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ON FOUR-DIMENSIONAL EINSTEIN AFFINE HYPERSPHERES

ZEJUN HU, HAIZHONG LI, AND LUC VRANCKEN

ABSTRACT. It is well-known that Vrancken, Li and Simon classified locally strongly convex affine hyperspheres whose affine metric are of constant sectional curvatures. In this paper, four-dimensional affine hyperspheres with affine metric being Einstein are shown to have constant sectional curvatures, and therefore they are locally affinely equivalent to an open part of either one of the hyperquadrics or the hyperbolic affine hypersphere $x_1x_2x_3x_4x_5 = 1$.

1. INTRODUCTION

We denote by \mathbb{R}^{n+1} the real unimodular-affine space equipped with its canonical flat connection D and a parallel volume form ω . Let $M := M^n$ be a differentiable, connected C^∞ -manifold of dimension $n \geq 2$, and let $F : M^n \rightarrow \mathbb{R}^{n+1}$ be a non-degenerate hypersurface immersion with equiaffine (unimodular) normal ξ . We denote by h its affine Blaschke-Berwald metric which is semi-Riemannian, by S the affine shape operator and by ∇ its induced affine connection. Let $\hat{\nabla}$ be the Levi-Civita connection of the affine metric h . The difference tensor K is defined by $K(X, Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$; it is symmetric as both connections are torsion free.

Here in this paper, we will always assume that the hypersurface is locally strongly convex, i.e., the affine metric is definite. In this case, if necessary by changing the sign of the affine normal, we may always assume that the affine metric is positive definite.

The hypersurface is called an affine hypersphere if $S = L_1 \text{id}$. In that case, one easily proves the affine mean curvature $\frac{1}{n} \text{trace } S = L_1 = \text{const}$. More precisely, F is called a proper affine hypersphere if $L_1 \neq 0$; if $L_1 > 0$ (resp. $L_1 < 0$), the proper affine hypersphere is called elliptic (resp. hyperbolic). If $L_1 = 0$, the affine hypersphere is called improper or parabolic. For a proper affine hypersphere the affine normal satisfies $\xi(p) = L_1 (F(p) - c)$, where c is a constant vector, called the center of $F(M^n)$; for simplicity, we choose c as origin. For an improper affine hypersphere the affine normal field is constant.

The affine hyperspheres form a very important class of affine hypersurfaces. From a global point of view locally strongly convex hyperbolic affine hyperspheres have been widely studied, see amongst others the works of [3], [9], [10], [12, 13], [18] or the book of Li, Simon and Zhao [14], also see the recent survey paper [15]. Even assuming global conditions, the class of hyperbolic affine hyperspheres

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is surprisingly large. Even more, locally, in arbitrary dimensions one is still far away from a complete understanding of such hypersurfaces.

Worthwhile to mention from a local point of view are the classification of the affine hyperspheres with constant sectional curvature, see [11, 21] for the locally strongly convex case, or [20] for the general non-degenerate case and the Calabi construction ([2], [4]) of hyperbolic affine hyperspheres which allows to associate with two hyperbolic affine hyperspheres $\psi_1 : M_1^{n_1} \rightarrow \mathbb{R}^{n_1+1}$ and $\psi_2 : M_2^{n_2} \rightarrow \mathbb{R}^{n_2+1}$, two new immersions: φ and $\tilde{\varphi}$: for $p \in M_1, t \in \mathbb{R}$,

$$\varphi(p, t) = (c_1 e^{\frac{t}{\sqrt{n_1+1}}} \psi_1(p), c_2 e^{-\sqrt{n_1+1}t}) \in \mathbb{R}^{n_1+2},$$

and, for $p \in M_1^{n_1}, q \in M_2^{n_2}, t \in \mathbb{R}$,

$$\tilde{\varphi}(p, q, t) = (c_1 e^{\sqrt{\frac{n_2+1}{n_1+1}}t} \psi_1(p), c_2 e^{-\sqrt{\frac{n_1+1}{n_2+1}}t} \psi_2(q)) \in \mathbb{R}^{n_1+n_2+2},$$

which are both again hyperbolic affine hyperspheres. Here, φ and $\tilde{\varphi}$ are respectively called the Calabi product of an affine hypersphere and a point, and the Calabi product of two hyperbolic affine hyperspheres. Note that characterizations of these Calabi products are established in [6] which becomes a crucial step for the complete classification of locally strongly convex affine hypersurfaces with parallel difference tensor (i.e. cubic form) [7, 8].

From a local point of view on locally strongly convex affine hyperspheres, after having the classification of the affine hyperspheres with constant sectional curvature and the classification of the affine hypersphere with parallel cubic form, an obviously natural and interesting problem might be the classification of affine hyperspheres whose affine metric is Einstein and possesses non-constant sectional curvatures.

In this paper, we will deal with the case $n = 4$. When compared with higher dimensions, the 4-dimensional Einstein affine hyperspheres are spectacular (cf. Remark 7.1). The main result of this paper is the following

Main Theorem. *Let $x : M^4 \hookrightarrow \mathbb{R}^5$ be a locally strongly convex affine hypersphere with its affine metric being Einstein, then it is locally affine equivalent to the open part of either one of the hyperquadrics, or the hyperbolic affine hypersphere $Q(4, 1) : x_1 x_2 x_3 x_4 x_5 = 1$, where (x_1, \dots, x_5) are the coordinates of \mathbb{R}^5 .*

This paper is organized as follows. In Section 2, we briefly recall the theory of local affine hypersurfaces. In Section 3, we review the construction of a typical orthonormal basis at a fixed point, by which we need to consider three independent cases. In Sections 4-6, we settle each of the three cases. Then finally we complete the proof of Main Theorem in Section 7.

