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# Well-posedness and stability results for nonlinear abstract evolution equations with time delays

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## Abstract

We consider abstract evolution equations with a nonlinear term depending on the state and on delayed states. We show that, if the  $C_0$ -semigroup describing the linear part of the model is exponentially stable, then the whole system retains this property under some Lipschitz continuity assumptions on the nonlinearity. More precisely, we give a general exponential decay estimate for small time delays if the nonlinear term is globally Lipschitz and an exponential decay estimate for solutions starting from small data when the nonlinearity is only locally Lipschitz and the linear part is a negative selfadjoint operator. In the latter case we do not need any restriction on the size of the time delays. In both cases, concrete examples are presented that illustrate our abstract results.

## 1 Introduction

Let  $\mathcal{H}$  be a fixed Hilbert space with inner product  $(\cdot, \cdot)_{\mathcal{H}}$  and norm  $\|\cdot\|_{\mathcal{H}}$ , and consider an operator  $\mathcal{A}$  from  $\mathcal{H}$  into itself that generates a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  that is exponentially stable, i.e., there exist two positive constants  $M$  and  $\omega$  such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \quad \forall t \geq 0, \quad (1.1)$$

where, as usual,  $\mathcal{L}(\mathcal{H})$  denotes the space of bounded linear operators from  $\mathcal{H}$  into itself.

We consider the evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \sum_{i=1}^I F_i(U(t), U(t - \tau_i)) & \text{in } (0, +\infty), \\ U(t - \tau) = U_0(t), \quad \forall t \in (0, \tau], \end{cases} \quad (1.2)$$

where  $I$  is a positive natural number and  $\tau_i > 0, i = 1, \dots, I$ , are time delays. Without loss of generality we can suppose that the delays are different from each other and that

$$\tau_i < \tau = \tau_1, \quad \forall i = 2, \dots, I.$$

The nonlinear terms  $F_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$  satisfy some Lipschitz conditions, while the initial datum  $U_0$  satisfies  $U_0 \in C([0, \tau]; \mathcal{H})$ . We denote by  $U^0$  the initial datum at  $t = 0$ , namely

$$U^0 = U_0(\tau) \in \mathcal{H}. \quad (1.3)$$

Time delay effects frequently appear in many practical applications and physical models. On the other hand, it is well-known (cfr. [3, 6, 7, 19, 29]) that they can induce some instability. Then, we are interested in giving an exponential stability result for such a problem under a suitable smallness condition on the delay  $\tau$  or on the initial data. In [21] we have studied the case of linear dependency on the delayed state  $U(t - \tau)$ , i.e., we have considered the model

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + G(U(t)) + k\mathcal{B}U(t - \tau) & \text{in } (0, +\infty), \\ U(0) = U^0, \quad \mathcal{B}U(t - \tau) = g(t), \quad \forall t \in (0, \tau), \end{cases} \quad (1.4)$$

where  $\mathcal{B}$  is a fixed bounded operator from  $\mathcal{H}$  into itself,  $G : \mathcal{H} \rightarrow \mathcal{H}$  satisfies some Lipschitz conditions, the initial datum  $U^0$  belongs to  $\mathcal{H}$  and  $g \in C([0, \tau]; \mathcal{H})$ .

For some concrete examples, mainly for  $G \equiv 0$ , it was known that the above problem, under certain *smallness* conditions on the delay feedback  $k\mathcal{B}$ , is exponentially stable, the proof being from time to time quite technical because some observability inequalities or perturbation methods are used. For instance the case of wave equation with interior feedback delay and dissipative boundary condition was considered in [2] by constructing suitable Lyapunov functionals while the case of a locally damped wave equation with distributed delay has been analyzed in [26] by introducing an auxiliary model easier to deal with and using a perturbative argument. The last approach has then been extended to a general class of second order evolution equations in [20]. The case of a Timoshenko system with delay has been studied in [28] by using appropriate Lyapunov functionals. Also wave type equations with viscoelastic dissipative damping and time delay feedback have been considered (see e.g. [12, 1]). Moreover we quote the book [3] for several examples, also in the parabolic case.

Therefore, in [21] our main goal was to furnish a general stability result based on a direct and more simple proof obtained by using the so-called Duhamel's formula (or variation of parameters formula). Indeed we proved an exponential stability result under a suitable condition between the constant  $k$  and the constants  $M, \omega, \tau$ , the nonlinear term  $G$  and the norm of the bounded delay operator  $\mathcal{B}$ .

Now, we extend the results in [21] by analyzing the more general model (1.2). In particular, while in [21] we have considered only linear dependency on the delayed state  $U(t - \tau)$  (see (1.4)), we now consider a nonlinear dependency on the state and also on the delayed states. We first treat the case when  $F_i$ ,  $i = 1, \dots, I$ , are globally Lipschitz continuous and give an exponential stability result under a smallness assumption on the time delays. The proof, when adapted to model (1.4), is easier with respect to the one that we gave in [21] for the globally Lipschitz case. However, the result from [21] allowed a bit larger size on the time delay (see Remark 2.5) in order to have an exponential stability estimate for problem (1.4).

Then, we assume that the nonlinearity is only locally Lipschitz continuous (with a stronger topology) and we prove global existence and an exponential decay estimates for small initial data, independently of the size of the time delays. In this last case, we need to restrict ourselves to the case of a negative self-adjoint operator  $\mathcal{A}$ . Nevertheless, this last case has various applications in population dynamics (see section 4).

The paper is organized as follow. In section 2 we deal with the globally Lipschitz case; we give a well-posedness result and an exponential stability estimate for small delays. Some concrete examples are also presented. In section 3 we treat the case where the reaction term is only locally Lipschitz. We prove the local well-posedness and then, for small initial conditions, a global existence result and an exponential decay estimate. Finally, in section 4 we give some illustrative applications of the abstract results of section 3 to models in population dynamics. In particular we consider model for single species and competition models for two species.

## 2 The case of small delays with $F_i$ globally Lipschitz

In this section we assume that the nonlinear terms are globally Lipschitz and study existence and asymptotic behavior of solutions, generalizing Proposition 2.1 and Theorem 2.2 of [21].

### 2.1 Well-posedness and stability estimate

Let the functions  $F_i$ ,  $i = 1, \dots, I$ , be globally Lipschitz continuous, namely for every  $i = 1, \dots, I$ ,

$$\begin{aligned} \exists \gamma_i > 0 \quad \text{such that} \quad & \|F_i(U_1, U_2) - F_i(U_1^*, U_2^*)\|_{\mathcal{H}} \leq \gamma_i (\|U_1 - U_1^*\|_{\mathcal{H}} + \|U_2 - U_2^*\|_{\mathcal{H}}), \\ \forall (U_1, U_2), (U_1^*, U_2^*) \in & \mathcal{H} \times \mathcal{H}. \end{aligned} \tag{2.1}$$

The following well-posedness result holds.

**Proposition 2.1** *Assume that the functions  $F_i$  satisfy (2.1), for all  $i = 1, \dots, I$ . For any initial datum  $U_0 \in C([0, \tau]; \mathcal{H})$ , there exists a unique (mild) solution  $U \in C([0, +\infty), \mathcal{H})$*

of problem (1.2). Moreover,

$$U(t) = S(t)U^0 + \int_0^t S(t-s) \sum_{i=1}^I F_i(U(s), U(s-\tau_i)) ds. \quad (2.2)$$

**Proof.** Like in [21], we use an iterative argument. Namely in the interval  $(0, \tau_{\min})$ , where  $\tau_{\min} = \min_{i=1, \dots, I} \tau_i$ , problem (1.2) can be seen as a standard evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g_1(U(t)) & \text{in } (0, \tau_{\min}) \\ U(0) = U^0, \end{cases} \quad (2.3)$$

where  $g_1(U(t)) = \sum_{i=1}^I F_i(U(t), U(t-\tau_i))$ . Note that the terms  $U(t-\tau_i)$  can be regarded as known data for  $t \in [0, \tau_{\min})$ . This problem has a unique solution  $U \in C([0, \tau_{\min}], \mathcal{H})$  (see Th. 1.2, Ch. 6 of [25]) satisfying

$$U(t) = S(t)U^0 + \int_0^t S(t-s)g_1(U(s)) ds.$$

This yields  $U(t)$ , for  $t \in [0, \tau_{\min}]$ . Therefore on  $(\tau_{\min}, 2\tau_{\min})$ , problem (1.2) can be seen as the evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g_2(U(t)) & \text{in } (\tau_{\min}, 2\tau_{\min}) \\ U(\tau_{\min}) = U(\tau_{\min}-), \end{cases} \quad (2.4)$$

where  $g_2(U(t)) = \sum_{i=1}^I F_i(U(t), U(t-\tau_i))$ , since the terms  $U(t-\tau_i)$  can be regarded as data being known from the first step. Hence, this problem has a unique solution  $U \in C([\tau_{\min}, 2\tau_{\min}], \mathcal{H})$  given by

$$U(t) = S(t-\tau_{\min})U(\tau_{\min}-) + \int_{\tau_{\min}}^t S(t-s)g_2(U(s)) ds, \forall t \in [\tau_{\min}, 2\tau_{\min}].$$

By iterating this procedure, we obtain a global solution  $U$  satisfying (2.2). ■

Now we will prove the following exponential stability result.

