) + B_1B_1^*\tilde{w}_t(t) = -\xi B_2B_2^*\varphi_t(t) - B_2B_2^*\varphi_t(t - \tau) \quad t > 0 \quad (3.18)$$

$$\tilde{w}(0) = 0 \quad \text{and} \quad \tilde{w}_t(0) = 0 \quad (3.19)$$

By (3.8),

$$\begin{aligned} F(0) &= E_S(w, 0) + \frac{\xi}{2} \int_{-\tau}^0 \|B_2^*\varphi_t(s)\|_{U_2}^2 ds \\ &= E_S(w, 0) + \frac{\xi}{2} \int_0^{\tau} \|B_2^*\varphi_t(t - \tau)\|_{U_2}^2 dt. \end{aligned}$$

Therefore, from (3.12), if  $T > \max\{\bar{T}, \tau\}$  we obtain

$$\begin{aligned} F(0) &\leq c \int_0^T \|B_1^*w_t(t)\|_{U_1}^2 dt + \frac{\xi}{2} \int_0^T \|B_2^*\varphi_t(t - \tau)\|_{U_2}^2 dt \\ &\leq 2c \int_0^T (\|B_1^*\varphi_t(t)\|_{U_1}^2 + \|B_1^*\tilde{w}_t(t)\|_{U_1}^2) dt + \frac{\xi}{2} \int_0^T \|B_2^*\varphi_t(t - \tau)\|_{U_2}^2 dt, \end{aligned} \quad (3.20)$$

where  $c$  is the observability constant for the damped system (3.10), (3.11).

Now, observe that from (3.18),

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} (\|\tilde{w}_t(t)\|_H^2 + \|A^{\frac{1}{2}}\tilde{w}(t)\|_H^2) + \|B_1^*\tilde{w}_t\|_{U_1}^2 &= \langle \tilde{w}_t, \tilde{w}_{tt} + A\tilde{w} + B_1B_1^*\tilde{w}_t \rangle \\ &= \langle \tilde{w}_t, -\xi B_2B_2^*\varphi_t(t) - B_2B_2^*\varphi_t(t - \tau) \rangle_H. \end{aligned}$$

Integrating in time from 0 to  $t$ , for  $t \in (0, 2T]$ , and using (3.19) we have

$$\begin{aligned} \frac{1}{2} (\|\tilde{w}_t(t)\|_H^2 + \|A^{\frac{1}{2}}\tilde{w}(t)\|_H^2) + \int_0^t \|B_1^*\tilde{w}_t(s)\|_{U_1}^2 ds \\ = \int_0^t \langle \tilde{w}_t, -\xi B_2B_2^*\varphi_t(s) - B_2B_2^*\varphi_t(s - \tau) \rangle_H ds, \end{aligned}$$

and then

$$\begin{aligned} \|\tilde{w}_t(t)\|_H^2 + 2 \int_0^t \|B_1^*\tilde{w}_t(s)\|_{U_1}^2 ds &\leq \frac{1}{8TC_2^2} \int_0^t \|B_2^*\tilde{w}_t(s)\|_{U_2}^2 ds + 8TC_2^2\xi^2 \int_0^t \|B_2^*\varphi_t(s)\|_{U_2}^2 ds \\ &+ \frac{1}{8TC_2^2} \int_0^t \|B_2^*\tilde{w}_t(s)\|_{U_2}^2 ds + 8TC_2^2 \int_0^t \|B_2^*\varphi_t(s - \tau)\|_{U_2}^2 ds, \end{aligned} \quad (3.21)$$

where  $C_2$  was defined in (3.13).

This estimate directly implies that for all  $t \in [0, 2T]$ , one has

$$\begin{aligned} \|\tilde{w}_t(t)\|_H^2 + 2 \int_0^t \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds &\leq \frac{1}{4TC_2^2} \int_0^{2T} \|B_2^* \tilde{w}_t(s)\|_{U_2}^2 ds + 8TC_2^2 \xi^2 \int_0^{2T} \|B_2^* \varphi_t(s)\|_{U_2}^2 ds \\ &+ 8TC_2^2 \int_0^{2T} \|B_2^* \varphi_t(s - \tau)\|_{U_2}^2 ds, \end{aligned} \quad (3.22)$$

and so, integrating in  $[0, 2T]$ ,

$$\begin{aligned} \int_0^{2T} \|\tilde{w}_t(t)\|_H^2 dt + 2 \int_0^{2T} \int_0^t \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds dt &\leq \frac{1}{2C_2^2} \int_0^{2T} \|B_2^* \tilde{w}_t(s)\|_{U_2}^2 ds \\ &+ 16T^2 C_2^2 \xi^2 \int_0^{2T} \|B_2^* \varphi_t(s)\|_{U_2}^2 ds + 16T^2 C_2^2 \int_0^{2T} \|B_2^* \varphi_t(s - \tau)\|_{U_2}^2 ds. \end{aligned} \quad (3.23)$$