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2. PRELIMINARIES

In this section, we briefly recall the theory of local equiaffine hypersurfaces in [14, 17]. Let \mathbb{R}^{n+1} be the standard $(n+1)$ -dimensional real affine space, i.e. \mathbb{R}^{n+1} endowed with the standard flat connection D and its parallel volume form w , given by the determinant. Let $F : M \hookrightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface, and ξ be any

transversal vector field on M , i.e. $T_p\mathbb{R}^{n+1} = T_pM \oplus \text{span}\{\xi_p\}$, $\forall p \in M$. For any tangent vector fields X, Y, X_1, \dots, X_n , we write

$$(2.1) \quad D_X F_*(Y) = F_*(\nabla_X Y) + h(X, Y)\xi,$$

$$(2.2) \quad \theta(X_1, \dots, X_n) = w(F_*(X_1), \dots, F_*(X_n), \xi),$$

thus defining a torsion-free affine connection ∇ , a symmetric bilinear form h , and a volume element θ on M . M is said to be non-degenerate if h is non-degenerate (this condition is independent of the choice of the transversal vector field). If M is non-degenerate, up to sign there exists a unique choice of transversal vector field such that $\nabla\theta = 0$ and $\theta = w_h$, where w_h is the metric volume element induced by h (see [16]). This special transversal vector field ξ , called the affine normal, induces the affine connection ∇ and a pseudo-Riemannian metric h on M . We call h the affine metric, or Blaschke-Berwald metric and $C := \nabla h$ the cubic form.

The condition $\nabla\theta = 0$ shows that $D_X\xi$ is tangent to M for all X . Hence we can define a $(1, 1)$ -type tensor S on M , called affine shape operator, by

$$(2.3) \quad D_X\xi = -F_*(SX),$$

and $L_1 = \frac{1}{n}\text{trace } S$ is called the affine mean curvature. Here S has the property of self-adjoint relative to h . As have been stated in the introduction section, the hypersurface M is called an affine hypersphere if $S = L_1 \text{id}$.

The classical Pick-Berwald theorem states that the affine connection coincides with the Levi-Civita connection $\hat{\nabla}$ of affine metric h if and only if the hypersurface is a hyperquadric. For that reason, the difference tensor $K_X Y = \nabla_X Y - \hat{\nabla}_X Y$, which related to the cubic form by

$$C(X, Y, Z) = -2h(K(X, Y), Z),$$

plays a fundamental role in affine differential geometry. Recall that the curvature tensor \hat{R} of the affine metric, affine shape operator S and the difference tensor K are related by the Gauss and Codazzi equations:

$$(2.4) \quad \begin{aligned} \hat{R}(X, Y)Z &= \frac{1}{2}[h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y] \\ &\quad - [K_X, K_Y]Z, \end{aligned}$$

$$(2.5) \quad \begin{aligned} (\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) &= \frac{1}{2}[h(Y, Z)SX - h(X, Z)SY \\ &\quad - h(SY, Z)X + h(SX, Z)Y], \end{aligned}$$

$$(2.6) \quad (\hat{\nabla}_X S)Y - (\hat{\nabla}_Y S)X = K_{SX}Y - K_{SY}X.$$

Moreover, h and C satisfy the apolarity condition

$$(2.7) \quad \text{trace}_h\{(X, Y) \mapsto C(Z, X, Y)\} = 0,$$

or equivalently $\text{trace } K_Z = 0$ for all Z .

3. THE CONSTRUCTION OF AN APPROPRIATE ORTHONORMAL BASIS

In this section, we consider an n -dimensional, locally strongly convex affine hypersphere M^n in \mathbb{R}^{n+1} with affine shape operator $S = L_1 \cdot \text{id}$.

Then the Gauss and Codazzi equations become:

$$(3.1) \quad \hat{R}(X, Y)Z = L_1[h(Y, Z)X - h(X, Z)Y] - [K_X, K_Y]Z,$$

$$(3.2) \quad (\hat{\nabla}_X K)(Y, Z) - (\hat{\nabla}_Y K)(X, Z) = 0.$$

Now we will review the construction of a typical orthonormal basis with respect to the affine metric h for $T_p M^n$, which was introduced by Ejiri and has been widely applied, and proved to be very useful for various purposes, see e.g. [5, 21]. The idea is to construct from the (1,2) tensor K a self adjoint operator at a point; then one extends the eigenbasis to a local field.

Let $p \in M^n$ and $U_p M = \{u \in T_p M^n \mid h(u, u) = 1\}$. Since M^n is locally strongly convex, $U_p M$ is compact. We define a function f on $U_p M$ by $f(u) = h(K_u u, u)$. Let e_1 be an element of $U_p M$ at which the function f attains an absolute maximum. Since $K \neq 0$, we easily see that $f(e_1) > 0$.

Let $u \in U_p M$ such that $h(u, e_1) = 0$, and define by $g(t) := f(\cos t e_1 + \sin t u)$ a function g . Then we have

$$(3.3) \quad \begin{aligned} g'(0) &= 3h(K_{e_1} e_1, u), \\ g''(0) &= 6h(K_{e_1} u, u) - 3f(e_1), \\ g'''(0) &= 6f(u) - 21h(K_{e_1} e_1, u). \end{aligned}$$

Since g attains an absolute maximum at $t = 0$, we have $g'(0) = 0$, i.e.

$$h(K_{e_1} e_1, u) = 0.$$

So e_1 is an eigenvector of K_{e_1} , say associated to the eigenvalue λ_1 . Let e_2, e_3, \dots, e_n be orthonormal vectors, orthogonal to e_1 , which are the remaining eigenvectors of the operator K_{e_1} , associated to the eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_n$. Further, since e_1 is an absolute maximum of f , we know that $g''(0) \leq 0$, and if $g''(0) = 0$, then $g'''(0) = 0$. This implies that for every $i \geq 2$, we have $\lambda_1 - 2\lambda_i \geq 0$ and if $\lambda_i = \frac{1}{2}\lambda_1$ for some i , then $f(e_i) = h(K_{e_1} e_i, e_i) = 0$.