**Theorem 2.2** *Assume that the functions  $F_i$  satisfy (2.1) and*

$$F_i(0, 0) = 0, \quad (2.5)$$

*for all  $i = 1, \dots, I$ . With  $M, \omega$  from (1.1), we assume that (see (2.1))*

$$\gamma = \sum_{i=1}^I \gamma_i < \frac{\omega}{2M}. \quad (2.6)$$

*If the time delay  $\tau$  satisfies the smallness condition*

$$\tau < \tau_0 := \frac{1}{\omega} \ln \left( \frac{\omega}{M\gamma} - 1 \right), \quad (2.7)$$

then there exists  $\omega' > 0$  such that the solution  $U \in C([0, +\infty), \mathcal{H})$  of problem (1.2), with  $U_0 \in C([0, \tau]; \mathcal{H})$ , satisfies

$$\|U(t)\|_{\mathcal{H}} \leq Me^{-\omega't} \left( \|U_0\|_{\mathcal{H}} + \sum_{i=1}^I \gamma_i \int_0^{\tau_i} e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds \right), \quad \forall t \geq 0. \quad (2.8)$$

**Proof.** From (2.2), we can estimate

$$\begin{aligned} \|U(t)\|_{\mathcal{H}} &\leq Me^{-\omega t} \left( \|U^0\|_{\mathcal{H}} + \sum_{i=1}^I \gamma_i \int_0^t e^{\omega s} (\|U(s)\|_{\mathcal{H}} + \|U(s - \tau_i)\|_{\mathcal{H}}) ds \right) \\ &\leq Me^{-\omega t} \left( \|U^0\|_{\mathcal{H}} + \gamma \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds + \sum_{i=1}^I \gamma_i \int_0^t e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds \right), \quad \forall t > 0. \end{aligned} \quad (2.9)$$

Then, for  $t \geq \tau$ ,

$$\begin{aligned} \|U(t)\|_{\mathcal{H}} &\leq Me^{-\omega t} \left( \|U^0\|_{\mathcal{H}} + \gamma \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds \right. \\ &\quad \left. + \alpha + \sum_{i=1}^I \gamma_i \int_{\tau_i}^t e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds \right), \end{aligned} \quad (2.10)$$

where

$$\alpha = \sum_{i=1}^I \gamma_i \int_0^{\tau_i} e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds. \quad (2.11)$$

Thus, for  $t \geq \tau$ , we obtain

$$\begin{aligned} \|U(t)\|_{\mathcal{H}} &\leq Me^{-\omega t} \left( \|U_0\|_{\mathcal{H}} + \gamma \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds + \alpha \right. \\ &\quad \left. + \sum_{i=1}^I \gamma_i \int_0^{t-\tau_i} e^{\omega(s+\tau_i)} \|U(s)\|_{\mathcal{H}} ds \right) \\ &\leq Me^{-\omega t} (\|U_0\|_{\mathcal{H}} + \alpha) + \gamma Me^{-\omega t} (1 + e^{\omega \tau}) \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds. \end{aligned} \quad (2.12)$$

For  $t \leq [0, \tau]$ , one can estimate

$$\begin{aligned}
& \sum_{i=1}^I \gamma_i \int_0^t e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds \\
& \leq \sum_{i=1}^I \gamma_i \int_0^{\tau_i} e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds + \sum_{i: \tau_i < t} \gamma_i \int_{\tau_i}^t e^{\omega s} \|U(s - \tau_i)\|_{\mathcal{H}} ds \\
& = \alpha + \sum_{i: \tau_i < t} \gamma_i \int_0^{t - \tau_i} e^{\omega(s + \tau_i)} \|U(s)\|_{\mathcal{H}} ds \\
& \leq \alpha + \sum_{i: \tau_i < t} \gamma_i e^{\omega \tau} \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds \\
& \leq \alpha + \gamma e^{\omega \tau} \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds.
\end{aligned} \tag{2.13}$$

Therefore, from (2.9), also for  $t \leq \tau$ , it results

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\omega t} (\|U^0\|_{\mathcal{H}} + \alpha) + \gamma M e^{-\omega t} (1 + e^{\omega \tau}) \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds. \tag{2.14}$$

Then, from (2.12) and (2.14),

$$e^{\omega t} \|U(t)\|_{\mathcal{H}} \leq M (\|U^0\|_{\mathcal{H}} + \alpha) + \gamma M (1 + e^{\omega \tau}) \int_0^t e^{\omega s} \|U(s)\|_{\mathcal{H}} ds, \quad \forall t \geq 0. \tag{2.15}$$

Therefore, Gronwall's lemma yields

$$e^{\omega t} \|U(t)\|_{\mathcal{H}} \leq M e^{M\gamma(1+e^{\omega\tau})t} (\|U^0\|_{\mathcal{H}} + \alpha), \quad t \geq 0. \tag{2.16}$$

Estimate (2.16) can be rewritten as

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-(\omega - M\gamma(1+e^{\omega\tau}))t} (\|U_0\|_{\mathcal{H}} + \alpha), \quad t \geq 0. \tag{2.17}$$

Then, exponential decay is ensured under the condition

$$\omega > M\gamma(1 + e^{\omega\tau}). \tag{2.18}$$

Under the assumption (2.6), inequality (2.18) is satisfied if and only if  $\tau$  satisfies the smallness condition (2.7). Then, the statement is proved. ■

We can restate the previous theorem for the particular case of model (1.4):

**Theorem 2.3** *Assume that the nonlinear term  $G$  satisfies the Lipschitz condition*

$$\exists \gamma > 0 \quad \text{such that} \quad \|G(U_1) - G(U_1^*)\|_{\mathcal{H}} \leq \gamma \|U_1 - U_1^*\|_{\mathcal{H}}, \quad \forall U_1, U_1^* \in \mathcal{H}. \tag{2.19}$$

Let  $M, \omega, \gamma$  as in (1.1) and (2.19). Moreover, let us assume that  $G(0) = 0$  and (2.6) hold. If the time delay  $\tau$  satisfies the smallness condition

$$\tau < \tau'_0 := \frac{1}{\omega} \ln \frac{1}{k\|\mathcal{B}\|_{\mathcal{H}}} \left( \frac{\omega}{M} - \gamma \right), \tag{2.20}$$

then there exists  $\omega^* > 0$  such that the solution  $U \in C([0, +\infty), \mathcal{H})$  of problem (1.4), with  $U^0 \in \mathcal{H}$  and  $g \in C([0, \tau]; \mathcal{H})$ , satisfies

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\omega^* t} \left( \|U_0\|_{\mathcal{H}} + k \int_0^\tau e^{\omega s} \|g(s)\|_{\mathcal{H}} ds \right), \quad \forall t \geq 0. \quad (2.21)$$

**Remark 2.4** Note that Theorem 2.2 is very general. For instance, it furnishes stability results for previously studied models for wave type equations (cfr. [2, 26, 20] and subsections 2.2-2.3 below) and Timoshenko models (cfr. [28]) eventually with the addition of a nonlinear term depending on the not delayed state and on a finite number of delayed states. Also, it includes recent stability results for problems with viscoelastic damping and time delay (cfr. [12, 1]).

**Remark 2.5** Note that the condition (2.20) is a bit less general with respect to the one obtained in [21]. Indeed, here we need

$$k \|\mathcal{B}\|_{\mathcal{H}} e^{\omega \tau} + \gamma < \frac{\omega}{M},$$

instead of the condition

$$k \|\mathcal{B}\|_{\mathcal{H}} e^{\omega \tau} + \gamma < \frac{e^{\omega \tau} - 1}{M \tau}$$

assumed there. On the other hand, the present approach allows to extend the class of the problems for which the stability result holds.

We now present a couple of examples, among the many, illustrating our previous abstract result.

## 2.2 The damped wave equation

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a boundary  $\Gamma$  of class  $C^2$ . Let  $f_j : \mathbb{R} \rightarrow \mathbb{R}$  be globally Lipschitz continuous functions,  $j = 1, 2$ , satisfying  $f_1(0) = f_2(0) = 0$ . Let us consider the following semilinear damped wave equation:

$$\begin{aligned} u_{tt}(x, t) - \Delta u(x, t) + a(x)u_t(x, t) &= f_1(u(x, t)) + f_2(u(x, t - \tau)) \quad \text{in } \Omega \times (0, +\infty), \\ u(x, t) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ u(x, t - \tau) &= u_0(x, t), \quad u_t(x, t - \tau) = u_1(x, t) \quad \text{in } \Omega \times (0, \tau], \end{aligned}$$

where  $\tau > 0$  is the time delay and the damping coefficient  $a \in L^\infty(\Omega)$  satisfies

$$a(x) \geq a_0 > 0, \quad \text{a.e. } x \in \omega, \quad (2.22)$$

for some nonempty open subset  $\omega$  of  $\Omega$  satisfying some control geometric properties (see e.g. [4]). The initial datum  $(u_0, u_1)$  is taken in  $C([0, \tau], H_0^1(\Omega) \times L^2(\Omega))$ .

