Radiative heating of a glass plate: the semi-discrete problem

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Radiative Heating of a Glass Plate: the semi-discrete problem (second revision)

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Absract. In a preceding paper (MSIA, 2012), we have studied the radiative heating of a glass plate. We have proved existence and uniqueness of the solution. Here, we want to study the semi-discrete problem and to prove a priori error estimates. Previously, in that purpose, we have to study the regularity of the solution to the exact problem and of its time derivative. A very important property, Proposition 13, is remarked concerning our elliptic projection, the milestone in deriving the a priori error estimates. A numerical test coroborating our theoretical a priori bounds is given.

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1 Statement of the problem and summary of our results

The setting is the same as in our previous paper [11], but the problem we study here is the related semi-discrete problem, our purpose being to prove some a priori error estimate on the semi-discrete solution. Let us remind only the equations of the exact problem to which we have been led in [11] and some results that we will need in the proof of the a priori error estimates. We consider an infinite plane horizontal glass plate of thickness l, laid down on its lower face $x_g = 0$, on a black sheet-metal maintained at absolute ambient temperature $T = T_a$. The x-axis is directed upward orthogonally to the glass plate so that the upper (resp. lower) face of the glass-plate has $x = l$ (resp. $x = 0$) for equation. An infinite plane horizontal black sheet metal S, at absolute temperature $u(t)$ at time t, placed above the glass plate, emits thermal radiation in every direction,

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whose spectral radiative intensity [18], [17], [10], [13] at wavelength λ in the (dry) air gap between S and the glass plate is given by the famous Planck function:

$$B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}} - 1)} \text{ with } T = u(t), \tag{1}$$

where $C_1 = hc_0^2 = 0.595531\ 10^{-16}$ W.m^2/sr and $C_2 = \frac{hc_0}{k_B} = 1.438786\ 10^{-2}$ m.°K ([5], p.98) (in [11], the temperature of the source of thermal radiation S at time t was denoted $T_S(t)$, but here we prefer to denote it $u(t)$ like in [12]). After refraction at the interface $\{x = l\}$ between air and glass, some part of the radiative energy emitted by the black source S, will be absorbed i.e. converted into heat in the glass producing in such a way an increase of the temperature $T(x,t)$ in the glass plate. Assuming the dependent quantities, independent of the cartesian coordinates y and z, and denoting by t_f the final time of the heating process, based on physical considerations, we have shown in [11], that the radiative heating of the glass plate, can be modelized by the following nonlinear initial boundary value problem in the unknown (absolute) temperature distribution in the glass plate $T(x,t)$ along the thickness of the glass plate $0 \leq x \leq l$ for time $0 \leq t \leq t_f$:

$$\left\{\begin{array}{r} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \frac{\partial^2 T}{\partial x^2}(x,t) + \psi(T(x,t)) + h_{T,u}(x,t), \\ 0 < x < l,\ 0 < t < t_f\ , \\ -k_h \frac{\partial T}{\partial x}(l,t) = h_c(T(l,t) - T_a) + (\Theta(T(l,t)) - \Theta(u(t))), \\ 0 < t < t_f\ ,\ (\text{B.C. at } x = l), \\ T(0,t) = T_a,\ 0 < t < t_f\ ,\ (\text{B.C. at } x = 0), \\ T(x,0) = T_0(x),\ \forall x \in [0,l],\ (\text{I.C. at } t = 0), \end{array}\right. \tag{2}$$

where c_p, m_g, k_h are assumed to be positive constants named respectively heat capacity, mass density and thermal conductivity of the glass [18], [17]. The quantities appearing in the right-hand side of the above nonlinear parabolic integro-differential equation and in the boundary condition (B.C.) at $x = l$, are defined by the following functions:

$$\begin{array}{c} \psi(T) := -\sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T),\ h_{T,u}(x,t) := \sum_{k=1}^{k=M} 2\pi\kappa_k h_{T,u}^k(x,t) \\ \text{and}\ \ \Theta(T) := \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T,\lambda) d\lambda, \end{array} \tag{3}$$

where (the meaning of M and $B_g^k(T)$ are explained below):

$$\begin{array}{c} h_{T,u}^k(x,t) := \int_0^l G_k(x,x')\ B_g^k(T(x',t))\ dx' + B_g^k(T_a)\ E_2(\kappa_k x) \\ +B_g^k(T_a)\ \Phi_2(\kappa_k(2l - x)) + B_g^k(u(t))[E_2(\kappa_k(l - x)) - \Phi_2(\kappa_k(l - x))] \\ +\kappa_k \int_0^l \Phi_1(\kappa_k(2l - x - x^{'}))B_g^k(T(x',t))\ dx'. \end{array} \tag{4}$$

Let us explain the meaning of these different quantities. In (4), $\forall\,(x,x') \in [0,l]^2$: $G_k(x,x') := \kappa_k E_1(\kappa_k\,|x-x'|)$, $\Phi_1(y) := \int_0^1 e^{-\frac{y}{\mu}} \rho_g(\mu) \frac{d\mu}{\mu}$, and $\Phi_2(y) := \int_0^1 e^{-\frac{y}{\mu}} \rho_g(\mu) d\mu$, $\forall y > 0$. $\rho_g(\mu)$ $(0 \leq \mu \leq 1)$ denotes the reflectivity coefficient given by Fresnel's relation ([10], formula (2.96) p. 47). $E_1(\cdot)$ and $E_2(\cdot)$ denote the exponential integral functions of order 1 and 2 respectively. In the definition of $\Theta(T)$ in (3), ϵ_λ is a positive constant called the spectral hemispherical emittance ([10], pp.62-63); like in ([16], p. 70), we have supposed that the spectral hemispherical emittance is equal to the spectral hemispherical absorptance and is independent of the temperature, for wavelength λ of the electromagnetic wave spectrum belonging to $[\lambda_0, +\infty[$ ($\lambda_0 \approx 5\mu m$ for glass, [16], p.70), wavelengths for which glass is opaque. On the other hand, the region of wavelengths in the electromagnetic spectrum where glass is semi-tranparent has been decomposed into M disjoint bands $[\lambda_k, \lambda_{k+1}[$ $(k = 1, \ldots, M)$ [15],[3],[8] and the constants κ_k appearing in the above formulas are the linear spectral absorption coefficients [18], [17], [7], [10], [13] in their respective band. $B^k(T) := \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)\; d\lambda$ and $B_g^k(T) := n_g^2 B^k(T)$ [11], [15], where n_g denotes the refractory index of the glass ($n_g \approx 1.46$). Let us also mention that $\psi(T(x,t)) + h_{T,u}(x,t) = -\frac{\partial q}{\partial x}(x,t)$ where $q(x,t)$ denotes the total radiative heat flux at time t through the horizontal plane at level $0 < x < l$ ([10], p. 276) in the positive x-direction. In the boundary condition at $x = l$ (see (2)), $h_c > 0$ denotes the convective heat transfer coefficient, the term $h_c(T(l,t) - T_a)$ representing the conducto-convective flux density at the infinite surface $\{x = l\}$ of the glass plate according to Newton's law ([18], p.16), ([17], p.13-16). We assume that the initial condition $T_0(\cdot)$ (see (2)) is a continuous positive function on the closed interval $[0,l]$, as an absolute temperature is always positive in classical physics and $T_0(\cdot)$ is a datum. In order to obtain regularity on the solution $T(.,.)$ of the nonlinear initial boundary value problem (2), we will suppose moreover in the present paper, at least that the initial condition $T_0(\cdot) \in H^1(]0,l[)$ and that the compatibility condition $T_0(0) = T_a$ between the initial condition T_0 and the inhomogeneous Dirichlet boundary condition $T(0,.) = T_a$ (see (2)) on the lower face $\{x = 0\}$ of the glass plate, is verified. We suppose given two different positive constants T_{low} and T_{up} such that $0 < T_{low} \leqslant T_0(x) \leqslant T_{up}$, $\forall x \in [0,l]$ (this implies that $0 < T_{low} \leqslant T_a \leqslant T_{up}$ due to $T_0(0) = T_a$) and that $T_{low} \leqslant u(t) \leqslant T_{up}$, $\forall t \in [0,t_f]$.
We have proved in [11] that for