Therefore,

$$\begin{aligned} \frac{1}{2} \int_0^{2T} \|\tilde{w}_t(t)\|_H^2 dt + 2 \int_0^{2T} \int_0^t \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds dt &\leq 16T^2 C_2^2 \xi^2 \int_0^{2T} \|B_2^* \varphi_t(s)\|_{U_2}^2 ds \\ &+ 16T^2 C_2^2 \int_0^{2T} \|B_2^* \varphi_t(s - \tau)\|_{U_2}^2 ds, \end{aligned}$$

from which follows

$$\begin{aligned} \int_0^{2T} \int_0^t \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds dt &\leq 8T^2 C_2^2 \xi^2 \int_0^{2T} \|B_2^* \varphi_t(s)\|_{U_2}^2 ds \\ &+ 8T^2 C_2^2 \int_0^{2T} \|B_2^* \varphi_t(s - \tau)\|_{U_2}^2 ds. \end{aligned}$$

Using the fact that

$$\begin{aligned} \int_0^{2T} \int_0^t \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds dt &= \int_0^{2T} \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 (2T - s) ds \\ &\geq \int_0^T \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 (2T - s) ds \\ &\geq T \int_0^T \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds, \end{aligned}$$

we deduce that

$$\begin{aligned} \int_0^T \|B_1^* \tilde{w}_t(s)\|_{U_1}^2 ds &\leq 8TC_2^2 \xi^2 \int_0^{2T} \|B_2^* \varphi_t(s)\|_{U_2}^2 ds \\ &+ 8TC_2^2 \int_0^{2T} \|B_2^* \varphi_t(s - \tau)\|_{U_2}^2 ds. \end{aligned} \quad (3.24)$$

Using (3.24) in (3.20) we obtain

$$\begin{aligned} F(0) &\leq 2c \int_0^T \|B_1^* \varphi_t(t)\|_{U_1}^2 dt + (16cTC_2^2 + \frac{\xi}{2}) \int_0^{2T} \|B_2^* \varphi(t - \tau)\|_{U_2}^2 dt \\ &+ 16cC_2^2 T \xi^2 \int_0^{2T} \|B_2^* \varphi_t(t)\|_{U_2}^2 dt, \end{aligned} \quad (3.25)$$

that we rewrite as

$$\begin{aligned}
F(0) \leq & 2c \int_0^{2T} \|B_1^* \varphi_t(t)\|_{U_1}^2 dt + \frac{32cTC_2^2 + \xi}{\xi - 1} \left( \frac{\xi - 1}{2} \int_0^{2T} \|B_2^* \varphi(t - \tau)\|_{U_2}^2 dt \right) \\
& + \frac{32cC_2^2 T \xi^2}{\xi - 1} \left( \frac{\xi - 1}{2} \int_0^{2T} \|B_2^* \varphi(t)\|_{U_2}^2 dt \right) \leq -C_0 \int_0^{2T} F'(t) dt,
\end{aligned} \tag{3.26}$$

with  $C_0$  as in (3.17).

Therefore, from (3.26), using also that  $F(\cdot)$  is decreasing we obtain

$$F(2T) \leq F(0) \leq C_0(F(0) - F(2T)).$$

Then,

$$F(2T) \leq \frac{C_0}{C_0 + 1} F(0),$$

and this implies the exponential estimate (3.14) with  $K, \tilde{\mu}$  as in (3.15), (3.16), due to the semi-group property together with the fact that  $F$  is non increasing.  $\blacksquare$

Now, let us recall the following classical result of Pazy (Theorem 1.1 in Ch. 3 of [24]).

**Theorem 3.5** *Let  $X$  be a Banach space and let  $A$  be the infinitesimal generator of a  $C_0$  semi-group  $T(t)$  on  $X$ , satisfying  $\|T(t)\| \leq Me^{\omega t}$ . If  $B$  is a bounded linear operator on  $X$  then  $A + B$  is the infinitesimal generator of a  $C_0$  semigroup  $S(t)$  on  $X$ , satisfying  $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$ .*

Using this perturbation result we are ready to give the asymptotic stability result for problem (1.1)–(1.3) when the norm of the delay feedback  $B_2^*$  is sufficiently small.