Setting  $U = (u, u_t)^T$ , this problem can be rewritten in the form (1.2) with  $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ ,

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -a \end{pmatrix}$$



and  $F_1(U(t), U(t - \tau)) = (0, f_1(u(t)) + f_2(u(t - \tau)))^T$ . It is well-known that  $\mathcal{A}$  generates a strongly continuous semigroup which is exponentially stable (see e.g. [30, 15]), thus the assumptions on  $f_1, f_2$  ensure that Proposition 2.1 and Theorem 2.2 apply to this model giving a well-posedness result and an exponential decay estimate of the energy for small values of the time delay  $\tau$ .

### 2.3 The wave equation with memory

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with a smooth boundary and let  $f_j, j = 1, 2$ , as in the previous example. Let us consider the following problem:

$$\begin{aligned} u_{tt}(x, t) - \Delta u(x, t) + \int_0^\infty \mu(s) \Delta u(x, t - s) ds \\ = f_1(u(x, t)) + f_2(u(x, t - \tau)) \quad \text{in } \Omega \times (0, +\infty), \end{aligned} \quad (2.23)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (2.24)$$

$$u(x, t) = u_0(x, t) \quad \text{in } \Omega \times (-\infty, 0], \quad (2.25)$$

where the initial datum  $u_0$  belongs to a suitable space, the constant  $\tau > 0$  is the time delay and the memory kernel  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a locally absolutely continuous function satisfying

- i)  $\mu(0) = \mu_0 > 0$ ;
- ii)  $\int_0^{+\infty} \mu(t) dt = \tilde{\mu} < 1$ ;
- iii)  $\mu'(t) \leq -\alpha\mu(t)$ , for some  $\alpha > 0$ .

As in [5], we denote

$$\eta^t(x, s) := u(x, t) - u(x, t - s). \quad (2.26)$$

Then we can restate (2.23)–(2.25) as

$$\begin{aligned} u_{tt}(x, t) = (1 - \tilde{\mu}) \Delta u(x, t) + \int_0^\infty \mu(s) \Delta \eta^t(x, s) ds \\ = f_1(u(x, t)) + f_2(u(x, t - \tau)) \quad \text{in } \Omega \times (0, +\infty) \end{aligned} \quad (2.27)$$

$$\eta_t^t(x, s) = -\eta_s^t(x, s) + u_t(x, t) \quad \text{in } \Omega \times (0, +\infty) \times (0, +\infty), \quad (2.28)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (2.29)$$

$$\eta^t(x, s) = 0 \quad \text{in } \partial\Omega \times (0, +\infty), \quad t \geq 0, \quad (2.30)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (2.31)$$

$$\eta^0(x, s) = \eta_0(x, s) \quad \text{in } \Omega \times (0, +\infty), \quad (2.32)$$

where

$$\begin{aligned} u_0(x) &= u_0(x, 0), \quad x \in \Omega, \\ u_1(x) &= \frac{\partial u_0}{\partial t}(x, t)|_{t=0}, \quad x \in \Omega, \\ \eta_0(x, s) &= u_0(x, 0) - u_0(x, -s), \quad x \in \Omega, \quad s \in (0, +\infty). \end{aligned} \quad (2.33)$$

Let us denote  $U := (u, u_t, \eta^t)^T$ . Then we can rewrite problem (2.27)–(2.32) in the abstract form

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F_1(U(t), U(t - \tau)), \\ U(0) = (u_0, u_1, \eta_0)^T, \end{cases} \quad (2.34)$$

where the operator  $\mathcal{A}$  is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} v \\ (1 - \tilde{\mu})\Delta u + \int_0^\infty \mu(s)\Delta w(s)ds \\ -w_s + v \end{pmatrix}, \quad (2.35)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ (u, v, \eta)^T \in H_0^1(\Omega) \times H_0^1(\Omega) \times L_\mu^2((0, +\infty); H_0^1(\Omega)) : \right. \\ \left. (1 - \tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \eta_s \in L_\mu^2((0, +\infty); H_0^1(\Omega)) \right\}, \quad (2.36)$$

where  $L_\mu^2((0, \infty); H_0^1(\Omega))$  is the Hilbert space of  $H_0^1$ -valued functions on  $(0, +\infty)$ , endowed with the inner product

$$\langle \varphi, \psi \rangle_{L_\mu^2((0, \infty); H_0^1(\Omega))} = \int_\Omega \left( \int_0^\infty \mu(s) \nabla \varphi(x, s) \nabla \psi(x, s) ds \right) dx.$$

Denote by  $\mathcal{H}$  the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L_\mu^2((0, \infty); H_0^1(\Omega)),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}} := (1 - \tilde{\mu}) \int_\Omega \nabla u \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx + \int_\Omega \int_0^\infty \mu(s) \nabla w \nabla \tilde{w} ds dx. \quad (2.37)$$

It is well-known (see e.g. [10]) that the operator  $\mathcal{A}$  generates an exponentially stable semi-group. Thus, Proposition 2.1 and Theorem 2.2 guarantee well-posedness and exponential stability, for small delays, also for the model (2.23)–(2.25).

### 3 The case of small data with general nonlinearities

Here we consider a more general class of nonlinearities but assume that  $\mathcal{A}$  is a negative selfadjoint operator in  $\mathcal{H}$ . In this case,  $\mathcal{A}$  generates an analytic semi-group (see [14, Example IX.1.25]) and existence results for problem (1.2) can be obtained for nonlinear terms satisfying the next hypothesis (3.2). More precisely, we recall that  $\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}})$  is a Hilbert space with the norm

$$\|U\|_{\mathcal{V}}^2 = ((-\mathcal{A})^{\frac{1}{2}}U, (-\mathcal{A})^{\frac{1}{2}}U), \quad \forall U \in \mathcal{V}.$$

Furthermore if  $\lambda_1$  is the smallest eigenvalue of  $-\mathcal{A}$ , we have

$$\lambda_1 \|U\|_{\mathcal{H}}^2 \leq \|U\|_{\mathcal{V}}^2, \quad \forall U \in \mathcal{V}. \quad (3.1)$$

Then we assume that there exist a positive real number  $\beta < \frac{1}{2}$ , a constant  $C_0$  and two continuous functions  $h_1, h_2$  from  $[0, \infty)^4$  to  $[0, \infty)$  such that, for all  $i = 1, \dots, I$ ,

$$\begin{aligned} \|F_i(U_1, V_1) - F_i(U_2, V_2)\|_{\mathcal{H}} &\leq C_0 \|U_1 - U_2\|_{\mathcal{H}} \\ &+ h_1(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \|V_2\|_{\mathcal{V}}) \|U_1 - U_2\|_{\mathcal{V}} \\ &+ h_2(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \|V_2\|_{\mathcal{V}}) \|V_1 - V_2\|_{D((-A)^\beta)}, \end{aligned} \quad (3.2)$$

for all  $(U_1, U_2), (V_1, V_2) \in \mathcal{V} \times \mathcal{V}$ .

The following local existence result holds (compare with Theorem 1 of [22]).

**Proposition 3.1** *Assume that the functions  $F_i$  satisfy (3.2). Then for any initial datum  $U_0 \in C([0, \tau]; \mathcal{V}) \cap C^{0,\theta}([0, \tau], D((-A)^\beta))$ , with  $\beta < \frac{1}{2}$  from the assumption (3.2) and  $\theta = \min\{\beta, \frac{1}{2} - \beta\}$ , there exist a time  $T_\infty \in (0, +\infty]$  and a unique solution  $U \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$  of problem (1.2).*

**Proof.** Like in the proof of Proposition 2.1, we look at the problem (1.2) in the time interval  $[0, \tau_{\min})$  where, as before,  $\tau_{\min} = \min_{i=1, \dots, I} \tau_i$ . Then the model can be rewritten as

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g(t, U(t)) & \text{in } (0, \tau_{\min}), \\ U(0) = U^0, \end{cases} \quad (3.3)$$

where  $g(t, U(t)) = \sum_{i=1}^I F_i(U(t), U(t - \tau_i))$ . Recall that the terms  $U(t - \tau_i)$  can be regarded as known data for  $t \in [0, \tau_{\min})$ . Then owing to (3.2) the nonlinear part satisfies

$$\begin{aligned} \|g(t_1, U_1) - g(t_2, U_2)\|_{\mathcal{H}} &= \left\| \sum_{i=1}^I [F_i(U_1, U_0(t_1 - \tau_i + \tau)) - F_i(U_2, U_0(t_2 - \tau_i + \tau))] \right\|_{\mathcal{H}} \\ &\leq \sum_{i=1}^I \|F_i(U_1, U_0(t_1 - \tau_i + \tau)) - F_i(U_2, U_0(t_2 - \tau_i + \tau))\|_{\mathcal{H}} \\ &\leq IC_0 \|U_1 - U_2\|_{\mathcal{H}} \\ &\quad + \sum_{i=1}^I h_1(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|U_0(t_1 - \tau_i + \tau)\|_{\mathcal{V}}, \|U_0(t_2 - \tau_i + \tau)\|_{\mathcal{V}}) \|U_1 - U_2\|_{\mathcal{V}} \\ &\quad + \sum_{i=1}^I h_2(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|U_0(t_1 - \tau_i + \tau)\|_{\mathcal{V}}, \|U_0(t_2 - \tau_i + \tau)\|_{\mathcal{V}}) \\ &\quad \cdot \|U_0(t_1 - \tau_i + \tau) - U_0(t_2 - \tau_i + \tau)\|_{D(-A^\beta)}. \end{aligned} \quad (3.4)$$