From now on, if not stated otherwise, we restrict to $n = 4$. For later's convenience we will change the previous notations $\{\lambda_i\}$ and summarize the above result in the following lemma.

Lemma 3.1. *For $K \neq 0$, given the existence of an orthonormal basis $\{e_i\}_{i=1}^4$ of $T_p M^4$ as above, by changing the notations and taking into account the apolarity condition we may assume that K_{e_1} takes the following form:*

$$(3.4) \quad \begin{aligned} K_{e_1} e_1 &= (\lambda_1 + \lambda_2 + \lambda_3)e_1, \quad \lambda_1 + \lambda_2 + \lambda_3 > 0, \\ K_{e_1} e_2 &= -\lambda_1 e_2, \quad K_{e_1} e_3 = -\lambda_2 e_3, \quad K_{e_1} e_4 = -\lambda_3 e_4, \end{aligned}$$

where $\{\lambda_i\}_{i=1}^4$ satisfy

$$(3.5) \quad \begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\geq -2\lambda_i, \quad i = 1, 2, 3; \\ \text{if } \lambda_1 + \lambda_2 + \lambda_3 &= -2\lambda_i \text{ for some } i, \text{ then } h(K_{e_i} e_i, e_i) = 0. \end{aligned}$$

Besides (3.4), we can further use the apolarity condition to assume that, at $p \in M^4$:

$$(3.6) \quad \begin{aligned} K_{e_2} e_2 &= -\lambda_1 e_1 + (\mu_4 + \mu_6)e_2 - \mu_2 e_3 - \mu_3 e_4, \\ K_{e_2} e_3 &= -\mu_2 e_2 - \mu_4 e_3 + \mu_5 e_4, \\ K_{e_2} e_4 &= -\mu_3 e_2 + \mu_5 e_3 - \mu_6 e_4, \\ K_{e_3} e_3 &= -\lambda_2 e_1 - \mu_4 e_2 + (\mu_1 + \mu_2)e_3 - \mu_7 e_4, \\ K_{e_3} e_4 &= \mu_5 e_2 - \mu_7 e_3 - \mu_1 e_4, \\ K_{e_4} e_4 &= -\lambda_3 e_1 - \mu_6 e_2 - \mu_1 e_3 + (\mu_3 + \mu_7)e_4, \end{aligned}$$

where the coefficients are numbers to be determined by the context.

Without loss of generality, we are sufficient to deal with the following three possibilities:

Case (I). $\lambda_1 > \lambda_2 > \lambda_3$.

Case (II). $\lambda_1 = \lambda_2 \neq \lambda_3$.

Case (III). $\lambda_1 = \lambda_2 = \lambda_3$.

In later sections, we will assume that as a Riemannian manifold, (M^4, h) is Einstein.

4. THE NONEXISTENCE OF CASE (I)

In this section, we prove

Proposition 4.1. *Let $x : M^4 \rightarrow \mathbb{R}^5$ be a locally strongly convex affine hypersphere whose affine metric is Einstein, then for any $p \in M^4$ and with respect to the orthonormal basis $\{e_i\}_{i=1}^4$ of $T_p M^4$ as stated in Lemma 3.1, Case (I) does not occur.*

Proof. The assertion is derived from the Gauss equation (3.1). As for the details, we write

$$\begin{aligned} R_{jk} &= \sum_i h(\hat{R}(e_i, e_j)e_k, e_i) \\ (4.1) \quad &= 3L_1\delta_{jk} + \sum_i h(K_{e_j}K_{e_k}e_i - K_{e_i}K_{e_j}e_k, e_i). \end{aligned}$$

Using (3.4), (3.6) and (4.1), the Einstein condition $R_{12} = R_{13} = R_{14} = 0$ gives

$$(4.2) \quad (\lambda_2 - \lambda_1)\mu_4 + (\lambda_3 - \lambda_1)\mu_6 = 0,$$

$$(4.3) \quad (\lambda_1 - \lambda_2)\mu_2 + (\lambda_3 - \lambda_2)\mu_1 = 0,$$

$$(4.4) \quad (\lambda_1 - \lambda_3)\mu_3 + (\lambda_2 - \lambda_3)\mu_7 = 0.$$

Hence we can write

$$\begin{aligned} (4.5) \quad \mu_4 &= (\lambda_1 - \lambda_3)\sigma_1, \quad \mu_1 = (\lambda_2 - \lambda_1)\sigma_2, \quad \mu_3 = (\lambda_3 - \lambda_2)\sigma_3; \\ \mu_6 &= (\lambda_2 - \lambda_1)\sigma_1, \quad \mu_2 = (\lambda_3 - \lambda_2)\sigma_2, \quad \mu_7 = (\lambda_1 - \lambda_3)\sigma_3. \end{aligned}$$

Putting (4.5) into (3.6), again the Einstein condition gives

$$\begin{aligned} 0 &= R_{23} = 2(\lambda_1 - \lambda_2)[(\lambda_1 - \lambda_2)\sigma_1\sigma_2 - \mu_5\sigma_3], \\ 0 &= R_{24} = 2(\lambda_1 - \lambda_3)[(\lambda_1 - \lambda_3)\sigma_1\sigma_3 + \mu_5\sigma_2], \\ 0 &= R_{34} = 2(\lambda_2 - \lambda_3)[(\lambda_2 - \lambda_3)\sigma_2\sigma_3 - \mu_5\sigma_1]. \end{aligned}$$

It follows that

$$(4.6) \quad (\lambda_1 - \lambda_2)\sigma_1\sigma_2 - \mu_5\sigma_3 = 0,$$

$$(4.7) \quad (\lambda_1 - \lambda_3)\sigma_1\sigma_3 + \mu_5\sigma_2 = 0,$$

$$(4.8) \quad (\lambda_2 - \lambda_3)\sigma_2\sigma_3 - \mu_5\sigma_1 = 0.$$

From $(4.7) \times (\lambda_1 - \lambda_2)\sigma_2 - (4.6) \times (\lambda_1 - \lambda_3)\sigma_3$, we obtain

$$(4.9) \quad \mu_5[(\lambda_1 - \lambda_2)(\sigma_2)^2 + (\lambda_1 - \lambda_3)(\sigma_3)^2] = 0.$$

If $\mu_5 \neq 0$, then (4.6)-(4.9) show that $\sigma_1 = \sigma_2 = \sigma_3 = 0$. If on the other hand, $\mu_5 = 0$, then (4.6)-(4.8) implies that at most one of $\{\sigma_1, \sigma_2, \sigma_3\}$ is nonzero. So by symmetry we need only to consider the following three subcases:

Case (I)-(i): $\mu_5 = \sigma_1 = \sigma_2 = 0$ and $\sigma_3 \neq 0$.