**Theorem 3.6** *Assume that the observability estimate (3.12) holds for problem (3.10) – (3.11). For all  $\xi > 1$  in the definition (3.2), there is  $\beta > 0$  depending on  $\bar{T}, \tau, \xi$  and on the operator  $B_1$ , such that if the delay feedback satisfies  $\|B_2^*\| < \beta$ , then there exist positive constants  $K, \mu$  for which we have*

$$E(t) \leq Ke^{-\mu t} E(0), \quad t > 0, \tag{3.27}$$

for any solution of (1.1) – (1.3).

**Proof.** We can see problem (1.1)–(1.3) as a perturbation of the auxiliary one. Therefore,

$$\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} = (\mathcal{A}^0 + \mathcal{B}) \begin{pmatrix} u \\ v \\ z \end{pmatrix}$$

with

$$\mathcal{B} \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -\xi B_2 B_2^* v \\ 0 \end{pmatrix}.$$

From Theorem 3.4 and Theorem 3.5, we know that if

$$-\tilde{\mu} + K\|\mathcal{B}\| < 0, \tag{3.28}$$

where  $\tilde{\mu}$  and  $K$  are defined by (3.15), (3.16) and (3.17), then

$$E(t) \leq K e^{-\mu t} E(0),$$

with  $\mu = \tilde{\mu} - K\|\mathcal{B}\|$ , for any solution of problem (1.1)–(1.3).

It remains to prove that (3.28) is satisfied for  $\|B_2^*\|$  sufficiently small. We can rewrite (3.28) as

$$\xi \|B_2\|^2 < \frac{\tilde{\mu}}{K},$$

that is

$$\xi C_2^2 < \frac{1}{2T} \frac{C_0}{C_0 + 1} \ln \frac{C_0 + 1}{C_0}. \quad (3.29)$$

The difficulty is that the constant  $C_0$  (defined by (3.17)) appearing in the right-hand side of this estimate depends on  $\xi$  and  $C_2$  as well. So let us consider the continuous function  $h : (0, +\infty) \rightarrow (0, +\infty)$ ,

$$h(s) := \frac{s}{s+1} \ln \frac{s+1}{s}.$$

Then,  $h$  tends to zero for  $s \rightarrow 0^+$  and for  $s \rightarrow +\infty$ . Moreover,  $h$  assumes the maximal value  $1/e$  at  $\frac{1}{e-1}$ , is increasing before  $\frac{1}{e-1}$  and decreasing after.

Now it is easy to check that  $\frac{\xi}{\xi-1} > \frac{1}{e-1}$ . Considering that  $\xi$  is fixed  $> 1$  and  $T$  is fixed as well, we consider  $C_0$  as a function of  $C_2 \geq 0$ , that we write  $C_0(C_2)$ . But we remark that from its definition,  $C_0$  is non decreasing (in  $C_2$ ) and

$$C_0(0) = \max \left\{ 2c, \frac{\xi}{\xi-1} \right\} > \frac{1}{e-1}.$$

Hence  $h(C_0(C_2))$  is non increasing as a function of  $C_2$  with  $h(C_0(0)) > 0$  and since the left-hand side of (3.29) is increasing in  $C_2$  and is zero at  $C_2 = 0$ , there exists a point  $\beta > 0$  such that

$$2\xi\beta^2 T = h(C_0(\beta)),$$

and for which (3.29) holds for all  $C_2 \in [0, \beta)$ .

Obviously  $\beta$  depends on  $T$  (and then on  $\bar{T}$  and  $\tau$ ), on  $\xi$  and, through the observability constant  $c$  and the time  $T$ , on the feedback operator  $B_1$ . ■

**Remark 3.7** If  $B_1$  is bounded, namely if  $B_1 \in \mathcal{L}(U_1, H)$ , then by Proposition 1 of [11], the system (3.10) – (3.11) is exponentially stable (or equivalently the observability estimate (3.12) holds for problem (3.10) – (3.11)) if and only if the observability estimate

$$E_S(0) = \frac{1}{2} (\|A^{\frac{1}{2}} w_0\|_H^2 + \|w_1\|_H^2) \leq c \int_0^T \|B_1^* \varphi_t(t)\|_{U_1}^2 dt, \quad (3.30)$$

holds for some  $T > 0$  and  $c > 0$ , for every weak solution  $\varphi$  of the conservative system