$$u \in U_{ad} = \{v \in H^1(]0,t_f[); T_{low} \leq v(t) \leq T_{up},\ \forall t \in]0,t_f[\}, \tag{5}$$

that the initial boundary value problem (2) possesses a unique bounded weak solution

$$T_u \in \{T \in L^2(0,t_f;H^1(]0,l[));\dot{T} \in L^2(0,t_f;\left[H_L^1(]0,l[)\right]^*)\} \tag{6}$$

which is also continuous on $\overline{Q} := [0,l] \times [0,t_f]$, Q denoting the "cylinder" $]0,l[\times]0,t_f[$. In (6), $H_L^1(]0,l[)$ denotes the closed subspace of $H^1(]0,l[)$ formed by those functions of $H^1(]0,l[)$ which vanish at the left extremity $x=0$ of the interval $]0,l[$. By a weak solution T_u of the initial boundary value problem (2), we mean that T_u belongs to

$$\{T \in L^2(0,t_f;H^1(]0,l[));\dot{T} \in L^2(0,t_f;\left[H_L^1(]0,l[)\right]^*)\}$$

and satisfies $\forall' t \in]0,t_f[$:

$$\left\{\begin{array}{c} c_p m_g \left\langle \frac{dT_u}{dt}(\cdot,t),\varphi\right\rangle_{H_L^1(]0,l[)^*,H_L^1(]0,l[)} + k_h \int_0^l \frac{\partial T_u}{\partial x}(x,t)\varphi'(x)dx \\ -\int_0^l \psi(T_u(x,t))\varphi(x)dx - \int_0^l h_{T_u,u}(x,t)\varphi(x)dx \\ -\left[\Theta(u(t)) - \Theta(T_u(l,t))\right]\cdot\varphi(l) - \left[h_c\left(T_a - T_u(l,t)\right)\right]\varphi(l) = 0,\ \forall \varphi \in H_L^1(]0,l[), \\ T_u(0,t) = T_a,\ \forall' t \in]0,t_f[, \\ T_u(x,0) = T_0(x),\ \forall' x \in]0,l[. \end{array}\right. \tag{7}$$

We have written T_u to underline the dependence of the weak solution of (2) with respect to u. Using Stampacchia's truncation method [1], we have also proved in [11], that the weak solution of the initial boundary value problem (2) $T_u \equiv T(u)$ satisfies:

$$T_{low} \leqslant T_u(x,t) \leqslant T_{up},\ \forall (x,t) \in \overline{Q}. \tag{8}$$

Let us mention, that though an absolute temperature is always positive in classical physics, that for mathematical purposes only, we have extended the definition of the Planck's function (1) to negative real numbers T by setting $B(T,\lambda) = 0$, if $T \leq 0$; in that way for fixed $\lambda > 0$, the function $T \mapsto B(T,\lambda)$ is defined on the whole real line and is lipschitzian of Lipschitz constant $\frac{2C_1}{C_2\lambda^4}$ (cf. Lemma 3.6 of [11]).

Now these things having been recalled, let us explain what are the results in the present paper. Firstly, we study the regularity of the exact solution T_u i.e. of the weak solution of our nonlinear initial boundary value problem (2). Under certain regularity hypotheses on the initial condition T_0 and compatibility conditions, we prove that $T_u \in H^{2,1}(Q)$ (see Proposition 3), and that $\frac{dT_u}{dt} \in H^{2,1}(Q)$ (see Corollary 8), regularity properties that we will need for our a priori error estimates (68), (72) and (77) to be valid. We introduce the semi-discrete problem (47), its matrix differential form being given by (44) from which the existence and unicity of its solution $T_{u,h}$ is nearly immediate (see Proposition 10). We

introduce the elliptic projection $\tilde{T}_{u,h}(.,t)$ of $T_u(.,t)$ by formula (49). If we would have a Dirichlet boundary condition at the point $x = l$, then the technique is well known ([19], p.7). But in our case, we have the nonlinear Robin boundary condition $(2)_{(ii)}$ at the point $x = l$. The key property (see Proposition 13) of our elliptic projection $\tilde{T}_{u,h}(.,t)$ of $T_u(.,t)$ (49) relative to our setting (2), which allows us to derive our a priori error estimates, is that $\tilde{T}_{u,h}(l,t) = T_u(l,t)$. After having written the error equation (48) and splitted the error $e := T_{u,h} - T_u$ on $T_{u,h}$ in $e = \theta + \rho$ where $\theta = T_{u,h} - \tilde{T}_{u,h}$ and $\rho = \tilde{T}_{u,h} - T_u$, we are on the road which will lead us to the proofs of our a priori error estimates in h^2 for $\|T_{u,h}(l,.) - T_u(l,.)\|_{L^2(]0,t_f[)}$ (Corollary 24), for $\|T_{u,h} - T_u\|_{C([0,t_f];L^2(]0,l[))}$ (Theorem 25) and in h for $\|\nabla_x(T_{u,h} - T_u)\|_{L^2(0,t_f;L^2(]0,l[)^2)}$ (Proposition 27). A numerical test is given coroborating our theoretical bounds and we take also this opportunity to show the influence on the solution of our problem of the sum of terms $\psi(T(x,t)) + h_{T,u}(x,t)$, modelizing the participating medium character of the glass into the integro-differential equation $(2)_{(i)}$.

2 Regularity results

At the present time, we know only that $T_u \in \{T \in L^2(0,t_f;H^1(]0,l[)); \dot{T} \in L^2(0,t_f;\left[H^1_L(]0,l[)\right]^*)\}$. But we need more on the exact solution T_u.

Proposition 1 *The right hand-side $(x,t) \mapsto \psi(T_u(x,t)) + h_{T_u,u}(x,t)$ of equation $(2)_{(i)}$ belongs to $L^2(Q)$.*

Proof. As $T_u \in L^2(Q)$, $\psi \circ T_u \in L^2(Q)$ by Corollary 3.3 p.10 of [11]. By the same result $h_{T_u,u}$ belongs also to $L^2(Q)$. ■

Proposition 2 *The right-hand side $t \longmapsto h_c(T_u(l,t) - T_a) + (\Theta(T_u(l,t)) - \Theta(u(t)))$ in the boundary condition at $x = l$ of equation $(2)_{(ii)}$ belongs to $L^2(]0,t_f[)$.*

Proof. As $T_u \in L^2(0,t_f;H^1(]0,l[))$ and $H^1(]0,l[)) \hookrightarrow C([0,l])$, the mapping $t \longmapsto T_u(l,t)$ belongs to $L^2(0,t_f)$. Using Corollary 3.4 p.12 of [11], it follows that the function $t \longmapsto \Theta(T_u(l,t))$ belongs also to $L^2(0,t_f)$. By the same result $\Theta \circ u \in L^2(0,t_f)$. This proves the result. ■

Proposition 3 $T_u \in H^{2,1}(Q) := L^2(]0,t_f[;H^2(]0,l[)) \cap H^1(]0,t_f[;L^2(]0,l[))$. *([9], p.6)*