From (3.4), using (3.1), easily follows

$$\|g(t_1, U_1) - g(t_2, U_2)\|_{\mathcal{H}} \leq L(R)(|t_1 - t_2|^\theta + \|U_1 - U_2\|_{\mathcal{V}}),$$

for all  $(U_1, U_2) \in \mathcal{V} \times \mathcal{V}$  with  $\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}} \leq R$  and for all  $t_1, t_2 \in [0, \tau_{\min}]$ . Therefore, we can apply Theorem 6.3.1 of [25] and deduce the existence of a unique local solution

$U \in C([0, \bar{t}], \mathcal{V}) \cap C^1((0, \bar{t}), \mathcal{H})$  defined in a time interval  $[0, \bar{t}]$  with  $\bar{t} \leq \tau_{\min}$ . If  $\bar{t} = \tau_{\min}$ , and  $\|U(\tau_{\min})\|_{\mathcal{V}} < +\infty$ , then one can extend the solution  $U$  for times  $t > \tau_{\min}$ , by considering on the time interval  $(\tau_{\min}, 2\tau_{\min})$  the problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g_1(t, U(t)) & \text{in } (\tau_{\min}, 2\tau_{\min}) \\ U(0) = U(\tau_{\min}^-), \end{cases} \quad (3.5)$$

where  $g_1(t, U(t)) = \sum_{i=1}^I F_i(U(t), U(t - \tau_i))$ . Note that, since we know the solution  $U(t)$  for  $t \in [0, \tau_{\min}]$  from the first step, the terms  $U(t - \tau_i)$  can be regarded as known data for  $t \in [\tau_{\min}, 2\tau_{\min})$ .

Observe also that

$$U \in C^{0,\theta}([0, \tau_{\min}], D(-\mathcal{A})^\beta). \quad (3.6)$$

Indeed from the proof of Theorem 6.3.1 of [25], we see that

$$U(t) = (-\mathcal{A})^{-\frac{1}{2}}y(t),$$

with  $y \in C([0, \tau_{\min}], \mathcal{H})$  given by

$$y(t) = S(t)(-\mathcal{A})^{\frac{1}{2}}U^0 + \int_0^t (-\mathcal{A})^{\frac{1}{2}}S(t-s)g(s, (-\mathcal{A})^{-\frac{1}{2}}y(s)) ds, \forall t \in [0, \tau_{\min}].$$

Hence for any  $t \in [0, \tau_{\min})$  and  $h > 0$  such that  $t + h \leq \tau_{\min}$ , we have

$$y(t+h) - y(t) = (S(h) - I)S(t)(-\mathcal{A})^{\frac{1}{2}}U^0 + R_2(t, h) + R_3(t, h),$$

where

$$\begin{aligned} R_2(t, h) &= \int_0^t (S(h) - I)(-\mathcal{A})^{\frac{1}{2}}S(t-s)g(s, (-\mathcal{A})^{-\frac{1}{2}}y(s)) ds \\ R_3(t, h) &= \int_t^{t+h} (-\mathcal{A})^{\frac{1}{2}}S(t+h-s)g(s, (-\mathcal{A})^{-\frac{1}{2}}y(s)) ds. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} (-\mathcal{A})^\beta(U(t+h) - U(t)) &= (-\mathcal{A})^{\beta-\frac{1}{2}}(y(t+h) - y(t)) \\ &= (S(h) - I)(-\mathcal{A})^\beta S(t)U^0 + (-\mathcal{A})^{\beta-\frac{1}{2}}(R_2(t, h) + R_3(t, h)), \end{aligned}$$

and since  $(-\mathcal{A})^{\beta-\frac{1}{2}}$  is a bounded operator from  $\mathcal{H}$  into itself (see [25, Lemma 2.6.3]), one gets

$$\|(-\mathcal{A})^\beta(U(t+h) - U(t))\|_{\mathcal{H}} \lesssim \|(S(h) - I)(-\mathcal{A})^\beta S(t)U^0\|_{\mathcal{H}} + \|R_2(t, h)\|_{\mathcal{H}} + \|R_3(t, h)\|_{\mathcal{H}}. \quad (3.7)$$

The estimates (6.3.15) and (6.3.16) of [25] give

$$\|R_2(t, h)\|_{\mathcal{H}} + \|R_3(t, h)\|_{\mathcal{H}} \lesssim h^\beta \lesssim h^\theta, \quad (3.8)$$

hence to get the Hölder continuity (3.6), it remains to show that

$$\|(S(h) - I)(-\mathcal{A})^\beta S(t)U^0\|_{\mathcal{H}} \lesssim h^\theta. \quad (3.9)$$

But we first notice that the estimates (2.6.26) of [25] yields

$$\|(S(h) - I)(-\mathcal{A})^\beta S(t)U^0\|_{\mathcal{H}} \lesssim h^{\frac{1}{2}-\beta} \|(-\mathcal{A})^{\frac{1}{2}} S(t)U^0\|_{\mathcal{H}},$$

and since  $(-\mathcal{A})^{\frac{1}{2}} S(t) = S(t)(-\mathcal{A})^{\frac{1}{2}}$  and the semigroup is of contractions, we get

$$\|(S(h) - I)(-\mathcal{A})^\beta S(t)U^0\|_{\mathcal{H}} \lesssim h^{\frac{1}{2}-\beta} \|(-\mathcal{A})^{\frac{1}{2}} U^0\|_{\mathcal{H}} \lesssim h^{\frac{1}{2}-\beta} \|U^0\|_{\mathcal{V}},$$

which proves (3.9).

Then, as before, one can apply Theorem 6.3.1 of [25] extending the previously found solution. One can eventually iterate this procedure by obtaining a solution

$$U \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$$

of problem (1.2), satisfying  $\lim_{t \rightarrow T_\infty^-} \|U(t)\|_{\mathcal{V}} = +\infty$  if  $T_\infty < +\infty$ . ■

We now give an exponential stability result for *small* initial data. Note that we do not require here any restriction on the size of the time delays. For that purpose, we need the additional assumption on our nonlinear functions, namely we suppose that there exist a positive constant  $C_1$  and a continuous function  $h_3$  from  $[0, \infty)^{I+1}$  to  $[0, \infty)$  satisfying  $h_3(0) = 0$  and such that

$$\left| \sum_{i=1}^I (W, F_i(U, V_i))_{\mathcal{H}} \right| \leq \|W\|_{\mathcal{H}} (C_1 \|U\|_{\mathcal{H}} + h_3(\|U\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \dots, \|V_I\|_{\mathcal{V}}) \|U\|_{\mathcal{V}}), \quad (3.10)$$

for all  $W \in \mathcal{H}$ ,  $U, V_i \in \mathcal{V}$ ,  $i = 1, \dots, I$ .

**Theorem 3.2** *Assume that (3.2) and (3.10) are satisfied. With  $C_1$  from the assumption (3.10), we assume that*

$$\frac{C_1}{\lambda_1} - 1 < 0. \quad (3.11)$$

*Then there exist  $K_0 > 0$  small enough and  $\gamma_0 < 1$  (depending on  $K_0$ ) such that for all  $K \in (0, K_0]$  and  $U_0 \in C([0, \tau]; \mathcal{V})$  satisfying*

$$\|U_0(t)\|_{\mathcal{V}} < \gamma_0 K, \forall t \in [-\tau, 0], \quad (3.12)$$

*problem (1.2) has a global solution  $U$  that satisfies the exponential decay estimate*

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\tilde{\omega}t} \quad \forall t \geq 0, \quad (3.13)$$

*for a positive constant  $M$  depending on  $U_0$  and a suitable positive constant  $\tilde{\omega}$ .*

**Proof.** By Proposition 3.1, there exists  $T_\infty > 0$  such that problem (1.2) has a unique solution  $U \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$ . As in [17], for  $K \in (0, K_0]$  with  $K_0$  fixed later on, we look at

$$T_0 = \sup\{\delta \in (0, \infty) : \|U(t)\|_{\mathcal{V}} \leq K, \forall t \in (0, \delta)\}. \quad (3.14)$$

Our assumption (3.12) clearly guarantees that  $T_0 > 0$ . We will now show that  $T_0 = +\infty$  by a contradiction argument. Indeed if we assume that  $T_0$  is finite, then by its definition, we will have

$$\|U(t)\|_{\mathcal{V}} < K, \forall t \in [-\tau, T_0), \quad (3.15)$$

and

$$\|U(T_0)\|_{\mathcal{V}} = K. \quad (3.16)$$

In a first step, for  $t \in [0, T_0)$ , we estimate  $\frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2$ . Indeed by (1.2), for all  $t \in [0, T_0)$ , we have

$$\begin{aligned} \frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2 &= 2(U(t), U_t(t))_{\mathcal{H}} \\ &= 2(U(t), \mathcal{A}U(t) + \sum_{i=1}^I F_i(U(t), U(t - \tau_i)))_{\mathcal{H}} \\ &= -2\|U(t)\|_{\mathcal{V}}^2 + 2\sum_{i=1}^I (U(t), F_i(U(t), U(t - \tau_i)))_{\mathcal{H}}. \end{aligned}$$