Case (I)-(ii): $\mu_5 = \sigma_1 = \sigma_2 = \sigma_3 = 0$.

Case (I)-(iii): $\mu_5 \neq 0$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

We first consider Case (I)-(i). Direct calculation gives

$$(4.10) \quad R_{11} = 3L_1 + 2[(\lambda_1)^2 + (\lambda_2)^2 + \lambda_2\lambda_3 + (\lambda_3)^2 + \lambda_1(\lambda_2 + \lambda_3)],$$

$$(4.11) \quad R_{22} = 3L_1 + 2[(\lambda_1)^2 + (\lambda_2 - \lambda_3)^2(\sigma_3)^2],$$

$$(4.12) \quad R_{33} = 3L_1 + 2[(\lambda_2)^2 + (\lambda_1 - \lambda_3)^2(\sigma_3)^2],$$

$$(4.13) \quad R_{44} = 3L_1 + 2\{(\lambda_3)^2 + [(\lambda_1)^2 + (\lambda_2)^2 - \lambda_2\lambda_3 + (\lambda_3)^2 - \lambda_1(\lambda_2 + \lambda_3)](\sigma_3)^2\}.$$

From the Einstein condition $R_{22} = R_{33}$ and (4.11), (4.12), we get

$$(4.14) \quad \lambda_1 + \lambda_2 = (\lambda_1 + \lambda_2 - 2\lambda_3)(\sigma_3)^2.$$

Similarly, from the condition $R_{22} = R_{44}$ and (4.11), (4.13), we get

$$(4.15) \quad \lambda_1 + \lambda_3 = (\lambda_1 - \lambda_2)(\sigma_3)^2.$$

From (4.14) and (4.15) we eliminate σ_3 to obtain

$$0 = (\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2 - 2\lambda_3) - (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2),$$

which implies that

$$(4.16) \quad \lambda_1 + \lambda_2 + 2\lambda_3 = 0.$$

On the other hand, (3.4) and (3.5) imply that

$$(4.17) \quad \lambda_1 + \lambda_2 + \lambda_3 > 0, \quad \lambda_1 + \lambda_2 + \lambda_3 \geq -2\lambda_3.$$

We get a contradiction.

Therefore, Case (I)-(i) does not occur.

Next, we consider Case (I)-(ii). In fact, now we have

$$R_{22} = 3L_1 + 2(\lambda_1)^2, \quad R_{33} = 3L_1 + 2(\lambda_2)^2, \quad R_{44} = 3L_1 + 2(\lambda_3)^2,$$

this contradicts the Einstein condition $R_{22} = R_{33} = R_{44}$ for $\lambda_1 > \lambda_2 > \lambda_3$. Thus, Case (I)-(ii) does not occur.

Finally, Case (I)-(iii) does not occur either. This follows from the following calculation:

$$\begin{aligned} R_{22} &= 3L_1 + 2(\lambda_1)^2 + 2(\mu_5)^2, \\ R_{33} &= 3L_1 + 2(\lambda_2)^2 + 2(\mu_5)^2, \\ R_{44} &= 3L_1 + 2(\lambda_3)^2 + 2(\mu_5)^2. \end{aligned}$$

We have completed the proof of Proposition 4.1. □

5. THE NONEXISTENCE OF CASE (II)

In this section, to deal with Case (II) we will prove

Proposition 5.1. *Let $x : M^4 \rightarrow \mathbb{R}^5$ be a locally strongly convex affine hypersphere whose affine metric is Einstein, then for any $p \in M^4$ and with respect to the orthonormal basis $\{e_i\}_{i=1}^4$ of $T_p M^4$ as stated in Lemma 3.1, Case (II) does not occur.*

First of all, we notice that in Case (II) we can choose $\{e_2, e_3\}$ in Lemma 3.1 such that $h(K_{e_2}e_2, e_3) = 0$, i.e., $\mu_2 = 0$ in (3.6). Then direct calculation gives

$$(5.1) \quad R_{12} = (\lambda_3 - \lambda_1)\mu_6, \quad R_{13} = (\lambda_3 - \lambda_1)\mu_1, \quad R_{14} = (\lambda_1 - \lambda_3)(\mu_3 + \mu_7),$$

so we get

$$(5.2) \quad \mu_6 = \mu_1 = 0, \quad \mu_7 = -\mu_3,$$

and then we further get

$$(5.3) \quad R_{24} = -2\mu_3\mu_4, \quad R_{34} = -2\mu_4\mu_5.$$

Since $R_{24} = R_{34} = 0$, we have two subcases:

Case (II)-(i): $\mu_4 = 0$. **Case (II)-(ii):** $\mu_4 \neq 0$ and $\mu_3 = \mu_5 = 0$.

If $\mu_4 = 0$, then direct calculation gives

$$\begin{aligned} R_{22} &= R_{33} = 3L_1 + 2[(\lambda_1)^2 + (\mu_3)^2 + (\mu_5)^2], \\ R_{44} &= 3L_1 + 2[(\lambda_3)^2 + (\mu_3)^2 + (\mu_5)^2], \end{aligned}$$

and the Einstein condition implies that $(\lambda_3)^2 = (\lambda_1)^2$, so $\lambda_3 = -\lambda_1$. By (4.17) we get a contradiction. This shows that Case (II)-(i) does not occur.