Proof. From the two previous propositions, it follows that T_u may be seen as the solution of:

$$\left\{\begin{array}{r} \frac{\partial T}{\partial t}(x,t) = a\frac{\partial^2 T}{\partial x^2}(x,t) + f(x,t), \\ 0 < x < l,\ 0 < t < t_f\ , \\ \frac{\partial T}{\partial x}(l,t) = g(t),\ 0 < t < t_f\ ,\ \text{(B.C. at } x = l), \\ T(0,t) = T_a,\ 0 < t < t_f\ ,\ \text{(B.C. at } x = 0), \\ T(x,0) = T_0(x),\ \forall x \in [0,l],\ \text{(I.C. at } t = 0). \end{array}\right. \tag{9}$$

where $a \in \mathbb{R}_+^*$, $f \in L^2(Q)$, $g \in L^2(]0,t_f[)$ and $T_0 \in H^1(]0,l[)$ (see the Introduction). A fortiori $f \in [H^{1/2+2\varepsilon,1/4+\varepsilon}(Q)]'$ and $g \in H^{-\varepsilon}(]0,t_f[)$, for $\varepsilon \in]0,1/4[$. Using the result at the bottom of page 78 of [9] with $s = -1/4 - \varepsilon$, we obtain that $T_u \in H^{3/2-2\varepsilon,3/4-\varepsilon}(Q)$ (multyplying T_u by a cutoff function depending only on the variable x, we reduce us respectively to the case of the Neumann boundary condition on the whole boundary $\{0,l\}$ or respectively to the case of the Dirichlet boundary conditions on the whole boundary $\{0,l\}$ and apply the cited result of [9] in these two cases). Using the "First Trace Theorem" p.9 of [9], we deduce that $T_u(l,.) \in H^{1/2-\varepsilon}(]0,t_f[)$ and thus a fortiori to $H^{1/4}(]0,t_f[)$. $\Theta(.)$ being a Lipschitz continuous function ([11], (3.13) p.17), it follows that

$$\int_0^{t_f}\int_0^{t_f} \frac{|\Theta(T_u(l,t)) - \Theta(T_u(l,s))|^2}{|t-s|^{3/2}} ds \otimes dt \lesssim \int_0^{t_f}\int_0^{t_f} \frac{|T_u(l,t) - T_u(l,s)|^2}{|t-s|^{3/2}} ds \otimes dt < +\infty,$$

which implies that $\Theta(T_u(l,.)) \in H^{1/4}(]0,t_f[)$. We have already seen in the proof of the previous proposition that $\Theta \circ u \in L^2(0,t_f)$. On the other hand $u \in H^1(]0,t_f[)$, so that $\frac{d(\Theta \circ u)}{dt}(t) = \Theta'(u(t))u'(t)$. $\Theta'(T) = \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda \frac{\partial B}{\partial T}(T,\lambda) d\lambda$ and $\left|\frac{\partial B}{\partial T}(T,\lambda)\right| \lesssim \frac{1}{\lambda^4}$ ([11], p.14) imply that $\Theta'(\cdot)$ is a bounded function. Thus $\Theta \circ u \in H^1(0,t_f)$ and a fortiori in $H^{1/4}(]0,t_f[)$. In conclusion, we know now in (9) that $f \in L^2(Q)$ and that $g \in H^{1/4}(]0,t_f[)$. Applying now Theorem 6.2 p.37 of [9] with $r = 0$, we obtain that $T_u \in H^{2,1}(Q)$. What was to be proved. ■

Lemma 4 *Let us consider the mapping*

$$f : Q \to \mathbb{R} : (x,t) \mapsto \psi(T_u(x,t)) + h_{T_u,u}(x,t). \tag{10}$$

Then $\frac{\partial f}{\partial t} \in L^2(Q)$.

Proof. Firstly $\psi(T_u(x,t)) = -\sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T_u(x,t))$. Thus $\frac{\partial(\psi \circ T_u)}{\partial t}(x,t) = -\sum_{k=1}^{k=M} 4\pi\kappa_k \frac{dB_g^k}{dT}(T_u(x,t)) \frac{\partial T_u}{\partial t}(x,t)$. But, as we have proved in [11] page 14, $\frac{dB_g^k}{dT}(\cdot)$ is bounded. By the previous proposition, we know that $\frac{\partial T_u}{\partial t} \in L^2(Q)$. Thus $\frac{\partial(\psi \circ T_u)}{\partial t} \in L^2(Q)$. Secondly, what can we say about $\frac{\partial h_{T_u,u}}{\partial t}$. In view of formula $h_{T_u,u}(x,t) := \sum_{k=1}^{k=M} 2\pi\kappa_k h_{T_u,u}^k(x,t)$ (see (3)), it suffices to prove that $\frac{\partial h_{T_u,u}^k}{\partial t}$

$\in L^2(Q)$. But

$$\begin{aligned}\frac{\partial h^k_{T_u,u}}{\partial t}(x,t) := &\int_0^l G_k(x,x')\ \frac{dB^k_g}{dT}(T_u(x',t))\frac{\partial T_u}{\partial t}(x',t)\ dx' \\ &+\frac{dB^k_g}{dT}(u(t))u'(t)[E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))] \\ &+\kappa_k\int_0^l \Phi_1(\kappa_k(2l-x-x'))\frac{dB^k_g}{dT}(T_u(x',t))\frac{\partial T_u}{\partial t}(x',t)\ \ dx'.\end{aligned} \tag{11}$$

From that formula, $\frac{dB^k_g}{dT}(\cdot)$ bounded, $\frac{\partial T_u}{\partial t} \in L^2(Q)$, Corollary 3.19, Corollary 3.20 of [11] page 23 and $u' \in L^2(]0,t_f[)$, follows that $\frac{\partial h^k_{T_u,u}}{\partial t} \in L^2(Q),\ \forall k = 1,\dots,M$. Thus also $\frac{\partial h_{T_u,u}}{\partial t} \in L^2(Q)$. The proof is complete. ■

Lemma 5 *Let us suppose that $T_0 \in H^2(]0,l[)$. Then, the following initial boundary value problem*

$$\left\{\begin{array}{c} c_p m_g \frac{\partial v}{\partial t}(x,t) = k_h \frac{\partial^2 v}{\partial x^2}(x,t) + \frac{\partial f}{\partial t}(x,t), \\ 0 < x < l,\ 0 < t < t_f\ , \\ -k_h \frac{\partial v}{\partial x}(l,t) = h_c v(l,t) + \Theta'(\int_0^t v(l,s)ds + T_0(l))v(l,t) - \Theta'(u(t))u'(t), \\ 0 < t < t_f\ ,\ \textit{(B.C. at } x = l\textit{)}, \\ v(0,t) = 0,\ 0 < t < t_f\ ,\ \textit{(B.C. at } x = 0\textit{)}, \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h T_0''(x)),\ \forall' x \in]0,l[,\ \textit{(I.C. at } t = 0\textit{)}, \end{array}\right. \tag{12}$$

possesses at least one weak solution. In (12)*, f denotes the function defined by formula* (10)*.*

Proof. Let us firstly precise what we mean by a weak solution of the initial boundary value problem (12). By a weak solution, we mean an element $v \in \{v \in L^2(0,t_f;H^1_L(]0,l[)); \dot v \in L^2(0,t_f;\left[H^1_L(]0,l[)\right]^*)\}$ such that $\forall' t \in]0,t_f[, \forall \varphi \in H^1_L(]0,l[)$:

$$\left\{\begin{array}{c} c_p m_g \frac{d}{dt}\left\langle v(\cdot,t),\varphi\right\rangle_{H^1_L(]0,l[)^*,H^1_L(]0,l[)} = -k_h\int_0^l \frac{\partial v}{\partial x}(x,t)\varphi'(x)dx \\ +\int_0^l \frac{\partial f}{\partial t}(x,t)\varphi(x)dx - h_c v(l,t)\varphi(l) - \Theta'(\int_0^t v(l,s)ds + T_0(l))v(l,t)\varphi(l) \\ +\Theta'(u(t))\dot u(t)\varphi(l), \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h T_0''(x)),\ \forall' x \in]0,l[. \end{array}\right. \tag{13}$$

The homogeneous Dirichlet boundary condition at $x = 0$, is expressed by the fact that $v \in L^2(0,t_f;H^1_L(]0,l[))$ and the initial condition at time $t = 0$ has

sense due to the fact that

$$\{v \in L^2(0,t_f;H_L^1(]0,l[)); \dot{v} \in L^2(0,t_f;\left[H_L^1(]0,l[)\right]^*)\} \hookrightarrow C([0,t_f];L^2(]0,l[)).$$