Hence using the assumption (3.10), we get

$$\begin{aligned} \frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2 &\leq -2\|U(t)\|_{\mathcal{V}}^2 + 2C_1\|U(t)\|_{\mathcal{H}}^2 \\ &\quad + 2h_3(\|U(t)\|_{\mathcal{V}}, \|U(t - \tau_1)\|_{\mathcal{V}}, \dots, \|U(t - \tau_I)\|_{\mathcal{V}})\|U(t)\|_{\mathcal{H}}\|U(t)\|_{\mathcal{V}}. \end{aligned}$$

Therefore by (3.15), we deduce that

$$\begin{aligned} \frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2 &\leq -2\|U(t)\|_{\mathcal{V}}^2 + 2C_1\|U(t)\|_{\mathcal{H}}^2 \\ &\quad + 2C_4(K)\|U(t)\|_{\mathcal{H}}\|U(t)\|_{\mathcal{V}}, \end{aligned}$$

where  $C_4(K) = \max_{0 \leq y, z_i \leq K} h_3(y, z_1, \dots, z_I)$  is a constant that depends on  $K$  and is a non-decreasing function of  $K$ . Using (3.1), we arrive at

$$\frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2 \leq 2\left(-1 + \frac{C_1}{\lambda_1} + \frac{C_4(K)}{\sqrt{\lambda_1}}\right)\|U(t)\|_{\mathcal{V}}^2.$$

From our assumption (3.11), we can choose  $K_0$  small enough such that

$$-\omega = \left(-1 + \frac{C_1}{\lambda_1} + C_4(K_0)\right)$$

is negative. Hence for all  $K \in (0, K_0]$ , the previous estimate implies that

$$\frac{d}{dt} \|U(t)\|_{\mathcal{H}}^2 \leq -2\omega \|U(t)\|_{\mathcal{V}}^2 \leq -2\omega\lambda_1 \|U(t)\|_{\mathcal{H}}^2, \quad \forall t \in [0, T_0]. \quad (3.17)$$

This estimate obviously implies that

$$\|U(t)\|_{\mathcal{H}}^2 \leq e^{-2\omega\lambda_1 t} \|U(0)\|_{\mathcal{H}}^2, \quad \forall t \in [0, T_0]. \quad (3.18)$$

But the first estimate of (3.17) means that

$$\frac{d}{dt} \|U(t)\|_{\mathcal{H}}^2 + 2\omega \|U(t)\|_{\mathcal{V}}^2 \leq 0,$$

and if we multiply this estimate by  $e^{\omega' t}$  for some  $\omega' \in (0, \omega]$  fixed later on, we find that

$$\frac{d}{dt} (e^{\omega' t} \|U(t)\|_{\mathcal{H}}^2) + 2\omega' e^{\omega' t} \|U(t)\|_{\mathcal{V}}^2 \leq \omega' e^{\omega' t} \|U(t)\|_{\mathcal{H}}^2,$$

and using (3.18), we get

$$\frac{d}{dt} (e^{\omega' t} \|U(t)\|_{\mathcal{H}}^2) + 2\omega' e^{\omega' t} \|U(t)\|_{\mathcal{V}}^2 \leq \omega e^{(\omega' - 2\lambda_1 \omega)t} \|U(0)\|_{\mathcal{H}}^2. \quad (3.19)$$

Hence we fix  $\omega' \in (0, \omega]$  such that  $\omega' < 2\lambda_1 \omega$  (possible since  $\lambda_1$  and  $\omega$  are positive), and then integrate (3.19) between 0 and  $T_0$  to find

$$\begin{aligned} e^{\omega' T_0} \|U(T_0)\|_{\mathcal{H}}^2 + 2\omega' \int_0^{T_0} e^{\omega' t} \|U(t)\|_{\mathcal{V}}^2 dt &\leq \left(1 + \frac{\omega}{2\lambda_1 \omega - \omega'} (1 - e^{(\omega' - 2\lambda_1 \omega) T_0})\right) \|U(0)\|_{\mathcal{H}}^2 \\ &\leq \left(1 + \frac{\omega}{2\lambda_1 \omega - \omega'}\right) \|U(0)\|_{\mathcal{H}}^2. \end{aligned}$$

This clearly implies that there exists a positive constant  $C(\omega, \lambda_1)$  depending only on  $\omega$  and  $\lambda_1$  such that

$$\int_0^{T_0} e^{\omega' t} \|U(t)\|_{\mathcal{V}}^2 dt \leq C(\omega, \lambda_1) \|U(0)\|_{\mathcal{V}}^2. \quad (3.20)$$

We now estimate  $\frac{d}{dt} \|U(t)\|_{\mathcal{V}}^2$ . First we notice that

$$\begin{aligned} \frac{d}{dt} \|U(t)\|_{\mathcal{V}}^2 &= 2((-A)^{\frac{1}{2}} U, (-A)^{\frac{1}{2}} U_t)_{\mathcal{H}} \\ &= -2(\mathcal{A}U, U_t)_{\mathcal{H}} \\ &= -2(U_t(t) - \sum_{i=1}^I F_i(U(t), U(t - \tau_i)), U_t)_{\mathcal{H}} \\ &= -2\|U_t(t)\|_{\mathcal{H}}^2 + 2 \sum_{i=1}^I (F_i(U(t), U(t - \tau_i)), U_t)_{\mathcal{H}}. \end{aligned}$$

By the assumption (3.10), we get

$$\frac{d}{dt}\|U(t)\|_{\mathcal{V}}^2 \leq -2\|U_t(t)\|_{\mathcal{H}}^2 + 2\|U_t(t)\|_{\mathcal{H}}(C_1\|U(t)\|_{\mathcal{H}} + C_4(K)\|U(t)\|_{\mathcal{V}}).$$

Young's inequality and (3.1) lead to

$$\frac{d}{dt}\|U(t)\|_{\mathcal{V}}^2 \leq \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K)\right)^2\|U(t)\|_{\mathcal{V}}^2. \quad (3.21)$$

Now we proceed as in the proof of Lemma 2.1 of [17], namely we multiply this estimate by  $e^{\omega't}$  to get

$$\frac{d}{dt}(e^{\omega't}\|U(t)\|_{\mathcal{V}}^2) \leq (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K)\right)^2)e^{\omega't}\|U(t)\|_{\mathcal{V}}^2.$$

Integrating this estimate between 0 and  $t \in (0, T_0]$ , one obtains

$$e^{\omega't}\|U(t)\|_{\mathcal{V}}^2 \leq \|U(0)\|_{\mathcal{V}}^2 + (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K)\right)^2) \int_0^t e^{\omega's}\|U(s)\|_{\mathcal{V}}^2 ds,$$

and therefore owing to (3.20), one finally finds

$$\|U(t)\|_{\mathcal{V}}^2 \leq \left(1 + (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K)\right)^2)C(\omega, \lambda_1)\right) \|U(0)\|_{\mathcal{V}}^2 e^{-\omega't}, \quad \forall t \in (0, T_0]. \quad (3.22)$$

Therefore if we define  $\gamma_0 > 0$  by

$$\gamma_0^{-2} = 4 \left(1 + (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K_0)\right)^2)C(\omega, \lambda_1)\right),$$

we clearly see that  $\gamma_0 < 1$ . Furthermore for any  $U_0$  satisfying (3.12), and reminding that  $0 < K \leq K_0$ , we have

$$\begin{aligned} & \left(1 + (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K)\right)^2)C(\omega, \lambda_1)\right) \|U(0)\|_{\mathcal{V}}^2 \\ & \leq \left((1 + (\omega' + \left(\frac{C_1}{\sqrt{\lambda_1}} + C_4(K_0)\right)^2)C(\omega, \lambda_1))\right) \gamma_0^2 K^2 = \frac{1}{4}K^2. \end{aligned}$$

Consequently, the estimate (3.22) guarantees that

$$\|U(t)\|_{\mathcal{V}} \leq \frac{K}{2} e^{-\frac{\omega't}{2}}, \quad \forall t \in (0, T_0]. \quad (3.23)$$

In particular, on gets

$$\|U(T_0)\|_{\mathcal{V}} \leq \frac{K}{2},$$

that clearly contradicts (3.16). This means that  $T_0$  is infinite and we conclude by the estimate (3.23). ■



## 4 Examples with general nonlinearities

In the whole section,  $\Omega$  denotes an arbitrary bounded domain of  $\mathbb{R}^d$ ,  $d \geq 1$ , with a Lipschitz boundary  $\Gamma$ .

### 4.1 Delay-diffusion equations

We consider the semilinear diffusion equation with time delay

$$u_t(t) - \Delta u(t) = f(u(t), u(t - \tau)) \quad \text{in } \Omega \times (0, +\infty), \quad (4.1)$$

$$Bu(x, t) = 0 \quad \text{on } \Gamma \times (0, +\infty), \quad (4.2)$$

$$u(x, t) = u_0(x, t) \quad \text{in } \Omega \times [-\tau, 0], \quad (4.3)$$

where the constant  $\tau > 0$  is the time delay and the initial datum  $u_0$  belongs to the space  $C([0, \tau]; L^2(\Omega))$ . The operator  $B$  is in the form

$$Bu = \alpha \partial_n u + \alpha' u,$$

with either  $\alpha = 0$  and  $\alpha' = 1$  corresponding to the case of Dirichlet boundary conditions or  $\alpha = 1$  and  $\alpha' \geq 0$  (with  $\alpha' \in L^\infty(\partial\Omega)$ ) corresponding to the case of Neumann–Robin boundary conditions.