$$\varphi_{tt}(t) + A\varphi(t) = 0 \quad t > 0 \quad (3.31)$$

$$\varphi(0) = w_0 \quad \text{and} \quad \varphi_t(0) = w_1 \quad (3.32)$$

with initial data  $(w_0, w_1) \in V \times H$ . ■

## 4 The wave equation

### 4.1 Internal dampings

Our first application concerns the wave equation with locally distributed internal dampings. More precisely, let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with a Lipschitz boundary  $\partial\Omega$ . We suppose given  $b_1, b_2$  in  $L^\infty(\Omega)$  such that

$$b_1(x), b_2(x) \geq 0 \quad \text{a.e. } x \in \Omega.$$

Let us consider the initial boundary value problem

$$u_{tt}(x, t) - \Delta u(x, t) + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (4.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (4.2)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (4.3)$$

$$\sqrt{b_2}u_t(x, t) = f^0(x, t) \quad \text{in } \omega_2 \times (-\tau, 0), \quad (4.4)$$

$$(4.5)$$

with initial data  $(u_0, u_1, f^0) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))$ , where  $\omega_i = \{x \in \Omega : b_i(x) > 0\}$  is the support of  $b_i$ ,  $i = 1$  or  $2$ .

This problem enters into our previous framework, if we take  $H = L^2(\Omega)$  and the operator  $A$  defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow -\Delta u,$$

where  $\mathcal{D}(A) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$ . This operator  $A$  is a self-adjoint and positive operator with a compact inverse in  $H$  and is such that  $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$ . We then define  $U_1 = L^2(\omega_1)$ ,  $U_2 = L^2(\omega_2)$  and the operators  $B_i, i = 1, 2$ , as

$$B_i : U_i \rightarrow H : v \rightarrow \sqrt{b_i(x)}\tilde{v}, \quad (4.6)$$

where  $\tilde{v} \in L^2(\Omega)$  is the extension of  $v$  by zero outside  $\omega_i$ . It is easy to verify that

$$B_i^* \varphi = \sqrt{b_i} \varphi|_{\omega_i} \quad \text{for } \varphi \in H.$$

As  $B_i B_i^* \varphi = b_i \varphi$ , for any  $\varphi \in H$  and  $i = 1, 2$ , we deduce that problem (4.1)–(4.4) enters in the abstract framework (1.1)–(1.3).

In this setting, the energy functional is

$$E(t) = \frac{1}{2} \int_{\Omega} \{u_t^2(x, t) + |\nabla u(x, t)|^2\} dx + \frac{\xi}{2} \int_{t-\tau}^t \int_{\Omega} b_2(x) u_t^2(x, s) dx ds, \quad (4.7)$$

which is the standard energy for wave equation

$$E_S(t) = E_S(w, t) := \frac{1}{2} \int_{\Omega} (w_t^2 + |\nabla w|^2) dx,$$

plus an integral term due to the presence of a time delay.

Since  $B_1$  is bounded, according to Remark 3.7, our main assumption concerns the existence of an observability estimate for the standard wave equation:

$$\varphi_{tt}(x, t) - \Delta\varphi(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (4.8)$$

$$\varphi(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (4.9)$$

$$\varphi(x, 0) = w_0(x) \quad \text{and} \quad \varphi_t(x, 0) = w_1(x) \quad \text{in } \Omega \quad (4.10)$$

with  $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$ .

We then assume that there exists a time  $\bar{T} > 0$  such that for every time  $T > \bar{T}$  there is a constant  $c$ , depending on  $T$  but independent of the initial data, such that

$$E_S(0) \leq c \int_0^T \int_{\Omega} b_1(x) \varphi_t^2(x, s) dx ds, \quad (4.11)$$

for every weak solution of problem (4.8) – (4.10).

According to (3.13) we have

$$\|B_2\| = \|b_2\|_{\infty}^{1/2}, \quad (4.12)$$

where, for  $v \in L^\infty(\Omega)$ , we denote  $\|v\|_{\infty} = \sup_{x \in \Omega} |v(x)|$ , the  $L^\infty$  norm of  $v$ .

Therefore, according to Theorem 3.6, we have the next result:

**Theorem 4.1** *Assume that the observability estimate (4.11) holds for the wave equation (4.8) – (4.10). For all  $\xi > 1$  in the definition (4.7), there is  $\beta > 0$  depending on  $\bar{T}$ ,  $\tau$ ,  $\xi$  and  $b_1$  such that if  $\|b_2\|_{\infty} < \beta$ , then there exist positive constants  $K, \mu$  for which we have*

$$E(t) \leq K e^{-\mu t} E(0), \quad t > 0,$$

for any solution of (4.1) – (4.4).