We know also by the previous lemma that $\frac{\partial f}{\partial t} \in L^2(Q)$ so that $\int_0^l \frac{\partial f}{\partial t}(x,t)\varphi(x)dx$ exists for $\forall' t \in]0,t_f[$. Also by Proposition 1 and Lemma 4, f and $\frac{df}{dt}$ belong to the space $L^2(0,t_f;L^2(]0,l[))$ which implies that $f(\cdot,0) \in L^2(]0,l[)$. Thus the right-hand side in the initial condition of problem (13) belongs to $L^2(]0,l[)$. Let us apply Faedo-Galerkin's method. Let us consider $(w_j)_{j\geq 1}$ an orthonormal basis in $L^2(]0,l[)$ also orthogonal basis in $H_L^1(]0,l[)$. In particular, these basis functions are continuous functions on the closed interval $[0,l]$ and thus bounded. Let us consider the vectorial subspace generated by $w_1,\ldots,w_m$. Let us try to find $v_m(t) = \sum_{j=1}^m (v_m(t)|\ w_j)_{L^2(]0,l[)} w_j$ solution of the following Cauchy problem for the nonlinear system of (integro-) differential equations $\forall j = 1,\ldots,m$:

$$\begin{cases} c_p m_g(\dot{v}_m(t)|w_j)_{L^2(]0,l[)} + k_h(v_m(t)|w_j)_{H_L^1(]0,l[)} = (\frac{\partial f}{\partial t}(\cdot,t)|w_j)_{L^2(]0,l[)} - h_c \\ v_m(t)(l)w_j(l) - \Theta'(\int_0^t v_m(s)(l)ds + T_0(l))v_m(t)(l)w_j(l) + \Theta'(u(t))\dot{u}(t)w_j(l), \\ (v_m(\cdot,0)|w_j)_{L^2(]0,l[)} = \frac{1}{c_p m_g}(f(\cdot,0) + k_h T_0''(\cdot)|w_j)_{L^2(]0,l[)}. \end{cases} \tag{14}$$

As is usual, to prove existence and uniqueness of a solution to a system of differential equations, we tranform this system of differential equations in a system of nonlinear integral equations:

$$\begin{cases} c_p m_g(v_m(t)|w_j)_{L^2(]0,l[)} + k_h \int_0^t (v_m(s)|w_j)_{H_L^1(]0,l[)} ds = (f(\cdot,t)|w_j)_{L^2(]0,l[)} \\ -(f(\cdot,0)|w_j)_{L^2(]0,l[)} - h_c \left(\int_0^t v_m(s)(l)ds\right) w_j(l) - \Theta(\int_0^t v_m(s)(l)ds \\ +T_0(l))w_j(l) + \Theta(T_0(l))w_j(l) + \Theta(u(t))w_j(l) - \Theta(u(0))w_j(l) \\ +c_p m_g(v_m(0)|w_j)_{L^2(]0,l[)},\ \forall j = 1,\ldots,m. \end{cases} \tag{15}$$

Let us note that

$$c_p m_g(v_m(0)|w_j)_{L^2(]0,l[)} = (f(\cdot,0) + k_h T_0''(\cdot)|w_j)_{L^2(]0,l[)},\ \forall j = 1,\ldots,m. \tag{16}$$

As $v_m(s) = \sum_{k=1}^m (v_m(s)|\ w_k)_{L^2(]0,l[)} w_k$,

$$\begin{aligned} k_h(v_m(s)|w_j)_{H_L^1(]0,l[)} &= k_h \sum_{k=1}^m (v_m(s)|w_k)_{L^2(]0,l[)}(w_k|w_j)_{H_L^1(]0,l[)} \\ &= k_h(v_m(s)|w_j)_{L^2(]0,l[)} \left\|w_j\right\|^2_{H_L^1(]0,l[)}. \end{aligned}$$

In particular $k_h \int_0^t (v_m(s)|w_j)_{H_L^1(]0,l[)} ds = k_h \left\|w_j\right\|^2_{H_L^1(]0,l[)} \int_0^t (v_m(s)|w_j)_{L^2(]0,l[)} ds$.
Let us set $v_m^{(j)}(t) := (v_m(t)|w_j)_{L^2(]0,l[)}$, $\forall j = 1, \ldots, m$. The above nonlinear integral equation may be rewritten:

$$
\begin{aligned}
c_p m_g \begin{pmatrix} v_m^{(1)}(t) \\ \vdots \\ v_m^{(m)}(t) \end{pmatrix} = -k_h \begin{pmatrix} \left\|w_1\right\|^2_{H_L^1(]0,l[)} \int_0^t v_m^{(1)}(s) ds \\ \vdots \\ \left\|w_m\right\|^2_{H_L^1(]0,l[)} \int_0^t v_m^{(m)}(s) ds \end{pmatrix} \\
+ \begin{pmatrix} (f(t)|w_1)_{L^2(]0,l[)} \\ \vdots \\ (f(t)|w_m)_{L^2(]0,l[)} \end{pmatrix} - \begin{pmatrix} (f(0)|w_1)_{L^2(]0,l[)} \\ \vdots \\ (f(0)|w_m)_{L^2(]0,l[)} \end{pmatrix} \\
-h_c \left(\sum_{k=1}^m \int_0^t v_m^{(k)}(s) ds \cdot w_k(l) \right) \begin{pmatrix} w_1(l) \\ \vdots \\ w_m(l) \end{pmatrix} \\
-\Theta \left(\sum_{k=1}^m \int_0^t v_m^{(k)}(s) ds \cdot w_k(l) + T_0(l) \right) \begin{pmatrix} w_1(l) \\ \vdots \\ w_m(l) \end{pmatrix} + \left(\Theta\left(T_0(l)\right) + \right. \\
\left. \Theta\left(u(t)\right) - \Theta\left(u(0)\right)\right) \begin{pmatrix} w_1(l) \\ \vdots \\ w_m(l) \end{pmatrix} + \begin{pmatrix} (f(\cdot,0) + k_h T_0''(\cdot)|w_1)_{L^2(]0,l[)} \\ \vdots \\ (f(\cdot,0) + k_h T_0''(\cdot)|w_m)_{L^2(]0,l[)} \end{pmatrix}.
\end{aligned}
\tag{17}
$$

Now, to prove that this system of m nonlinear scalar integral equations in the m unknowns $v_m^{(j)}(\cdot)$, $j = 1, \ldots, m$ possesses one and only one solution, we proceed as in the proof of Cauchy's theorem. Firstly, let us introduce the Banach space $E = C([0, t_f]; \mathbb{R}^m)$ endowed with the sup norm. Let us define the mapping $T : E \to E : (v_m^{(1)}, \ldots, v_m^{(m)}) \mapsto T(v_m^{(1)}, \ldots, v_m^{(m)})$, where $T(v_m^{(1)}, \ldots, v_m^{(m)})$ denotes the tranposed of the right-hand side of equation (17). Now, we are going to prove that some power of the mapping T for the composition law is a contraction. Inequality (3.6) page 13 of [11] about Planck function, implies that $\Theta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Thus there exists a positive constant C such that:

$$
\left| T(v_m^{(1)}, \ldots, v_m^{(m)})\left(t\right) - T(\tilde{v}_m^{(1)}, \ldots, \tilde{v}_m^{(m)})\left(t\right) \right|_{\mathbb{R}^m} \leq C \sum_{j=1}^m \int_0^t \left| v_m^{(j)}(s) - \tilde{v}_m^{(j)}(s) \right| ds.
\tag{18}
$$

Applying Cauchy-Schwarz inequality in $\mathbb{R}^m$, we obtain:

$$\left|T(v_m^{(1)},\dots,v_m^{(m)})(t)-T(\tilde{v}_m^{(1)},\dots,\tilde{v}_m^{(m)})(t)\right|_{\mathbb{R}^m}\leq \\ C\sqrt{m}\int_0^t |v_m(s)-\tilde{v}_m(s)|_{\mathbb{R}^m}\,ds\leq C\sqrt{m}\;t\;\|v_m-\tilde{v}_m\|_E\,. \tag{19}$$

This inequality implies that:

$$\left\|T(v_m^{(1)},\dots,v_m^{(m)})-T(\tilde{v}_m^{(1)},\dots,\tilde{v}_m^{(m)})\right\|_E\leq C\sqrt{m}\;t_f\;\|v_m-\tilde{v}_m\|_E\,. \tag{20}$$

Iterating, we obtain $\forall l\in\mathbb{N}^*$ that

$$\left\|T^{\circ l}(v_m^{(1)},\dots,v_m^{(m)})-T^{\circ l}(\tilde{v}_m^{(1)},\dots,\tilde{v}_m^{(m)})\right\|_E\leq \frac{\left(C\sqrt{m}t_f\right)^l}{l!}\|v_m-\tilde{v}_m\|_E\,.$$