Analogous problems have been considered by Friesecke in [9], where an exponential stability result is obtained for small time delay, under some growth conditions on the locally Lipschitz function  $f$ , by using a Lyapunov functional approach. We may also quote the paper of Pao [23] where a coupled system of parabolic semilinear equations with delays is studied. Using the method of upper and lower solutions to investigate existence and asymptotic behavior, sufficient conditions for stability and instability are given, under some monotonicity properties on the reaction term  $f$ , independent of the time delays. In the same spirit, in [8] the global asymptotic behavior of some quasimonotone reaction–diffusion system with delays is analyzed. A trichotomy of the global dynamics is established via linearization.

For further uses, in the case of Neumann conditions ( $\alpha = 1, \alpha' = 0$ ), we fix a positive real parameter  $\varepsilon$  (that may depend on  $f$ ), otherwise we set  $\varepsilon = 0$ . Now we assume that the nonlinearity  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the following assumption: there exist a non-negative constant  $\alpha_0$  (that may depend on  $\varepsilon$ ) and two polynomials  $P_1$  and  $P_2$  (of one real variable) of degree  $n_1$  and  $n_2$  in the form

$$P_i(X) = \sum_{j=1}^{n_i} \alpha_{i,j} X^j,$$

with non negative real numbers  $\alpha_{i,j}$  such that

$$\begin{aligned} |f(x_1, y_1) - f(x_2, y_2) + \varepsilon(x_1 - x_2)| \leq & \alpha_0 |x_1 - x_2| + P_1(|x_1| + |y_1| + |x_2| + |y_2|) |x_1 - x_2| \\ & + P_2(|x_1| + |y_1| + |x_2| + |y_2|) |y_1 - y_2|, \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2. \end{aligned} \quad (4.4)$$

In particular this assumption means that  $f$  is only locally Lipschitz.

Under this assumption, let us now show that problem (4.1)–(4.3) enters in the framework of section 3. Indeed in such a situation, we take  $\mathcal{H} = L^2(\Omega)$  and define  $\mathcal{A}$  as follows:

$$D(\mathcal{A}) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \text{ and satisfying } Bu = 0 \text{ on } \Gamma\},$$

and

$$\mathcal{A}u = \Delta u - \varepsilon u, \quad \forall u \in D(\mathcal{A}).$$

Note that by Lemmas 1.5.3.7 and 1.5.3.9 of [11], for any  $u \in \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ ,  $Bu$  has a meaning (as element of  $H^{-\frac{1}{2}}(\Gamma)$  if  $\alpha > 0$ ). It is not difficult to show that  $-\mathcal{A}$  is a positive selfadjoint operator in  $\mathcal{H}$  since it is the Friedrichs extension of the triple  $(\mathcal{H}, \mathcal{V}, a)$  where  $\mathcal{V} = H_0^1(\Omega)$  in case of Dirichlet boundary conditions otherwise  $\mathcal{V} = H^1(\Omega)$ , and the sesquilinear form  $a$  is defined by

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} + \varepsilon u \cdot \bar{v}) dx + \int_{\Gamma} \alpha' u \cdot \bar{v} d\sigma(x),$$

that is symmetric, continuous and strongly coercive on  $\mathcal{V}$ . Before going on, let us notice that in the case of Neumann boundary conditions, the smallest eigenvalue  $\lambda_1$  of  $-\mathcal{A}$  is  $\varepsilon$ , otherwise it does not depend on  $\varepsilon$  and corresponds to the smallest eigenvalue of the Dirichlet problem or to the Robin eigenvalue problem.

By introducing the function

$$F_1(x, y) = f(x, y) + \varepsilon x,$$

we see that problem (4.1)–(4.3) can be written as (1.2) with  $I = 1$ ,  $U(t) = u(\cdot, t)$  and  $U_0(t) = u_0(\cdot, t)$ . Consequently the next local existence result follows from Proposition 3.1.

**Proposition 4.1** *Assume that (4.4) holds and that  $d$  satisfies*

$$d \leq 2\left(1 + \frac{1}{n_1}\right) \quad \text{and} \quad d < 2\left(1 + \frac{1}{n_2}\right). \quad (4.5)$$

*Then there exists  $\beta \in (0, \frac{1}{2})$  such that for any initial datum*

$$u_0 \in C([0, \tau]; \mathcal{V}) \cap C^{0, \theta}([0, \tau], D((-\mathcal{A})^\beta)),$$

*with  $\theta = \min\{\beta, \frac{1}{2} - \beta\}$ , there exist a time  $T_\infty \in (0, +\infty]$  and a unique solution  $u \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$  of problem (4.1) – (4.3).*

**Proof.** It suffices to check that  $F_1$  defined above satisfies (3.2). For that purpose, we see that (4.4) implies that (for shortness we write  $F$  instead of  $F_1$ )

$$\begin{aligned} |F(x_1, y_1) - F(x_2, y_2)| &\leq \alpha_0 |x_1 - x_2| + P_1(|x_1| + |y_1| + |x_2| + |y_2|) |x_1 - x_2| \\ &\quad + P_2(|x_1| + |y_1| + |x_2| + |y_2|) |y_1 - y_2|, \quad \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2. \end{aligned}$$

and consequently for any  $u_i, v_i \in \mathcal{V}$ ,

$$\begin{aligned} \|F(u_1, v_1) - F(u_2, v_2)\|_{\mathcal{H}} &\leq \alpha_0 \|u_1 - u_2\|_{\mathcal{H}} + \|P_1(|u_1| + |v_1| + |u_2| + |v_2|) |u_1 - u_2|\|_{\mathcal{H}} \\ &\quad + \|P_2(|u_1| + |v_1| + |u_2| + |v_2|) |v_1 - v_2|\|_{\mathcal{H}}. \end{aligned}$$

Hence by the form of  $P_i$  and Hölder's inequality, we arrive at

$$\begin{aligned} \|F(u_1, v_1) - F(u_2, v_2)\|_{\mathcal{H}} &\leq \alpha_0 \|u_1 - u_2\|_{\mathcal{H}} \\ &+ C \sum_{j=1}^{n_1} \|(|u_1| + |v_1| + |u_2| + |v_2|)^j\|_{L^{q_1}(\Omega)} \|u_1 - u_2\|_{L^{p_1}(\Omega)} \\ &+ C \sum_{j=1}^{n_2} \|(|u_1| + |v_1| + |u_2| + |v_2|)^j\|_{L^{q_2}(\Omega)} \|v_1 - v_2\|_{L^{p_2}(\Omega)}, \end{aligned}$$

for some  $C > 0$  and  $p_i, q_i > 2$  chosen below and such that

$$\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}, \quad i = 1, 2.$$

Now as  $D((-\mathcal{A})^{\frac{1}{2}}) \hookrightarrow H^1(\Omega)$ <sup>1</sup> and  $D((-\mathcal{A})^0) = L^2(\Omega)$ , by interpolation we get that

$$D((-\mathcal{A})^\beta) \hookrightarrow H^{2\beta}(\Omega), \quad \forall \beta \in [0, \frac{1}{2}].$$

Then we notice that the Sobolev embedding theorem guarantees that  $H^1(\Omega) \hookrightarrow L^{p_1}(\Omega)$  and  $H^{2\beta}(\Omega) \hookrightarrow L^{p_2}(\Omega)$  as soon as

$$1 - \frac{d}{2} \geq -\frac{d}{p_1} \quad \text{and} \quad 2\beta - \frac{d}{2} \geq -\frac{d}{p_2}.$$

Since  $\beta$  can be taken as close to  $\frac{1}{2}$  as we want, we get equivalently

$$1 - \frac{d}{2} \geq -\frac{d}{p_1} \quad \text{and} \quad 1 - \frac{d}{2} > -\frac{d}{p_2}. \quad (4.6)$$

In a second step we now need to estimate terms like

$$\|(|u_1| + |v_1| + |u_2| + |v_2|)^j\|_{L^{q_i}(\Omega)}$$

for all  $1 \leq j \leq n_i$ . But clearly we have

$$\|(|u_1| + |v_1| + |u_2| + |v_2|)^j\|_{L^{q_i}(\Omega)} \leq C_2 (\|u_1\|_{L^{jq_i}(\Omega)}^j + \|v_1\|_{L^{jq_i}(\Omega)}^j + \|u_2\|_{L^{jq_i}(\Omega)}^j + \|v_2\|_{L^{jq_i}(\Omega)}^j),$$

for some  $C_2 > 0$  (that depends on  $j$ ). Therefore our last assumptions are

$$1 - \frac{d}{2} \geq -\frac{d}{jq_i}, \quad \forall 1 \leq j \leq n_i,$$

or equivalently (since  $p_i$  and  $q_i$  are conjugates)

$$1 - \frac{d}{2} \geq \frac{d}{j} \left( \frac{1}{p_i} - \frac{1}{2} \right), \quad \forall 1 \leq j \leq n_i. \quad (4.7)$$

---

<sup>1</sup>by  $X \hookrightarrow Y$ , we mean that  $X$  is continuously embedded into  $Y$

This condition and (4.6) guarantee that

$$\begin{aligned} \|F(u_1, v_1) - F(u_2, v_2)\|_{\mathcal{H}} &\leq \alpha_0 \|u_1 - u_2\|_{\mathcal{H}} + C \sum_{j=1}^{n_1} (\|u_1\|_{\mathcal{V}}^j + \|v_1\|_{\mathcal{V}}^j + \|u_2\|_{\mathcal{V}}^j) \|u_1 - u_2\|_{\mathcal{V}} \\ &+ C \sum_{j=1}^{n_2} (\|u_1\|_{\mathcal{V}}^j + \|v_1\|_{\mathcal{V}}^j + \|u_2\|_{\mathcal{V}}^j) \|v_1 - v_2\|_{D((-A)^\beta)}, \end{aligned}$$

for some  $C > 0$  and some  $\beta \in (0, \frac{1}{2})$ , which yields (3.2) where  $C_0 = \alpha_0$  and  $h_i$  are polynomials with positive coefficients.