**Remark 4.2** 1. From Lemma VII.2.4 of [18] (see also [14, 15, 17, 19, 28]), the observability estimate (4.11) holds for the wave equation (4.8) – (4.10) if the boundary of  $\Omega$  is of class  $C^2$ , if  $T$  is bigger than the diameter of  $\Omega$  and if

$$b_1(x) \geq b^0 > 0, \quad \text{a.e. } x \in \omega, \quad (4.13)$$

when the open subset  $\omega$  of  $\Omega$  is a neighborhood of  $\bar{\Gamma}_0$ , where

$$\Gamma_0 = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}, \quad (4.14)$$

for some  $x_0 \in \mathbb{R}^n$  and  $\nu(x)$  is the outer unit normal vector at  $x \in \partial\Omega$ .

2. From [2], the observability estimate (4.11) also holds for the wave equation (4.8) – (4.10) if the boundary of  $\Omega$  is of class  $C^\infty$  and if (4.13) holds when the open subset  $\omega$  of  $\Omega$  satisfies the geometric control property. ■

**Remark 4.3** According to point 1 of the previous remark, Theorem 4.1 allows to recover the results from Theorem 1.2 of [25] in a larger setting. ■

## 4.2 Internal and boundary dampings

We assume here that the boundary  $\partial\Omega$  of  $\Omega$  is splitted up as  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0, \Gamma_1$  are closed subsets of  $\partial\Omega$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Moreover we assume that  $\Gamma_0$  and  $\Gamma_1$  have an non empty interior (on  $\partial\Omega$ ). We suppose given  $k \in L^\infty(\Gamma_0)$  and  $b \in L^\infty(\Omega)$  such that  $b(x) \geq 0$  a.e.  $x \in \Omega$  and

$$k(x) \geq k_0 > 0 \text{ a.e. } x \in \Gamma_0.$$

We here consider the problem

$$u_{tt}(x, t) - \Delta u(x, t) + b(x)u_t(x, t - \tau) = 0, \quad x \in \Omega, \quad t > 0, \quad (4.15)$$

$$u(x, t) = 0, \quad x \in \Gamma_1, \quad t > 0 \quad (4.16)$$

$$\frac{\partial u}{\partial \nu}(x, t) = -k(x)u_t(x, t), \quad x \in \Gamma_0, \quad t > 0 \quad (4.17)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (4.18)$$

$$\sqrt{b}u_t(x, t) = f^0(x, t), \quad x \in \Omega, \quad t \in (-\tau, 0), \quad (4.19)$$

with initial data in a suitable space.

This problem enters into our previous framework, if we take  $H = L^2(\Omega)$  and the operator  $A$  defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow -\Delta u,$$

where

$$\mathcal{D}(A) := \{ u \in H_{\Gamma_1}^1(\Omega) : \Delta u \in L^2(\Omega) \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0 \},$$

with

$$H_{\Gamma_1}^1 := \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1 \}.$$

We then define  $U_1 := L^2(\Gamma_0)$ ,  $U_2 := L^2(\omega_2)$  ( $\omega_2$  being the support of  $b$ ) and the operators  $B_1, B_2$  as

$$B_2 \in \mathcal{L}(U_2; H), \quad B_2 u = \sqrt{b(x)}\tilde{u}, \quad \forall u \in L^2(\omega_2),$$

and

$$B_1 \in \mathcal{L}(U_1; V'), \quad B_1 u = \sqrt{k} A_{-1} N u, \quad \forall u \in L^2(\Gamma_0), \quad B_1^* w = \sqrt{k} w|_{\Gamma_0}, \quad \forall w \in V := \mathcal{D}(A^{1/2}),$$

where  $A_{-1}$  is the extension of  $A$  to  $H$ , namely for all  $h \in H$  and  $\varphi \in \mathcal{D}(A)$ ,  $A_{-1}h$  is the unique element in  $(\mathcal{D}(A))'$  (the duality is in the sense of  $H$ ), such that (see for instance [26])

$$\langle A_{-1}h; \varphi \rangle_{(\mathcal{D}(A))', \mathcal{D}(A)} = \int_{\Omega} h A \varphi \, dx.$$

Here and below  $N \in \mathcal{L}(L^2(\Gamma_0); L^2(\Omega))$  is defined as follows: for all  $v \in L^2(\Gamma_0)$ ,  $Nv$  is the unique solution (transposition solution) of

$$\Delta Nv = 0, \quad Nv|_{\Gamma_1} = 0, \quad \frac{\partial Nv}{\partial \nu} \Big|_{\Gamma_0} = v.$$

With these definitions, we can show that problem (4.15)–(4.19) enters in the abstract framework (1.1)–(1.3).