Next, we consider Case (II)-(ii). By direct calculation, we get

$$\begin{aligned} R_{11} &= 3L_1 + 6(\lambda_1)^2 + 4\lambda_1\lambda_3 + 2(\lambda_3)^2 \\ R_{22} &= 3L_1 + 2[(\lambda_1)^2 + (\mu_4)^2] = R_{33}, \\ R_{44} &= 3L_1 + 2(\lambda_3)^2. \end{aligned}$$

From the Einstein condition, we have $R_{11} = R_{44}$ and therefore

$$(5.4) \quad \lambda_1(3\lambda_1 + 2\lambda_3) = 0.$$

If $3\lambda_1 + 2\lambda_3 = 0$, i.e., $\lambda_3 = -\frac{3}{2}\lambda_1$, then by (4.17) we get a contradiction. So Case (II)-(ii) reduces to satisfy

$$(5.5) \quad (\mu_4)^2 = (\lambda_3)^2 \neq 0 \quad \text{and} \quad \lambda_1 = \lambda_2 = 0 = \mu_i, \quad i = 1, 2, 3, 5, 6, 7.$$

Replacing e_2 by $-e_2$ if necessary, we may assume $\mu_4 = \lambda_3 > 0$. Then the Ricci curvature R_{ij} satisfies

$$(5.6) \quad R_{ij} = [3L_1 + 2(\lambda_3)^2]\delta_{ij}, \quad 1 \leq i, j \leq 4,$$

and the difference tensor is given as follows:

$$(5.7) \quad \begin{cases} K_{e_1}e_1 = \lambda_3e_1, & K_{e_1}e_4 = -\lambda_3e_4, & K_{e_2}e_2 = \lambda_3e_2, \\ K_{e_2}e_3 = -\lambda_3e_3, & K_{e_3}e_3 = -\lambda_3e_2, & K_{e_4}e_4 = -\lambda_3e_1, \\ K_{e_1}e_2 = K_{e_1}e_3 = K_{e_2}e_4 = K_{e_3}e_4 = 0. \end{cases}$$

We notice from (5.6) that λ_3 is independent of the point p , it is in fact determined by the Einstein constant κ : If $R_{ij} = \kappa\delta_{ij}$, then $\lambda_3 = \sqrt{(\kappa - 3L_1)/2}$.

From the above expression of the difference tensor, we can use (3.1) to calculate the curvature tensor at $p \in M^4$ as follows:

$$(5.8) \quad \begin{cases} \hat{R}(e_1, e_2)e_1 = -L_1e_2, & \hat{R}(e_1, e_2)e_2 = L_1e_1; \\ \hat{R}(e_1, e_3)e_1 = -L_1e_3, & \hat{R}(e_1, e_3)e_3 = L_1e_1; \\ \hat{R}(e_1, e_4)e_1 = -(L_1 + 2(\lambda_3)^2)e_4, & \hat{R}(e_1, e_4)e_4 = (L_1 + 2(\lambda_3)^2)e_1; \\ \hat{R}(e_2, e_3)e_2 = -(L_1 + 2(\lambda_3)^2)e_3, & \hat{R}(e_2, e_3)e_3 = (L_1 + 2(\lambda_3)^2)e_2; \\ \hat{R}(e_2, e_4)e_2 = -L_1e_4, & \hat{R}(e_2, e_4)e_4 = L_1e_2; \\ \hat{R}(e_3, e_4)e_3 = -L_1e_4, & \hat{R}(e_3, e_4)e_4 = L_1e_2; \\ \hat{R}(e_i, e_j)e_k = 0, & \text{if } i, j, k \text{ are distinct.} \end{cases}$$

It follows that (M^4, h) is not of constant sectional curvature at the point p .

Next, we calculate the curvature tensor from the definition:

$$(5.9) \quad \hat{R}(X, Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X, Y]} Z.$$

For this purpose, we first recall the following result (cf. section 3).

Lemma 5.1 (cf. Lemma 3.1 of [21]). *Let h be a positive definite metric on \mathbb{R}^k and let $T : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a symmetric multilinear mapping. We denote by $S^{k-1} = \{v \in \mathbb{R}^k \mid h(v, v) = 1\}$ and define a function F on S^{k-1} by $F(v) = T(v, v, v)$. Let u be a vector at which the function F on S^{k-1} attains an extremal value and let $w \in S^{k-1}$ such that $h(u, w) = 0$. Then $T(u, u, w) = 0$. Moreover, if F attains a relative maximum in u , then we have also that $T(u, u, u) - 2T(u, w, w) \geq 0$ and in the case $T(u, u, u) - 2T(u, w, w) = 0$, we have $T(w, w, w) = 0$.*

Corollary 5.1. *Let M^n be a locally strongly convex affine hypersurface of \mathbb{R}^{n+1} . For $p \in M^n$, by taking $\mathbb{R}^n = T_p M^n$ in Lemma 5.1 and define T by*

$$T(u, v, w) = h(K_u v, w).$$

Then for a unit vector u , if $K_u u = \lambda u$, $T(u, u, u) = \lambda$ is an extremal value.