Now $e^{C\sqrt{m}t_f}=\sum_{l=1}^{+\infty}\frac{\left(C\sqrt{m}t_f\right)^l}{l!}$ and as the general term of a convergent series tends to zero, $\frac{\left(C\sqrt{m}t_f\right)^l}{l!}$ tends to zero as $l\to+\infty$. Thus for l sufficiently large $T^{\circ l}:E\to E$ is a contraction and possesses thus one and only one fixed point $(v_m^{(1)},\dots,v_m^{(m)})\in E=C([0,t_f];\mathbb{R}^m)$ which is also the unique fixed point of $T:E\to E$. Thus, we know now that the Cauchy problem for the system of m nonlinear differential equations (14) possesses one and only one solution in the time interval $[0,t_f]$. Now, we have to pass to the limit as $m\to+\infty$. In that purpose, we are going to prove some energy estimate. As $v_m(t)$ is for a fixed t a linear combination of the functions w_j, $j=1,\dots,m$, it follows from (14) that:

$$c_p m_g(\dot{v}_m(t)|v_m(t))_{L^2(]0,l[)}+k_h\,\|v_m(t)\|^2_{H^1_L(]0,l[)}=(\tfrac{\partial f}{\partial t}(\cdot,t)|v_m(t))_{L^2(]0,l[)}$$
$$-h_c v_m(t)(l)^2-\Theta'(\int_0^t v_m(s)(l)ds+T_0(l))v_m(t)(l)^2+\Theta'(u(t))\dot{u}(t)v_m(t)(l).$$

As $\Theta'(\cdot)\geq 0$, it follows from the previous equality, the inequality: $\forall\varepsilon>0$:

$$\tfrac{c_p m_g}{2}\tfrac{d}{dt}\|v_m(t)\|^2_{L^2(]0,l[)}+\left(k_h-l(1+\tfrac{l}{2})\varepsilon^2\right)\|v_m(t)\|^2_{H^1_L(]0,l[)} \\ \leq \tfrac{1}{\varepsilon^2}\left\|\tfrac{\partial f}{\partial t}(\cdot,t)\right\|^2_{L^2(]0,l[)}+\tfrac{1}{\varepsilon^2}\left|(\Theta\circ u)'(t)\right|^2. \tag{21}$$

Choosing in the previous inequality $\varepsilon>0$ sufficiently small for $l(1+\frac{l}{2})\varepsilon^2$ to be smaller than $\frac{k_h}{2}$, we have in particular

$$\frac{c_p m_g}{2}\frac{d}{dt}\|v_m(t)\|^2_{L^2(]0,l[)}\leq\frac{1}{\varepsilon^2}\left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|^2_{L^2(]0,l[)}+\frac{1}{\varepsilon^2}\left|(\Theta\circ u)'(t)\right|^2.$$

Integrating both sides from 0 to t, we obtain: $\forall t \in]0, t_f[$:

$$\begin{aligned}\tfrac{c_p m_g}{2} \left\|v_m(t)\right\|^2_{L^2(]0,l[)} &\leq \tfrac{1}{\varepsilon^2} \int_0^{t_f} \left\| \tfrac{\partial f}{\partial t}(\cdot,t) \right\|^2_{L^2(]0,l[)} dt + \tfrac{1}{\varepsilon^2} \int_0^{t_f} \left|(\Theta \circ u)'(t)\right|^2 dt \\ &\quad + \tfrac{1}{2c_p m_g} \left\| f(\cdot,0) + k_h T_0''(\cdot) \right\|^2_{L^2(]0,l[)} \\ &\leq \tfrac{1}{\varepsilon^2} \left\| \tfrac{\partial f}{\partial t} \right\|^2_{L^2(Q)} + \tfrac{1}{\varepsilon^2} \left\|\Theta'\right\|^2_\infty \left\|u\right\|^2_{H^1(]0,t_f[)} + \tfrac{1}{2c_p m_g} \left\| f(\cdot,0) + k_h T_0''(\cdot) \right\|^2_{L^2(]0,l[)}.\end{aligned}$$

This inequality shows that $\left\|v_m\right\|_{C([0,t_f];L^2(]0,l[))}$ is bounded. Going back to the inequality (21) and integrating both sides from 0 to t_f, we obtain:

$$\begin{aligned}&(k_h - l(1+\tfrac{l}{2})\varepsilon^2) \int_0^{t_f} \left\|v_m(t)\right\|^2_{H^1_L(]0,l[)} dt + \tfrac{c_p m_g}{2} \left\|v_m(t_f)\right\|^2_{L^2(]0,l[)} \\ &\leq \tfrac{1}{\varepsilon^2} \left\| \tfrac{\partial f}{\partial t} \right\|^2_{L^2(Q)} + \tfrac{1}{\varepsilon^2} \int_0^{t_f} \left|(\Theta \circ u)'(t)\right|^2 dt + \tfrac{c_p m_g}{2} \left\|v_m(0)\right\|^2_{L^2(]0,l[)}.\end{aligned}$$

As

$$\left\|v_m(0)\right\|_{L^2(]0,l[)} \leq \frac{1}{c_p m_g} \left\| f(\cdot,0) + k_h T_0'' \right\|_{L^2(]0,l[)}$$

and thus bounded independently of m, it follows from the previous inequality that $\left\|v_m\right\|_{L^2(0,t_f;H^1_L(]0,l[))}$ is also bounded independently of m. Now, we want to show that $(\dot{v}_m)_{m\geq 1}$ is bounded in $L^2(0,t_f;H^1_L(]0,l[)^*)$. Let $w \in H^1_L(]0,l[)$ such that $\left\|w\right\|_{H^1_L(]0,l[)} \leq 1$. Let us decompose w orthogonally into a part $\tilde{w}_1 \in span(w_1,\ldots,w_m)$ and another part $\tilde{w}_2$ belonging to the orthogonal of the finite dimensional vector subspace $span(w_1,\ldots,w_m)$ in $H^1_L(]0,l[)$. This implies that $\left\|\tilde{w}_1\right\|_{H^1_L(]0,l[)} \leq 1$. Now:

$$\begin{aligned}\langle \dot{v}_m(t), w \rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} &= (\dot{v}_m(t)|w)_{L^2(]0,l[)} \\ &= (\dot{v}_m(t)|\tilde{w}_1)_{L^2(]0,l[)} + (\dot{v}_m(t)|\tilde{w}_2)_{L^2(]0,l[)}.\end{aligned}$$

But $\tilde{w}_2 = \sum_{k=m+1}^{+\infty} (w|\frac{w_k}{\left\|w_k\right\|_{H^1_L(]0,l[)}})_{H^1_L(]0,l[)} \frac{w_k}{\left\|w_k\right\|_{H^1_L(]0,l[)}}$, this series being convergent in $H^1_L(]0,l[)$ and thus a fortiori in $L^2(]0,l[)$ as $H^1_L(]0,l[) \hookrightarrow L^2(]0,l[)$. This implies that for $j \in \{1,\ldots,m\}$:

$$(w_j|\tilde{w}_2)_{L^2(]0,l[)} = \sum_{k=m+1}^{+\infty} (w|\frac{w_k}{\left\|w_k\right\|_{H^1_L(]0,l[)}})_{H^1_L(]0,l[)} \frac{(w_j|w_k)_{L^2(]0,l[)}}{\left\|w_k\right\|_{H^1_L(]0,l[)}} = 0.$$