Now, the energy functional is

$$E(t) = \frac{1}{2} \int_{\Omega} \{u_t^2(x, t) + |\nabla u(x, t)|^2\} dx + \frac{\xi}{2} \int_{t-\tau}^t \int_{\Omega} b(x) u_t^2(x, s) dx ds. \quad (4.20)$$

As  $B_1$  is not bounded, we need to consider the non delayed system

$$w_{tt}(x, t) - \Delta w(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (4.21)$$

$$w(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty) \quad (4.22)$$

$$\frac{\partial w}{\partial \nu}(x, t) = -k(x)w_t(x, t), \quad x \in \Gamma_0, \quad t > 0 \quad (4.23)$$

$$w(x, 0) = w_0(x) \quad \text{and} \quad w_t(x, 0) = w_1(x) \quad \text{in } \Omega \quad (4.24)$$

with  $(w_0, w_1) \in H_{\Gamma_1}^1(\Omega) \times L^2(\Omega)$ .

Hence our main assumption will be: There exists a time  $\bar{T} > 0$  such that for every time  $T > \bar{T}$  there is a constant  $c$ , depending on  $T$  but independent of the initial data, such that

$$E_S(0) \leq c \int_0^T \int_{\Gamma_0} k(x) w_t^2(x, s) dx ds, \quad (4.25)$$

for every weak solution of problem (4.21) – (4.24).

Then, our previous results apply also to this model and we can restate Theorem 3.6.

**Theorem 4.4** *Assume that the observability estimate (4.25) holds for every weak solution of problem (4.21) – (4.24). For all  $\xi > 1$  in the definition (4.20), there is  $\beta > 0$  depending on  $\bar{T}$ ,  $\tau$ ,  $\xi$  and  $k$ , such that if  $\|b(x)\|_{\infty} < \beta$ , then there exist positive constants  $K, \mu$  for which we have*

$$E(t) \leq K e^{-\mu t} E(0), \quad t > 0,$$

for any solution of (4.15) – (4.19).

**Remark 4.5** 1. From Theorem 1 and Remark 1 of [13] (see also [5, 6, 7, 16]), the observability estimate (4.25) holds for the damped wave equation (4.21) – (4.24) if the boundary of  $\Omega$  is of class  $C^2$ , if  $T$  is large enough and if  $\Gamma_0$  is given by (4.14) for some  $x_0 \in \mathbb{R}^n$ .

2. From [2], the observability estimate (4.25) also holds for the damped wave equation (4.21) – (4.24) if the boundary of  $\Omega$  is of class  $C^{\infty}$  and if the part  $\Gamma_0$  satisfies the geometric control property.

3. If we suppress the assumption  $\Gamma_0 \cap \Gamma_1 = \emptyset$ , then Theorem 1 of [13] shows that the observability estimate (4.25) holds for the damped wave equation (4.21) – (4.24) under the same assumptions than in point 1 but with the choice  $k(x) = (x - x_0) \cdot \nu(x)$  and if  $n \leq 3$  (see also Proposition 6.4 of [10] in dimension 2). For this example,  $k$  is no more uniformly positive on  $\Gamma_0$ , nevertheless it enters into our abstract framework. ■

**Remark 4.6** This result, namely exponential decay of the energy for solutions to problem (4.15)–(4.19) for “small” internal delay feedback, has been first proved in [1], for  $b$  and  $k$  constant and  $\Gamma_0$  given by (4.14), by constructing a suitable Lyapunov functional and using the multiplier method. We give here a simpler proof by using a more general method, allowing to weaken the assumptions on  $b$ ,  $k$  and  $\Gamma_0$ . ■



## 5 The elasticity system

### 5.1 Internal dampings

Here we consider the following elastodynamic system

$$u_{tt}(x, t) - \mu \Delta u(x, t) - (\lambda + \mu) \nabla \operatorname{div} u + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (5.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (5.2)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (5.3)$$

$$\sqrt{b_2}u_t(x, t) = f^0(t) \quad \text{in } \omega_2 \times (-\tau, 0), \quad (5.4)$$

with initial data  $(u_0, u_1, f^0) \in H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2((-\tau, 0); L^2(\omega_2)^n)$  and  $b_1, b_2$  satisfying the same assumptions as in subsection 4.1. Note that in this case the state variable  $u$  is vector-valued and  $\lambda, \mu$  are the Lamé coefficients that are positive real numbers.