Returning to (5.7), we now prove

Lemma 5.2. *There exists a neighbourhood U of p and a local unit vector field W on U such that*

$$(5.10) \quad K_W W = \lambda_3 W.$$

Proof. Let $\{E_1, E_2, E_3, E_4\}$ be an arbitrary local differentiable h -orthonormal frame fields of (M^4, h) defined on a neighbourhood U' of p such that $E_i(p) = e_i$, $1 \leq i \leq 4$. We want to look for functions $\{a_1, a_2, a_3, a_4\}$ defined on U' such that

$$a_1(p) = 1, \quad a_2(p) = a_3(p) = a_4(p) = 0,$$

and $W = \sum_{i=1}^4 a_i E_i$ has the property of (5.10). Define a mapping

$$\sigma : \mathbb{R}^4 \times U' \rightarrow \mathbb{R}^4 \quad \text{by} \quad \sigma(x_1, \dots, x_4, q) = (y_1, \dots, y_4),$$

where

$$(5.11) \quad y_k = \sum_{i,j} x_i x_j h(K_{E_i} E_j, E_k) - \lambda_3 x_k, \quad 1 \leq k \leq 4$$

are regarded as functions on $\mathbb{R}^4 \times U'$: $y_k = y_k(x_1, x_2, x_3, x_4, q)$.

Since $E_i(p) = e_i$, it is easily seen from (5.7) that at the point

$$A := (1, 0, 0, 0, p) \in \mathbb{R}^4 \times U',$$

we have $y_k(A) = 0$, $1 \leq k \leq 4$, and

$$(5.12) \quad \left(\frac{\partial y_k}{\partial x_\ell} \right) \Big|_A = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & -\lambda_3 & 0 & 0 \\ 0 & 0 & -\lambda_3 & 0 \\ 0 & 0 & 0 & -3\lambda_3 \end{pmatrix}$$

is invertible, hence the implicit function theorem shows that there exists differentiable functions $\{x_i(q), 1 \leq i \leq 4\}$ defined on a neighbourhood $U'' \subset U'$ of p such that

$$(5.13) \quad y_k(x_1(q), x_2(q), x_3(q), x_4(q), q) \equiv 0, \quad 1 \leq k \leq 4.$$

Define the local vector field V by

$$V(q) = x_1(q)E_1(q) + x_2(q)E_2(q) + x_3(q)E_3(q) + x_4(q)E_4(q), \quad q \in U''.$$

Then (5.11) and (5.13) imply that $K_V V = \lambda_3 V$ and $V(p) = e_1$.

Let us define $\|V\| = \sqrt{h(V, V)}$. Since $\|V\|(p) = 1$, there exists a neighbourhood $U \subset U''$ of p such that $V \neq 0$ on U . Define a local unit vector field $W = \frac{V}{\|V\|}$, then we have

$$(5.14) \quad K_W W = \frac{\lambda_3}{\|V\|} W.$$

According to Lemma 5.1 and Corollary 5.1, the expression (5.14) shows that, for any $q \in U$, the function F_q attains the relative extremal at $W(q)$ with extremal value $\frac{\lambda_3}{\|V\|}$.

Claim. *At any $q \in U$, the set of extremal values of the function F_q on $U_q M^4$ is finite.*

Proof of the Claim. We are sufficient to verify the assertion at p to see that how many values can $h(K_v v, v)$ take for $v \in U_p M$, under the additional condition that $K_v v$ is parallel to v , i.e., v is a relative extremal vector.

Let us denote $v = t_1 e_1 + t_2 e_2 + t_3 e_3 + t_4 e_4 \in U_p M^4$. Then by (5.7) we see that the condition $K_v v = cv$ for some constant c becomes, equivalently,

$$(5.15) \quad \begin{cases} (t_1^2 - t_4^2)\lambda_3 = ct_1, \\ (t_2^2 - t_3^2)\lambda_3 = ct_2, \\ -2t_2 t_3 \lambda_3 = ct_3, \\ -2t_1 t_4 \lambda_3 = ct_4. \end{cases}$$

To solve (5.15) under the condition $\sum_i (t_i)^2 = 1$ we see that there are only two possible values of c , i.e., $c_1 = \lambda_3$, $c_2 = \frac{\sqrt{2}}{2}\lambda_3$. This finishes the proof of the Claim.

From the above Claim and (5.14), and since $\frac{\lambda_3}{\|V\|}$ changes continuously, we obtain that

$$\|V\|(q) = \|V\|(p) = 1,$$

for all $q \in U$. We have proved Lemma 5.2. \square

Lemma 5.3. *There exists a neighbourhood U of $p \in M^4$ and a local orthonormal frame fields $\{E_1, E_2, E_3, E_4\}$ such that $E_i(p) = e_i$ and $\{K_{E_1}, K_{E_2}, K_{E_3}, K_{E_4}\}$ still have the same representation as in (5.7), namely*

$$(5.16) \quad \begin{cases} K_{E_1}E_1 = \lambda_3 E_1, & K_{E_1}E_4 = -\lambda_3 E_4, & K_{E_2}E_2 = \lambda_3 E_2, \\ K_{E_2}E_3 = -\lambda_3 E_3, & K_{E_3}E_3 = -\lambda_3 E_2, & K_{E_4}E_4 = -\lambda_3 E_1, \\ K_{E_1}E_2 = K_{E_1}E_3 = K_{E_2}E_4 = K_{E_3}E_4 = 0, \end{cases}$$

where $\lambda_3 = \sqrt{(\kappa - 3L_1)/2}$.

Proof. By taking the first unit vector field $E_1 = W$ as in Lemma 5.2, it satisfies

$$K_{E_1}E_1 = \lambda_3 E_1,$$

and that $h(K_{E_1}E_1, E_1) = \lambda_3 = \sqrt{(\kappa - 3L_1)/2}$ is the maximum of the function $h(K_v, v, v)$ in $U_q M$ for any $q \in U$. Since at $p \in U$, the metric h is not of constant sectional curvature, without loss of generality, we may assume that at any $q \in U$ the sectional curvature is not constant. Similar to the previous discussion, the eigenvalues of $K_{E_1(q)} : T_q M^4 \rightarrow T_q M^4$ must be $\{\lambda_3, 0, 0, -\lambda_3\}$. Thus the eigenvalues functions of K_{E_1} are constant functions. According to Szabo's result [19], we have a neighbourhood of p , still denoted by U such that on U we have smooth eigenvector fields $\{E_2, E_3, E_4\}$ of K_{E_1} , corresponding to the eigenvalues 0 (multiplicities 2) and $-\lambda_3$, respectively, which together with E_1 form an orthonormal frame fields with