As $\dot{v}_m(t)$ is a linear combination of $w_1,\ldots,w_m$, it follows that $(\dot{v}_m(t)|\tilde{w}_2)_{L^2(]0,l[)} = 0$. Thus $\langle \dot{v}_m(t), w \rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} = (\dot{v}_m(t)|\tilde{w}_1)_{L^2(]0,l[)}$. By the first equation

of (14),

$$\begin{aligned} & c_p m_g \left\langle \dot{v}_m(t), w \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} = c_p m_g (\dot{v}_m(t)|\tilde{w}_1)_{L^2(]0,l[)} \\ = & -k_h (v_m(t)|\tilde{w}_1)_{H_L^1(]0,l[)} + (\tfrac{\partial f}{\partial t}(\cdot,t)|\tilde{w}_1)_{L^2(]0,l[)} - h_c v_m(t)(l)\tilde{w}_1(l) \\ & -\Theta'(\int_0^t v_m(s)(l)ds + T_0(l))v_m(t)(l)\tilde{w}_1(l) + \Theta'(u(t))\dot{u}(t)\tilde{w}_1(l) \\ & \leq C(\left\| \tfrac{\partial f}{\partial t}(\cdot,t) \right\|_{L^2(]0,l[)} + \|v_m(t)\|_{H_L^1(]0,l[)} + |\dot{u}(t)|), \end{aligned}$$

for some constant C independent of m. Replacing w by $-w$, we obtain:

$$c_p m_g \left| \left\langle \dot{v}_m(t), w \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} \right| \leq C(\left\| \frac{\partial f}{\partial t}(\cdot,t) \right\|_{L^2(]0,l[)} + \|v_m(t)\|_{H_L^1(]0,l[)} + |\dot{u}(t)|),$$

for every $w \in H_L^1(]0,l[)$ such that $\|w\|_{H_L^1(]0,l[)} \leq 1$. Thus

$$c_p m_g \|\dot{v}_m(t)\|_{H_L^1(]0,l[)^*} \leq C(\left\| \frac{\partial f}{\partial t}(\cdot,t) \right\|_{L^2(]0,l[)} + \|v_m(t)\|_{H_L^1(]0,l[)} + |\dot{u}(t)|).$$

Integrating both sides from 0 to t_f, we obtain:

$$\int_0^{t_f} \|\dot{v}_m(t)\|^2_{H_L^1(]0,l[)^*} \, dt \leq C(\left\| \frac{\partial f}{\partial t} \right\|^2_{L^2(Q)} + \int_0^{t_f} \|v_m(t)\|^2_{H_L^1(]0,l[)} \, dt + \int_0^{t_f} |\dot{u}(t)|^2 \, dt).$$

Thus $(\dot{v}_m)_{m\geq 1}$ is bounded in $L^2(0,t_f;H_L^1(]0,l[)^*)$ independently of m as $\|v_m\|_{L^2(0,t_f;H_L^1(]0,l[))}$ is bounded independently of m. Thus, there exists a subsequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ of the sequence $(v_m)_{m\geq 1}$ which converges weakly to some element v in $L^2(0,t_f;H_L^1(]0,l[))$, weakly star in $L^\infty(0,t_f;L^2(]0,l[))$ and such that their derivatives $\left(\dot{v}_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converge weakly to $\dot{v}$ in $L^2(0,t_f;H_L^1(]0,l[)^*)$. Let ξ be an arbitrary function belonging to the space $L^2(]0,t_f[)$. $\forall j \in \mathbb{N}^*$:

$$\int_0^{t_f} c_p m_g (\dot{v}_{m_{\tilde{l}}}(t)|w_j)_{L^2(]0,l[)} \xi(t)dt \to \int_0^{t_f} c_p m_g \left\langle \dot{v}(t), w_j \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} \xi(t)dt$$

as $\tilde{l} \to +\infty$ as the mapping

$$L^2(0,t_f;H_L^1(]0,l[)^*) \to \mathbb{R} : \chi \mapsto \int_0^{t_f} c_p m_g \left\langle \chi(t), w_j \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} \xi(t)dt$$

is a continuous linear form on $L^2(0,t_f;H_L^1(]0,l[)^*)$.

$$\int_0^{t_f} k_h (v_{m_{\tilde{l}}}(t)|w_j)_{H_L^1(]0,l[)} \xi(t)dt \to \int_0^{t_f} k_h (v(t)|w_j)_{H_L^1(]0,l[)} \xi(t)dt$$

as the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1_L(]0,l[))$ and the mapping

$$L^2(0,t_f;H^1_L(]0,l[)) \to \mathbb{R} : \chi \mapsto \int_0^{t_f} k_h(\chi(t)|w_j)_{H^1_L(]0,l[)}\xi(t)dt$$

is a continuous linear form on $L^2(0,t_f;H^1_L(]0,l[))$. As the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1_L(]0,l[))$, and the mapping

$$L^2(0,t_f;H^1_L(]0,l[)) \to L^2(]0,t_f[) : w \mapsto w(.,l)$$

is linear and continuous, it follows that the sequence of functions $v_{m_{\tilde{l}}}(\cdot)(l)$ converge weakly to $v(\cdot)(l)$ in $L^2(]0,t_f[)$ as $\tilde{l} \to +\infty$. This implies that

$$-\int_0^{t_f} h_c v_{m_{\tilde{l}}}(t)(l)w_j(l)\xi(t)dt \to -\int_0^{t_f} h_c v(t)(l)w_j(l)\xi(t)dt$$

as $\tilde{l} \to +\infty$. Let us show that:

$$-\int_0^{t_f} \Theta'(\int_0^t v_{m_{\tilde{l}}}(s)(l)ds + T_0(l))v_{m_{\tilde{l}}}(t)(l)w_j(l)\xi(t)dt \to$$
$$-\int_0^{t_f} \Theta'(\int_0^t v(s)(l)ds + T_0(l))v(t)(l)w_j(l)\xi(t)dt$$

as $\tilde{l} \to +\infty$. As for every $t \in]0,t_f[$:

$$\delta_l \otimes \chi_{]0,t[} \in H^1_L(]0,l[)^* \otimes L^2(]0,t_f[)^* \simeq L^2(0,t_f;H^1_L(]0,l[))^*$$

and as the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1_L(]0,l[))$, it follows that $\int_0^t v_{m_{\tilde{l}}}(s)(l)ds \to \int_0^t v(s)(l)ds$ as $\tilde{l} \to +\infty$, $\forall t \in]0,t_f[$. Thus

$$\Theta'(\int_0^t v_{m_{\tilde{l}}}(s)(l)ds + T_0(l)) \to \Theta'(\int_0^t v(s)(l)ds + T_0(l))$$

as $\tilde{l} \to +\infty$, $\forall t \in]0, t_f[$. The sequence of functions $v_{m_{\tilde{l}}}(\cdot)(l)$ converging weakly to $v(\cdot)(l)$ in $L^2(]0, t_f[)$ as $\tilde{l} \to +\infty$ is bounded in $L^2(]0, t_f[)$. Thus:

$$\begin{aligned}\Bigg| \int_0^{t_f} \Theta'\Big(\int_0^t v_{m_{\tilde{l}}}(s)(l)ds + T_0(l)\Big) v_{m_{\tilde{l}}}(t)(l) w_j(l)\xi(t)dt - \\ \int_0^{t_f} \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big) v_{m_{\tilde{l}}}(t)(l) w_j(l)\xi(t)dt \Bigg| \\ \leq |w_j(l)| \Big(\sup_{\tilde{l}\geq 1} \big\| v_{m_{\tilde{l}}}(\cdot)(l) \big\|_{L^2(]0,t_f[)}\Big) \\ \cdot \Big[\int_0^{t_f} \Bigg| \Theta'\Big(\int_0^t v_{m_{\tilde{l}}}(s)(l)ds + T_0(l)\Big) - \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big) \Bigg|^2 |\xi(t)|^2 \, dt\Big]^{1/2},\end{aligned} \tag{22}$$