Finally it is an easy exercise to check that conditions (4.5) are equivalent to the existence of  $p_1 > 2$  and  $p_2 > 2$  satisfying (4.6) and (4.7). ■

Concerning global existence and exponential decay, owing to Theorem 3.2, we can state the next result.

**Theorem 4.2** *Assume that conditions (4.4), (4.5) as well as*

$$f(0, y) = 0, \quad \forall y \in \mathbb{R}, \quad (4.8)$$

*are satisfied. With  $\alpha_0$  from the assumption (4.4), we assume that*

$$\alpha_0 < \varepsilon, \quad (4.9)$$

*in the case of Neumann boundary conditions, and*

$$\alpha_0 < \lambda_1, \quad (4.10)$$

*otherwise. Then there exist  $K_0 > 0$  small enough and  $\gamma_0 < 1$  (depending on  $K_0$ ) such that for all  $K \in (0, K_0]$  and  $u_0 \in C([0, \tau]; \mathcal{V})$  satisfying*

$$\|u_0(t)\|_{\mathcal{V}} < \gamma_0 K, \quad \forall t \in [-\tau, 0], \quad (4.11)$$

*problem (4.1) – (4.3) has a global solution  $u$  that satisfies the exponential decay estimate*

$$\|u(t)\|_{\mathcal{H}} \leq M e^{-\tilde{\omega}t} \quad \forall t \geq 0, \quad (4.12)$$

*for a positive constant  $M$  depending on  $u_0$  and a suitable positive constant  $\tilde{\omega}$ .*

**Proof.** It suffices to check that (3.10) and (3.11) hold. The first condition directly follows from Cauchy-Schwarz's inequality and the assumptions (4.8) and (4.4) where  $C_1 = \alpha_0$ . Then the second condition (3.11) simply becomes either (4.9) in the Neumann case or (4.10) in the two other cases. ■

Our general setting covers a very large number of concrete examples. Let us mention the following cases:

1. **Diffusive logistic equations with delay.** In that case,  $f$  is given by

$$f(x, y) = ax - bx^2 + cxy,$$

with  $a, b, c \in L^\infty(\Omega)$ . This example covers the Hutchinson equation by taking  $a = \alpha \in \mathbb{R}$ ,  $c = -\alpha$  and  $b = 0$ . In such a case, the condition (4.4) holds with  $\alpha_0 = \sup_\Omega |a + \varepsilon|$ , and  $n_1 = n_2 = 1$ . Hence local existence (i.e. Theorem 4.1) holds for any  $d \leq 3$ , while exponential decay for sufficiently small initial data (i.e. Theorem 4.2) holds under the additional assumption that

$$\sup_\Omega a < 0, \tag{4.13}$$

in the case of Neumann boundary conditions (by choosing  $\varepsilon > 0$  large enough), and

$$\sup_\Omega |a| < \lambda_1, \tag{4.14}$$

in the other cases.

**2. The modified Hutchinson equation.** In that case,  $f$  is given by (see [18, 9])

$$f(x, y) = \alpha x(1 + \beta y + \gamma y^2 + \delta y^3),$$

with  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ . In such a case, the condition (4.4) holds with  $\alpha_0 = |\alpha + \varepsilon|$ , and  $n_1 = n_2 = 3$ . Hence local existence (i.e. Theorem 4.1) holds for any  $d \leq 2$ , while exponential decay for sufficiently small initial data (i.e. Theorem 4.2) holds under the additional assumption  $\alpha < 0$  in the case of Neumann boundary conditions, and  $-\lambda_1 < \alpha < \lambda_1$  in the other cases.

**2. A cubic nonlinearity.** The case where  $f$  is given by

$$f(x, y) = -x^2 y$$

considered in [13, 16, 9] is also covered by our setting in the case of Dirichlet or Robin boundary conditions, since (4.4) holds with  $\alpha_0 = 0$  and  $n_1 = n_2 = 2$ . Therefore local existence and exponential decay for sufficiently small initial data hold for any  $d \leq 2$ .

**Remark 4.3** If  $f$  is globally Lipschitz, i.e.,  $P_1 = 0$  and  $P_2$  is a positive constant, we can alternatively use the theory from section 2 and obtain exponential decay for small delays.

## 4.2 Predator-prey or two species competition systems with delays

Here we consider the semilinear diffusion system with time delays

$$u_{1,t}(x, t) = d_1 \Delta u_1(x, t) + u_1(x, t) \left( a_1 + a_{11} u_1(x, t) + \sum_{j=1}^2 a'_{1j} u_j(x, t - \tau_{1j}) \right) \quad \text{in } \Omega \times (0, +\infty), \tag{4.15}$$

$$u_{2,t}(x, t) = d_2 \Delta u_2(x, t) + u_2(x, t) \left( a_2 + a_{22} u_2(x, t) + \sum_{j=1}^2 a'_{2j} u_j(x, t - \tau_{2j}) \right) \quad \text{in } \Omega \times (0, +\infty), \tag{4.16}$$

$$B_1 u_1(x, t) = B_2 u_2(x, t) = 0 \quad \text{on } \Gamma \times (0, +\infty), \tag{4.17}$$

$$(u_1(x, t), u_2(x, t)) = (u_{0,1}(x, t), u_{0,2}(x, t)) \quad \text{in } \Omega \times [-\tau, 0], \tag{4.18}$$

where the constants  $\tau_{ij} > 0$  are the time delays,  $\tau = \max \tau_{ij}$  and the initial datum  $(u_{0,1}, u_{0,2})^\top$  belongs to  $C([0, \tau]; L^2(\Omega)^2)$ . The operator  $B_i$  are in the form

$$B_i u = \alpha_i \partial_n u_i + \beta_i u_i,$$

with either  $\alpha_i = 0$  and  $\beta_i = 1$  corresponding to the case of Dirichlet boundary conditions or  $\alpha_i = 1$  and  $\beta_i \geq 0$  (with  $\beta_i \in L^\infty(\partial\Omega)$ ) corresponding to the case of Neumann-Robin boundary conditions. Here  $d_i$  are positive constants, while  $a_i, a_{ij}$  and  $a'_{ij}$  are simply functions in  $L^\infty(\Omega)$ .

Systems for competitive species are studied in [23] under some monotonicity properties on the nonlinear functions. Conditions for stability or instability of solutions are given independently of the time delays. Two species prey-predator models and competition system with delays are also analyzed in [27]. By using the infinite-dimensional dissipative system theory and comparison arguments, persistence criteria and also global extinction criteria are established.

System (4.15)-(4.18) enters in the framework of section 3. Indeed in such a situation, we take  $\mathcal{H} = L^2(\Omega)^2$  and define  $\mathcal{A}$  as follows:

$$D(\mathcal{A}) := \{u = (u_1, u_2)^\top \in H^1(\Omega)^2 : \Delta u_i \in L^2(\Omega) \text{ and satisfying } B_i u_i = 0 \text{ on } \Gamma, i = 1, 2\},$$

and

$$\mathcal{A}u = (d_1 \Delta u_1 - \varepsilon_1 u_1, d_2 \Delta u_2 - \varepsilon_2 u_2)^\top, \quad \forall u \in D(\mathcal{A}),$$

where  $\varepsilon_i > 0$  if Neumann condition is imposed on  $u_i$  (fixed later on), and  $\varepsilon_i = 0$  otherwise. As before,  $-\mathcal{A}$  is a positive selfadjoint operator in  $\mathcal{H}$  with

$$\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}}) = H_0^1(\Omega)^2,$$

if Dirichlet boundary conditions are imposed on  $u_1$  and  $u_2$ ,

$$\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}}) = H_0^1(\Omega) \times H^1(\Omega)$$

if Dirichlet boundary condition is imposed on  $u_1$  and Neumann or Robin type on  $u_2$ ,

$$\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}}) = H^1(\Omega) \times H_0^1(\Omega)$$

if Dirichlet boundary condition is imposed on  $u_2$  and Neumann or Robin type on  $u_1$ , and finally

$$\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}}) = H^1(\Omega)^2,$$

otherwise.