As before this problem enters into our abstract setting, once we take  $H = L^2(\Omega)^n$ , and  $A$  defined by

$$A : \mathcal{D}(A) \rightarrow H : u \rightarrow -\mu \Delta u(x, t) - (\lambda + \mu) \nabla \operatorname{div} u,$$

where  $\mathcal{D}(A) = \{u \in H_0^1(\Omega)^n : \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u \in L^2(\Omega)^n\}$ .

The operator  $A$  is a self-adjoint and positive operator with a compact inverse in  $H$  and is such that  $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)^n$  equipped with the inner product

$$(u, v)_V = \int_{\Omega} \left( \mu \sum_{i,j=1}^n \partial_i u_j \partial_i v_j + (\lambda + \mu) \operatorname{div} u \operatorname{div} v \right) dx, \quad \forall u, v \in H_0^1(\Omega)^n.$$

We then define  $U_i = L^2(\omega_i)^n$  and the operators  $B_i, i = 1, 2$ , as

$$B_i : U_i \rightarrow H : v \rightarrow \sqrt{b_i} \tilde{v},$$

where  $\tilde{v}$  is the extension of  $v$  by zero outside  $\omega_i$ . As before

$$B_i^*(\varphi) = \sqrt{b_i} \varphi|_{\omega_i} \quad \text{for } \varphi \in H,$$

and thus  $B_i B_i^*(\varphi) = b_i \varphi$ , for any  $\varphi \in H$  and  $i = 1, 2$ . So, problem (5.1)–(5.4) enters in the abstract framework (1.1)–(1.3).

Therefore in order to apply the abstract results of section 3, we only need to check the observability estimate for the associated conservative system: There exists a time  $T > 0$  and a constant  $c > 0$  such that

$$\frac{1}{2}((w_0, w_0)_V + \int_{\Omega} |w_1|^2 dx) \leq c \int_0^T \int_{\Omega} b_1(x) |\varphi_t|^2(x, s) dx ds, \quad (5.5)$$

for every weak solution  $\varphi$  of

$$\varphi_{tt}(x, t) - \mu \Delta \varphi(x, t) - (\lambda + \mu) \nabla \operatorname{div} \varphi(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty),$$

$$\varphi(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

$$\varphi(x, 0) = w_0(x) \quad \text{and} \quad \varphi_t(x, 0) = w_1(x) \quad \text{in } \Omega,$$

with initial data  $(w_0, w_1) \in H_0^1(\Omega)^n \times L^2(\Omega)^n$ .

If such an estimate holds, the stability result from section 3 can be applied to the above system.

**Remark 5.1** Under the assumptions of point 1 of Remark 4.2, the observability estimate (5.5) is obtained in the proof of Theorem 3.1 of [4] (estimate (3.2) of [4]). ■

## 5.2 Internal and boundary dampings

Under the assumptions of subsection 4.2 we consider the following elastodynamic system

$$\begin{aligned} u_{tt}(x, t) - \mu \Delta u(x, t) - (\lambda + \mu) \nabla \operatorname{div} u + b(x) u_t(x, t - \tau) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ u(x, t) &= 0, \quad x \in \Gamma_1, \quad t > 0 \end{aligned} \quad (5.6)$$

$$\sigma(u(x, t)) \cdot \nu(x) = -k(x) u_t(x, t), \quad x \in \Gamma_0, \quad t > 0 \quad (5.7)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (5.8)$$

$$\sqrt{b} u_t(x, t) = f^0(t) \quad \text{in } \omega_2 \times (-\tau, 0), \quad (5.9)$$

with initial data  $(u_0, u_1, f^0) \in H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2((-\tau, 0); L^2(\omega_2)^n)$  and

$$\sigma(u) = \mu \left( \sum_{i=1}^n \partial_i (u_j) \nu_i \right)_{j=1}^n + (\lambda + \mu) (\operatorname{div} u) \nu \quad \text{on } \Gamma_0.$$

This problem enters into our abstract setting, once we take  $H = L^2(\Omega)^n$ ,  $A$  defined in the previous subsection, and  $B_1$  and  $B_2$  defined as in subsection 4.2.