$\to 0$ as $\tilde{l} \to +\infty$ by Lebesgue bounded convergence theorem. As $\Theta'(\cdot)$ is bounded, the function

$$]0, t_f[\to \mathbb{R} : t \mapsto \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big)$$

is bounded and thus the function

$$]0, t_f[\to \mathbb{R} : t \mapsto \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big)\xi(t)$$

belongs to $L^2(]0, t_f[)$. As the sequence of functions $v_{m_{\tilde{l}}}(\cdot)(l)$ converges weakly to $v(\cdot)(l)$ in $L^2(]0, t_f[)$ as $\tilde{l} \to +\infty$, we have also that

$$\int_0^{t_f} \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big)\xi(t) v_{m_{\tilde{l}}}(t)(l)\Big)dt \to \int_0^{t_f} \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big)\xi(t) v(t)(l)\Big)dt \tag{23}$$

as $\tilde{l} \to +\infty$. In conclusion, from (22) and (23) follows that:

$$\begin{aligned}\int_0^{t_f} \Theta'\Big(\int_0^t v_{m_{\tilde{l}}}(s)(l)ds + T_0(l)\Big) v_{m_{\tilde{l}}}(t)(l) w_j(l)\xi(t)dt \\ \to \int_0^{t_f} \Theta'\Big(\int_0^t v(s)(l)ds + T_0(l)\Big) v(t)(l) w_j(l)\xi(t)dt,\end{aligned}$$

as $\tilde{l} \to +\infty$. We are now in a position to pass to the limit as $\tilde{l} \to +\infty$ in the first equation of (14) and we obtain:

$$\int_0^{t_f} c_p m_g \left\langle \dot{v}(t), w_j \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} \xi(t)dt + \int_0^{t_f} k_h (v(t)|w_j)_{H_L^1(]0,l[)} \xi(t)dt$$
$$= \int_0^{t_f} (\tfrac{\partial f}{\partial t}(\cdot,t)|w_j)_{L^2(]0,l[)} \xi(t)dt - \int_0^{t_f} h_c v(t)(l) w_j(l) \xi(t)dt$$
$$- \int_0^{t_f} \Theta'(\int_0^t v(s)(l)ds + T_0(l)) v(t)(l) w_j(l) \xi(t)dt + \int_0^{t_f} \Theta'(u(t)) \dot{u}(t) w_j(l) \xi(t)dt, \tag{24}$$

$\forall j \in \mathbb{N}^*$ and $\forall \xi \in L^2(]0,t_f[)$. As the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H_L^1(]0,l[))$, and their derivatives $\left(\dot{v}_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ weakly to $\dot{v}$ in $L^2(0,t_f;H_L^1(]0,l[)^*)$, the sequence $\left(v_{m_{\tilde{l}}}(\cdot,0)\right)_{\tilde{l}\geq 1}$ converges weakly to $v(\cdot,0)$ in $L^2(]0,l[)$. From, the second equation of (14) follows that $v(.,0) = \frac{1}{c_p m_g}(f(\cdot,0) + k_h T_0'')$. This fact and (24) prove the existence of at least one weak solution to the initial boundary value problem (12). ■

Lemma 6 *The initial boundary value problem* (12) *possesses one and only one weak solution.*

Proof. The existence of a solution has been proved in the preceding lemma. Thus it remains to prove uniqueness. Let us consider two weak solutions v and $\hat{v}$ of the initial boundary value problem (12). Thus v and $\hat{v}$ satisfies (13). In particular $v(x,0) = \hat{v}(x,0),\ \forall' x \in]0,l[$. We must prove that $v = \hat{v}$. Considering $v - \hat{v}$, we have:

$$c_p m_g \tfrac{d}{dt} \left\langle v(\cdot,t) - \hat{v}(\cdot,t), \varphi \right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} = -k_h \int_0^l \tfrac{\partial (v-\hat{v})}{\partial x}(x,t) \varphi'(x) dx$$
$$-h_c \left(v(l,t) - \hat{v}(l,t)\right) \varphi(l)$$
$$- \left[\Theta'(\int_0^t v(l,s)ds + T_0(l)) v(l,t) - \Theta'(\int_0^t \hat{v}(l,s)ds + T_0(l)) \hat{v}(l,t) \right] \varphi(l).$$

Taking $\varphi = v(\cdot,t) - \hat{v}(\cdot,t)$ for some fixed $t \in]0, t_f[$ in the previous equation, we obtain:

$$\begin{aligned}
&\tfrac{c_p m_g}{2}\tfrac{d}{dt}\int_0^l (v(x,t)-\hat{v}(x,t))^2 dx + k_h \int_0^l (\tfrac{\partial(v-\hat{v})}{\partial x}(x,t))^2 dx + h_c \left(v(l,t)-\hat{v}(l,t)\right)^2 \\
&\qquad = -\Theta'(\int_0^t v(l,s)ds + T_0(l)) \left(v(l,t)-\hat{v}(l,t)\right)^2 \\
&\qquad -\Theta'(\int_0^t v(l,s)ds + T_0(l))\hat{v}(l,t) \left(v(l,t)-\hat{v}(l,t)\right) \\
&\qquad +\Theta'(\int_0^t \hat{v}(l,s)ds + T_0(l))\hat{v}(l,t) \left(v(l,t)-\hat{v}(l,t)\right).
\end{aligned} \tag{25}$$

This equality can be rewritten:

$$\begin{aligned}
&\tfrac{c_p m_g}{2}\tfrac{d}{dt}\int_0^l (v(x,t)-\hat{v}(x,t))^2 dx + k_h \int_0^l (\tfrac{\partial(v-\hat{v})}{\partial x}(x,t))^2 dx + h_c \left(v(l,t)-\hat{v}(l,t)\right)^2 \\
&\qquad +\Theta'(\int_0^t v(l,s)ds + T_0(l)) \left(v(l,t)-\hat{v}(l,t)\right)^2 \\
&= \left[\Theta'(\int_0^t \hat{v}(l,s)ds + T_0(l)) - \Theta'(\int_0^t v(l,s)ds + T_0(l))\right] \hat{v}(l,t) \left(v(l,t)-\hat{v}(l,t)\right).
\end{aligned}$$

Now, it is easy to see that Θ' is a Lipschitz function as Θ'' is bounded. Denoting by C the Lipschitz constant of Θ', we have:

$$\begin{aligned}
&\left[\Theta'(\int_0^t \hat{v}(l,s)ds + T_0(l)) - \Theta'(\int_0^t v(l,s)ds + T_0(l))\right] \hat{v}(l,t) \left(v(l,t)-\hat{v}(l,t)\right) \\
&\qquad \leq C \left|\int_0^t (\hat{v}(l,s)-v(l,s))ds\right| |\hat{v}(l,t)| \, |v(l,t)-\hat{v}(l,t)|.
\end{aligned}$$

Setting $C' = C\,|\hat{v}(l,t)|$, $a = |v(l,t)-\hat{v}(l,t)|$, $b = \left|\int_0^t (\hat{v}(l,s)-v(l,s))ds\right|$ and applying Young's inequality:

$$ab \leq \frac{h_c}{2C'}a^2 + \frac{C'}{2h_c}b^2,$$

we have that:

$$C\left|\int_0^t(\hat{v}(l,s)-v(l,s))ds\right||\hat{v}(l,t)|\,|v(l,t)-\hat{v}(l,t)|=C'ab$$
$$\leq \tfrac{h_c}{2}\left(v(l,t)-\hat{v}(l,t)\right)^2+\tfrac{(C')^2}{2h_c}\left(\int_0^t(\hat{v}(l,s)-v(l,s))ds\right)^2.$$

Thus:

$$\tfrac{c_p m_g}{2}\tfrac{d}{dt}\int_0^l(v(x,t)-\hat{v}(x,t))^2dx+k_h\int_0^l(\tfrac{\partial(v-\hat{v})}{\partial x}(x,t))^2dx+\tfrac{h_c}{2}\left(v(l,t)-\hat{v}(l,t)\right)^2$$
$$+\Theta'(\int_0^t v(l,s)ds+T_0(l))\left(v(l,t)-\hat{v}(l,t)\right)^2$$
$$\leq \tfrac{(C')^2}{2h_c}\left(\int_0^t(\hat{v}(l,s)-v(l,s))ds\right)^2\leq\tfrac{(C')^2}{2h_c}t\int_0^t(\hat{v}(l,s)-v(l,s))^2ds.$$

Let us set $\psi(t)=\int_0^t(\hat{v}(l,s)-v(l,s))^2ds$. We have thus a fortiori the inequality:

$$\frac{c_p m_g}{2}\frac{d}{dt}\int_0^l(v(x,t)-\hat{v}(x,t))^2dx+\frac{h_c}{2}\psi'(t)\leq\frac{C^2t}{2h_c}\hat{v}(l,t)^2\psi(t).$$