We now distinguish four different cases:

**Case 1:** when Neumann boundary conditions are imposed on both  $u_1$  and  $u_2$ , then we take  $\varepsilon_1 = \varepsilon_2 = \varepsilon$  and the smallest eigenvalue  $\lambda_1$  of  $-\mathcal{A}$  is  $\varepsilon$ .

**Case 2:** when Neumann boundary condition is imposed on  $u_2$  and Dirichlet or Robin type on  $u_1$ , then the smallest eigenvalue  $\lambda_1$  of  $-\mathcal{A}$  is equal to  $\min\{\mu_1, \varepsilon_2\}$ , where  $\mu_1 > 0$  corresponds to the smallest eigenvalue of  $-d_1 \Delta$  with Dirichlet or Robin boundary conditions.

**Case 3:** when Neumann boundary condition is imposed on  $u_1$  and Dirichlet or Robin type on  $u_2$ , then the smallest eigenvalue  $\lambda_1$  of  $-\mathcal{A}$  is equal to  $\min\{\mu_2, \varepsilon_1\}$ , where  $\mu_2 > 0$  corresponds to the smallest eigenvalue of  $-d_2\Delta$  with Dirichlet or Robin boundary conditions.

**Case 4:** when Dirichlet or Robin boundary conditions are imposed on both  $u_1$  and  $u_2$ , then the smallest eigenvalue  $\lambda_1$  of  $-\mathcal{A}$  is equal to  $\min\{\mu_1, \mu_2\}$ .

Without loss of generality we can suppose that  $\tau = \max\{\tau_{11}, \tau_{12}\}$  and therefore we set

$$F_1(t, u, v) = ((a_1 + \varepsilon_1)u_1 + a_{11}u_1^2 + a'_{11}u_1v_1, 0)^\top,$$

if  $\tau_{11} > \tau_{12}$ , and

$$F_1(t, u, v) = ((a_1 + \varepsilon_1)u_1 + a_{11}u_1^2 + a'_{12}u_1v_2, 0)^\top,$$

if  $\tau_{11} < \tau_{12}$ . Then in the first case, we set  $\tau_2 = \tau_{21}$ ,  $\tau_3 = \tau_{12}$ ,  $\tau_4 = \tau_{22}$  and for all  $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{C}^2$ :

$$\begin{aligned} F_2(t, u, v) &= (0, (a_2 + \varepsilon_2)u_2 + a_{22}u_2^2 + a'_{21}u_2v_1)^\top, \\ F_3(t, u, v) &= (a'_{12}u_1v_2, 0)^\top, \\ F_4(t, u, v) &= (0, a'_{22}u_2v_2)^\top. \end{aligned}$$

The second case is similar by simply changing  $\tau_3$  and  $F_3$  accordingly. With these notations, we see that problem (4.15)-(4.18) can be written as (1.2) with  $I = 2$ ,  $U(t) = (u_1(\cdot, t), u_2(\cdot, t))^\top$  and  $U_0(t) = (u_{0,1}(\cdot, t), u_{0,2}(\cdot, t))^\top$ . Consequently the next local existence result follows from Proposition 3.1.

**Proposition 4.4** *Assume that  $d \leq 3$ , then there exists  $\beta \in (0, \frac{1}{2})$  such that for any initial datum  $(u_{0,1}, u_{0,2})^\top \in C([0, \tau]; \mathcal{V}) \cap C^{0,\theta}([0, \tau], D((-\mathcal{A})^\beta))$ , with  $\theta = \min\{\beta, \frac{1}{2} - \beta\}$ , there exist a time  $T_\infty \in (0, +\infty]$  and a unique solution  $(u_1, u_2)^\top \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$  of problem (4.15) – (4.18).*

**Proof.** It is a direct consequence of Proposition 3.1 since it is easy to check that  $F_i$  satisfies (3.2) with  $n_1 = n_2 = 1$  and  $C_0 = \max_{i=1,2} \sup_\Omega |a_i + \varepsilon_i|$ . ■

As before global existence and exponential decay follow from Theorem 3.2.

**Theorem 4.5** *Assume that  $d \leq 3$  and that*

$$\sup_\Omega |a_i| < \lambda_1, i = 1, 2, \tag{4.19}$$

*in case 1, that*

$$\sup_\Omega |a_1| < \mu_1, \sup_\Omega a_2 < 0, 2 \sup_\Omega a_2 - \inf_\Omega a_2 < 2\mu_1, \tag{4.20}$$

*in case 2, that*

$$\sup_\Omega |a_2| < \mu_2, \sup_\Omega a_1 < 0, 2 \sup_\Omega a_1 - \inf_\Omega a_1 < 2\mu_2, \tag{4.21}$$

*in case 3, and that*

$$\sup_\Omega a_1 < 0, \sup_\Omega a_2 < 0, \tag{4.22}$$

in case 4. Then there exist  $K_0 > 0$  small enough and  $\gamma_0 < 1$  (depending on  $K_0$ ) such that for all  $K \in (0, K_0]$  and  $(u_{0,1}, u_{0,2})^\top \in C([0, \tau]; \mathcal{V})$  satisfying

$$\|(u_{0,1}, u_{0,2})^\top(t)\|_{\mathcal{V}} < \gamma_0 K, \quad \forall t \in [-\tau, 0], \quad (4.23)$$

problem (4.1) – (4.3) has a global solution  $(u_1, u_2)^\top$  that satisfies the exponential decay estimate

$$\|(u_1(\cdot, t), u_2(\cdot, t))^\top\|_{\mathcal{H}} \leq M e^{-\tilde{\omega}t} \quad \forall t \geq 0, \quad (4.24)$$

for a positive constant  $M$  depending on  $u_0$  and a suitable positive constant  $\tilde{\omega}$ .

**Proof.** It suffices to check that (3.10) and (3.11) hold. The first condition directly follows from Cauchy-Schwarz's inequality where  $C_1 = C_0 = \max_{i=1,2} \sup_{\Omega} |a_i + \varepsilon_i|$ . In the first and fourth cases, the second condition (3.11) simply becomes either (4.19) or (4.22). Hence it remains to check that in cases 2 and 3, one can find  $\varepsilon_1$  and  $\varepsilon_2$  appropriately. By symmetry, it suffices to look at case 2. In such a situation, (3.11) becomes

$$\max\left\{\sup_{\Omega} |a_1|, \sup_{\Omega} |a_2 + \varepsilon_2|\right\} < \min\{\mu_1, \varepsilon_2\},$$

that is clearly equivalent to

$$\begin{aligned} \sup_{\Omega} |a_1| < \mu_1 \text{ and } \sup_{\Omega} |a_1| < \varepsilon_2, \\ \sup_{\Omega} |a_2 + \varepsilon_2| < \mu_1 \text{ and } \sup_{\Omega} |a_2 + \varepsilon_2| < \varepsilon_2. \end{aligned}$$

These conditions hold if and only if there exists  $\delta > 0$  small enough such that

$$\begin{aligned} -\mu_1 + \delta \leq a_1 \leq \mu_1 - \delta \text{ and } -\varepsilon_2 + \delta \leq a_1 < \varepsilon_2 - \delta \text{ in } \Omega, \\ -\mu_1 + \delta \leq a_2 + \varepsilon_2 \leq \mu_1 - \delta \text{ and } -\varepsilon_2 + \delta \leq a_2 + \varepsilon_2 \leq \varepsilon_2 - \delta \text{ in } \Omega. \end{aligned}$$

By re-arranging those conditions, we find

$$-\mu_1 + \delta \leq a_1 \leq \mu_1 - \delta \text{ and } a_2 \leq -\delta \text{ in } \Omega, \quad (4.25)$$

$$\max\left\{a_1 + \delta, -a_1 + \delta, \frac{\delta - a_2}{2}, -a_2 - \mu_1 + \delta\right\} \leq \varepsilon_2 \leq \mu_1 - \delta - a_2 \text{ in } \Omega. \quad (4.26)$$

A necessary condition to find such a  $\varepsilon_2$  is that

$$\max\left\{a_1 + \delta, -a_1 + \delta, \frac{\delta - a_2}{2}, -a_2 - \mu_1 + \delta\right\} \leq \mu_1 - \delta - a_2 \text{ in } \Omega,$$

which is indeed true if  $\delta$  is small enough and if (4.25) holds. Therefore we fix

$$\varepsilon_2 = \inf_{\Omega} (\mu_1 - \delta - a_2) = \mu_1 - \sup_{\Omega} a_2 - \delta,$$

with  $\delta > 0$  small enough so that this quantity is positive. We then easily check that (4.26) holds under the assumption (4.25) and the condition

$$3\delta + 2 \sup_{\Omega} a_2 - \inf_{\Omega} a_2 \leq 2\mu_1.$$

Hence this condition and (4.25) hold for  $\delta$  small enough if and only if (4.20) is valid. ■



**Remark 4.6** Clearly, our approach can be used to study a general system of  $N$ -competing species with delays like the system (1.3) of [24] since the nonlinear terms appearing in this system are similar to the ones from system (4.15)-(4.18). We then let the details to the reader.

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