As  $B_1$  is not bounded, we need to assume that there exists a time  $\bar{T} > 0$  such that for every time  $T > \bar{T}$  there is a constant  $c$ , depending on  $T$  but independent of the initial data, such that

$$\frac{1}{2} ((w_0, w_0)_V + \int_{\Omega} |w_1|^2 dx) \leq c \int_0^T \int_{\Gamma_0} k(x) |w_t|^2(x, s) dx ds, \quad (5.10)$$

for every weak solution  $w$  of the non delayed system

$$\begin{aligned} w_{tt}(x, t) - \mu \Delta w(x, t) - (\lambda + \mu) \nabla \operatorname{div} w &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ w(x, t) &= 0, \quad x \in \Gamma_1, \quad t > 0 \end{aligned} \quad (5.11)$$

$$\sigma(w(x, t)) \cdot \nu(x) = -k(x) w_t(x, t), \quad x \in \Gamma_0, \quad t > 0 \quad (5.12)$$

$$w(x, 0) = w_0(x) \quad \text{and} \quad w_t(x, 0) = w_1(x) \quad \text{in } \Omega, \quad (5.13)$$

for initial data  $(w_0, w_1) \in H_0^1(\Omega)^n \times L^2(\Omega)^n$ .

Again if such an estimate holds, the stability result from section 3 can be applied to the system (5.6)–(5.9).

**Remark 5.2** 1. Under the assumptions of point 1 of Remark 4.5, the observability estimate (5.10) is proved in [3].

2. If we assume that the boundary of  $\Omega$  is smooth and that

$$(x - x_0) \cdot \nu(x) \leq 0 \quad \text{on } \Gamma_1,$$

then the observability estimate (5.10) is proved in Lemma 3.2 of [12]. ■

## 6 The Petrovsky system

### 6.1 Hinged boundary conditions

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with a boundary  $\partial\Omega$  of class  $C^4$  (as before this regularity could be weakened).

Let us consider the initial boundary value problem

$$u_{tt}(x, t) + \Delta^2 u(x, t) + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \text{ in } \Omega \times (0, +\infty), \quad (6.1)$$

$$u(x, t) = \Delta u(x, t) = 0 \text{ on } \partial\Omega \times (0, +\infty), \quad (6.2)$$

$$u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = u_1(x) \text{ in } \Omega, \quad (6.3)$$

$$u_t(x, t) = f^0(x, t) \text{ in } \omega_2 \times (-\tau, 0), \quad (6.4)$$

with initial data  $(u_0, u_1, f^0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))$  and  $b_1, b_2$  satisfying the same assumptions as in subsection 4.1.

Now, we take  $H = L^2(\Omega)$  and let  $A$  be the operator

$$A : \mathcal{D}(A) \rightarrow H : \quad u \rightarrow \Delta^2 u, \quad (6.5)$$

where

$$\mathcal{D}(A) = \{v \in H_0^1(\Omega) \cap H^4(\Omega) : \Delta u = 0 \text{ on } \partial\Omega\}.$$

The operator  $A$  is self-adjoint and positive, has a compact inverse in  $H$  and satisfies  $\mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$ . We then define  $U_i = L^2(\omega_i)$  and the operators  $B_i, i = 1, 2$ , by (4.6). So, problem (6.1)–(6.4) enters in the abstract framework (1.1)–(1.3).

If an observability estimate of the associated conservative system holds, then the results of section 3 apply also to the plate model.

**Remark 6.1** Under the assumptions of point 1 of Remark 4.2 and the additional regularity of the boundary, it is well-known that an observability estimate of the associated conservative system holds, see Proposition 7.5.7 (see also Example 11.2.4) of [26]. ■

### 6.2 Clamped boundary conditions

Let  $\Omega \subset \mathbb{R}^2$  be an open bounded set with a boundary  $\partial\Omega$  of class  $C^4$ .

Here we consider the initial boundary value problem

$$u_{tt}(x, t) + \Delta^2 u(x, t) + b_1(x)u_t(x, t) + b_2(x)u_t(x, t - \tau) = 0 \text{ in } \Omega \times (0, +\infty), \quad (6.6)$$

$$u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0 \text{ on } \partial\Omega \times (0, +\infty), \quad (6.7)$$

$$u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = u_1(x) \text{ in } \Omega, \quad (6.8)$$

$$u_t(x, t) = f^0(x, t) \text{ in } \omega_2 \times (-\tau, 0), \quad (6.9)$$

with initial data  $(u_0, u_1, f^0) \in H_0^2(\Omega) \times L^2(\Omega) \times L^2((-\tau, 0); L^2(\omega_2))$  where

$$H_0^2(\Omega) := \{ \varphi \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \},$$

and  $b_1, b_2$  satisfy the same assumptions than in the previous subsection.

As usual, if an observability estimate holds for the associated conservative system, the results of section 3 can be applied to this model.

**Remark 6.2** Under the assumptions of point 1 of Remark 4.2 and the additional assumptions of this subsection, the observability estimate for the associated conservative system has been recently proved by the authors (see [22], Theorem 6.1). ■

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