$$K_{E_1}E_2 = K_{E_1}E_3 = 0, \quad K_{E_1}E_4 = -\lambda_3 E_4.$$

By the uniqueness of such non-constant sectional curvature case, we also have

$$K_{E_4}E_2 = K_{E_4}E_3 = 0, \quad K_{E_4}E_4 = -\lambda_3 E_1.$$

Now, by the apolarity condition, we can assume that

$$(5.17) \quad \begin{aligned} K_{E_2}E_2 &= \alpha E_2 + \beta E_3, \\ K_{E_2}E_3 &= \beta E_2 - \alpha E_3, \\ K_{E_3}E_3 &= -\alpha E_2 - \beta E_3, \end{aligned}$$

where α and β are smooth functions. From the above property of the difference tensor, we immediately have the calculation

$$R_{11} = 3L_1 + 2(\lambda_3)^2, \quad R_{33} = 3L_1 + 2(\alpha^2 + \beta^2).$$

Then the Einstein condition implies that $\alpha^2 + \beta^2 = (\lambda_3)^2$. Obviously, by the choice of E_1, E_2 we can assume that $\alpha > 0, \beta \geq 0$.

If $\beta \neq 0$, we choose φ such that $\cot(3\varphi) = \frac{\alpha}{\beta}$, and define

$$(5.18) \quad \begin{cases} F_2 = \cos \varphi E_2 + \sin \varphi E_3, \\ F_3 = -\sin \varphi E_2 + \cos \varphi E_3. \end{cases}$$

Then it holds

$$h(K_{F_2}F_2, F_3) = -\alpha \sin 3\varphi + \beta \cos 3\varphi = 0.$$

Therefore, up to a transformation like (5.18), we may eventually assume that in (5.17) we have $\beta = 0$. Then the assertion of Lemma 5.3 follows. \square

Completion of the proof of Proposition 5.1.

As an application of (5.16), we look at the Codazzi equations:

$$(ijk) \quad (\hat{\nabla}_{E_i} K)(E_j, E_k) - (\hat{\nabla}_{E_j} K)(E_i, E_k) = 0.$$

Let us denote

$$\hat{\nabla}_{E_i} E_j = \Gamma_{ij}^k E_k, \quad \Gamma_{ij}^k + \Gamma_{ik}^j = 0.$$

Direct calculation of the equation (121), we obtain

$$\Gamma_{11}^2 = \Gamma_{21}^2 = 0, \quad \Gamma_{21}^3 = \Gamma_{11}^3, \quad \Gamma_{12}^4 = 3\Gamma_{21}^4.$$

If we calculate the equation (131), then we further obtain

$$\Gamma_{11}^3 = \Gamma_{31}^2 = \Gamma_{31}^3 = 0, \quad \Gamma_{13}^4 = 3\Gamma_{31}^4.$$

In this way, by calculating all the equations (ijk) we will finally obtain

$$\Gamma_{ij}^k = 0, \quad \forall i, j, k,$$

which imply that, in (5.9), $\hat{R}(X, Y)Z = 0$ for any X, Y, Z , i.e., the affine metric is flat. This is a contradiction to (5.8). From this contradiction we complete the proof of Proposition 5.1. \square

6. 4-DIMENSIONAL EINSTEIN AFFINE HYPERSPHERE IN CASE (III)

In this section we prove

Proposition 6.1. *Let $x : M^4 \rightarrow \mathbb{R}^5$ be a locally strongly convex affine hypersphere whose affine metric is Einstein, then for any $p \in M^4$ and with respect to the orthonormal basis $\{e_i\}_{i=1}^4$ of $T_p M^4$ as stated in Lemma 3.1, if Case (III) occurs then the affine metric is of constant sectional curvature at p .*

In Case (III), the eigenvalues of $K_{e_1} : T_p M^4 \rightarrow T_p M^4$ are $3\lambda_1, -\lambda_1, -\lambda_1, -\lambda_1$. Now, we first choose e_2 such that

$$(6.1) \quad h(K_{e_2} e_2, e_2) = \max_{u \in U\{e_1\}^\perp} h(K_u u, u),$$

where $U\{e_1\}^\perp$ denotes the set of all unit vectors perpendicular to e_1 in $U_p M^4$. From (6.1), we get

$$(6.2) \quad h(K_{e_2} e_2, u) = 0, \quad \forall u \in \{e_1, e_2\}^\perp,$$

where we denote by $\{e_1, e_2\}^\perp$ the subspace of $T_p M^4$ that is perpendicular to e_1 and e_2 .

Now, from (6.2) we have the observation that $\{e_1, e_2\}^\perp$ is an invariant subspace of K_{e_2} . From this observation we will separate the following discussions into two subcases.

Case (III)-(i). If $K_{e_2} : \{e_1, e_2\}^\perp \rightarrow \{e_1, e_2\}^\perp$ has an eigenvalue of multiplicity two, denoting $-\sigma$. Denote by $U\{e_1, e_2\}^\perp$ the set of all unit vectors in $\{e_1, e_2\}^\perp$, and we choose an orthonormal basis $\{e_3, e_4\}$ of $U\{e_1, e_2\}^\perp$ such that

$$(6.3) \quad h(K_{e_3} e_3, e_3) = \max_{u \in U\{e_1, e_2\}^\perp} h(K_u u, u).$$

It follows from (6.3) that $h(K_{e_3} e_3, e_4) = 0$.

Then we easily deduce from the above requirements and the apolarity condition the following expressions of the difference tensor:

$$(6.4) \quad \begin{cases} K_{e_1} e_1 = 3\lambda_1 e_1, & K_{e_1} e_2 = -\lambda_1 e_2, & K_{e_1} e_3 = -\lambda_1 e_3, & K_{e_1} e_4 = -\lambda_1 e_4, \\ K_{e_2} e_2 = -\lambda_1 e_1 + 2\sigma e_2, & K_{e_2} e_3 = -\sigma e_3, & K_{e_2} e_4 = -\sigma e_4, \\ K_{e_3} e_3 = -\lambda_1 e_1 - \sigma e_2 + \mu_1 e_3, & K_{e_3} e_4 = -\mu_1 e_4, \\ K_{e_4} e_4 = -\lambda_1 e_1 - \sigma e_2 - \mu_1 e_3, \end{cases}$$

where the coefficients satisfy the relation $3\lambda_1 \geq 2\sigma \geq \mu_1 \geq 0$. From (4.1) and (6.4), we have

$$\begin{aligned} R_{11} &= 3L_1 + 12(\lambda_1)^2, \\ R_{22} &= 3L_1 + 2(\lambda_1)^2 + 6\sigma^2, \\ R_{33} &= R_{44} = 3L_1 + 2(\lambda_1)^2 + 2\sigma^2 + 2(\mu_1)^2. \end{aligned}$$

Then the Einstein condition gives

$$\sigma = \sqrt{\frac{5}{3}}\lambda_1, \quad \mu_1 = \sqrt{\frac{10}{3}}\lambda_1,$$

and it easily follows that

$$\hat{R}(e_i, e_j)e_k = (L_1 + 4(\lambda_1)^2)(\delta_{jk}e_i - \delta_{ik}e_j),$$

i.e., (M^4, h) is of constant sectional curvatures $L_1 + 4(\lambda_1)^2$ at p .

Case (III)-(ii): $K_{e_2} : \{e_1, e_2\}^\perp \rightarrow \{e_1, e_2\}^\perp$ has two distinct eigenvalues $-\sigma_1$ and $-\sigma_2$. We choose the unit vectors $\{e_3, e_4\}$ such that $K_{e_2}e_3 = -\sigma_1e_3$, $K_{e_2}e_4 = -\sigma_2e_4$.

Then we easily deduce the following expressions of the difference tensor:

$$(6.5) \quad \begin{cases} K_{e_1}e_1 = 3\lambda_1e_1, & K_{e_1}e_2 = -\lambda_1e_2, & K_{e_1}e_3 = -\lambda_1e_3, & K_{e_1}e_4 = -\lambda_1e_4; \\ K_{e_2}e_2 = -\lambda_1e_2 + (\sigma_1 + \sigma_2)e_2, & K_{e_2}e_3 = -\sigma_1e_3, & K_{e_2}e_4 = -\sigma_2e_4; \\ K_{e_3}e_3 = -\lambda_1e_1 - \sigma_1e_2 + \mu_1e_3 + \mu_2e_4, & K_{e_3}e_4 = \mu_2e_3 - \mu_1e_4; \\ K_{e_4}e_4 = -\lambda_1e_1 - \sigma_2e_2 - \mu_1e_3 - \mu_2e_4, \end{cases}$$

where the coefficients satisfy the relation $3\lambda_1 \geq \sigma_1 + \sigma_2 > 0$.

To verify the assertion $\sigma_1 + \sigma_2 > 0$, we notice from (6.1) that

$$(6.6) \quad \sigma_1 + \sigma_2 = \max_{u \in U\{e_1\}^\perp} h(K_u u, u) \geq 0,$$

so if $\sigma_1 + \sigma_2 = 0$, it will hold $h(K_u u, u) = 0$ for all $u \in U\{e_1\}^\perp$. On the other hand, since $\sigma_1 \neq \sigma_2$, we may assume $\sigma_1 < 0$, then by taking $v = \frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3$, we get a contradiction $h(K_v v, v) = -\frac{9}{8}\sigma_1 > 0$.

From (4.1) and (6.5), we have

$$\begin{cases} R_{12} = R_{13} = R_{14} = R_{34} = 0, \\ R_{23} = (\sigma_2 - \sigma_1)\mu_1, & R_{24} = (\sigma_2 - \sigma_1)\mu_2; \\ R_{11} = 3L_1 + 12(\lambda_1)^2, \\ R_{22} = 3L_1 + 2(\lambda_1)^2 + 2(\sigma_1)^2 + 2(\sigma_2)^2 + 2\sigma_1\sigma_2, \\ R_{33} = 3L_1 + 2(\lambda_1)^2 + 2(\sigma_1)^2 + 2(\mu_1)^2 + 2(\mu_2)^2, \\ R_{44} = 3L_1 + 2(\lambda_1)^2 + 2(\sigma_2)^2 + 2(\mu_1)^2 + 2(\mu_2)^2. \end{cases}$$

Then the Einstein condition $R_{33} = R_{44}$ gives $(\sigma_2)^2 = (\sigma_1)^2$, a contradiction to $\sigma_2 \neq \pm\sigma_1$. Therefore, Case (III)-(ii) does not occur.

7. PROOF OF THE MAIN THEOREM

Let M^4 be a locally strongly convex Einstein affine hypersphere in \mathbb{R}^5 which is not a quadric, then we may assume $K \neq 0$ on M^4 . Propositions 4.1, 5.1 and 6.1 show that for any $p \in M^4$, the only possibility is that Case (III)-(i) occurs and (M^4, h) has constant sectional curvature at p . According to the classification of affine hyperspheres with constant affine sectional curvatures, due to Vrancken, Li

and Simon [21] (see also [14]), we finally see that M^4 is affinely equivalent to the hyperbolic affine hypersphere $Q(4, 1) : x_1x_2x_3x_4x_5 = 1$ in \mathbb{R}^5 . \square

Remark 7.1. For $n \geq 5$, there exists affine hypersphere in \mathbb{R}^{n+1} whose affine metric is Einstein and possesses non-constant sectional curvatures. A typical example is the standard embedding of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ into \mathbb{R}^6 (cf. [8]). For more typical examples, we refer to [7] (see also [1]).

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