Let us integrate both sides of this inequality from 0 to $\tilde{t}$ for an arbitrary $\tilde{t}\in]0,t_f[$. We obtain:

$$\frac{c_p m_g}{2}\int_0^l(v(x,\tilde{t})-\hat{v}(x,\tilde{t}))^2dx+\frac{h_c}{2}\psi(\tilde{t})\leq\frac{C^2}{2h_c}\int_0^{\tilde{t}}t\hat{v}(l,t)^2\psi(t)dt.$$

Setting $\eta\left(\tilde{t}\right)=\int_0^{\tilde{t}}t\hat{v}(l,t)^2\psi(t)dt$, we obtain:

$$\frac{c_p m_g}{2}\int_0^l(v(x,\tilde{t})-\hat{v}(x,\tilde{t}))^2dx+\frac{h_c}{2}\psi(\tilde{t})\leq\frac{C^2}{2h_c}\eta\left(\tilde{t}\right).$$

A fortiori:

$$\frac{h_c}{2}\psi(\tilde{t})\leq\frac{C^2}{2h_c}\eta\left(\tilde{t}\right),\ \forall\tilde{t}\in]0,t_f[.$$

Thus:

$$\eta'\left(\tilde{t}\right) \leq \frac{C^2}{h_c^2}\tilde{t}\hat{v}(l,\tilde{t})^2\eta\left(\tilde{t}\right),\ \forall\tilde{t} \in]0,t_f[.$$

Setting $g(\tilde{t}) := \frac{C^2}{h_c^2}\tilde{t}\hat{v}(l,\tilde{t})^2$, we obtain $\eta'\left(\tilde{t}\right) \leq g(\tilde{t})\eta\left(\tilde{t}\right),\ \forall\tilde{t} \in]0,t_f[$. We have also $\eta\left(0\right) = 0$. Thus by Gronwall's inequality ([6],(ii) p.624): $\eta\left(\tilde{t}\right) = 0,\ \forall\tilde{t} \in]0,t_f[$ i.e. $\int_0^{\tilde{t}} t\hat{v}(l,t)^2\psi(t)dt = 0$. But $\psi(t) = \int_0^t (\hat{v}(l,s) - v(l,s))^2 ds$. Thus:

$$\hat{v}(l,t)^2 \int_0^t (\hat{v}(l,s) - v(l,s))^2 ds = 0,\ \forall' t \in]0,t_f[. \tag{26}$$

By symmetry, we have also that:

$$v(l,t)^2 \int_0^t (\hat{v}(l,s) - v(l,s))^2 ds = 0,\ \forall' t \in]0,t_f[. \tag{27}$$

We want to prove that this implies

$$\hat{v}(l,s) = v(l,s), \forall' s \in [0,t_f]. \tag{28}$$

Let us consider:

$$\bar{t} := \sup\{t \in [0,t_f]; \int_0^t (\hat{v}(l,s) - v(l,s))^2 ds = 0\}.$$

By continuity, we have of course that:

$$\int_0^{\bar{t}} (\hat{v}(l,s) - v(l,s))^2 ds = 0.$$

$\forall' s \in [0,\bar{t}]$, we have thus $\hat{v}(l,s) = v(l,s)$. If $\bar{t} = t_f$, then (28) is proved. If $\bar{t} < t_f$, $\forall t \in]\bar{t},t_f[$: $\int_0^t (\hat{v}(l,s) - v(l,s))^2 ds > 0$. By (26), (27): $\hat{v}(l,t) = 0 = v(l,t)$, $\forall' t \in]\bar{t},t_f[$. Thus (28) is also proved in that case. From (28) and (25) follows that $\forall' t \in]0,t_f[$:

$$\frac{c_p m_g}{2}\frac{d}{dt}\int_0^l (v(x,t) - \hat{v}(x,t))^2 dx + k_h \int_0^l \left(\frac{\partial\left(v - \hat{v}\right)}{\partial x}(x,t)\right)^2 dx = 0.$$

The second term in the left-hand side of the previous equation is of course positive, so that: $\frac{d}{dt}\int_0^l (v(x,t)-\hat{v}(x,t))^2 dx \leq 0$, $\forall' t \in]0,t_f[$. Thus the absolutely continuous function:

$$[0,t_f] \to \mathbb{R}_+ : t \mapsto \int_0^l (v(x,t)-\hat{v}(x,t))^2 dx$$

is decreasing and null at $t=0$. Being positive, this function is null $\forall t \in [0,t_f]$ so that $v(\cdot,t)=\hat{v}(\cdot,t)$ in $L^2(]0,l[)$, $\forall t \in [0,t_f]$. Thus uniqueness is also proved. ■

Proposition 7 *Under the regularity condition on the initial condition T_0 that $T_0 \in H^2(]0,l[)$, the compatibility condition $T_0(0)=T_a$ at $x=0$, and the compatibility condition between the initial condition and the boundary condition (29)$_{(ii)}$ at $x=l$:*

$$-k_h T_0'(l) = h_c\,(T_0(l)-T_a) + \Theta(T_0(l)) - \Theta(u(0)),$$

the unique weak solution T_u of the initial boundary value problem (2) i.e.

$$\left\{\begin{array}{r} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \frac{\partial^2 T}{\partial x^2}(x,t) + \psi(T(x,t)) + h_{T,u}(x,t), \\ 0<x<l,\ 0<t<t_f\ , \\ -k_h \frac{\partial T}{\partial x}(l,t) = h_c(T(l,t)-T_a) + (\Theta(T(l,t)) - \Theta(u(t))), \\ 0<t<t_f\ ,\ \textit{(B.C. at } x=l\textit{)}, \\ T(0,t)=T_a,\ 0<t<t_f\ ,\ \textit{(B.C. at } x=0\textit{)}, \\ T(x,0)=T_0(x),\ \forall x \in [0,l],\ \textit{(I.C. at } t=0\textit{)}, \end{array}\right. \tag{29}$$

satisfies the regularity property: $\frac{dT_u}{dt} \in L^2(]0,t_f[;H^1(]0,l[))$.

Proof. Let us consider the weak solution $v \in \{w \in L^2(0,t_f;H^1_L(]0,l[));\dot{w} \in L^2(0,t_f;\left[H^1_L(]0,l[)\right]^*)\}$ of the initial boundary value problem (12); thus by (13) $\forall \varphi \in H^1_L(]0,l[)$:

$$\left\{\begin{array}{c} c_p m_g \frac{d}{dt}\left\langle v(\cdot,t),\varphi\right\rangle_{H^1_L(]0,l[)^*,H^1_L(]0,l[)} = -k_h \int_0^l \frac{\partial v}{\partial x}(x,t)\varphi'(x)dx + \int_0^l \frac{\partial f}{\partial t}(x,t)\varphi(x)dx \\ -h_c v(l,t)\varphi(l) - \Theta'(\int_0^t v(l,s)ds + T_0(l))v(l,t)\varphi(l) + \Theta'(u(t))\dot{u}(t)\varphi(l) \\ \text{and } v(x,0) = \frac{1}{c_p m_g}(f(x,0)+k_h T_0''(x)),\ \forall' x \in]0,l[, \end{array}\right. \tag{30}$$

where f denotes the function $f : Q \to \mathbb{R} : (x,t) \mapsto \psi(T_u(x,t)) + h_{T_u,u}(x,t)$ (10). We know by the two preceding lemmas, that this initial boundary value problem possesses one and only one weak solution. Let us set:

$$T(x,t) := \int_0^t v(x,s)ds + T_0(x).$$

This implies that $v(x,t) = \frac{\partial T}{\partial t}(x,t)$. Let us integrate both sides of (30) from 0 to t; we obtain:

$$\left\{\begin{array}{l} c_p m_g \int_0^l \frac{\partial T}{\partial t}(x,t)\varphi(x)dx - \int_0^l f(x,0)\varphi(x)dx + \int_0^l k_h T_0'(x)\varphi'(x)dx - k_h T_0'(l)\varphi(l) \\ \quad = -k_h \int_0^l \frac{\partial T}{\partial x}(x,t)\varphi'(x)dx + k_h \int_0^l T_0'(x)\varphi'(x)dx + \int_0^l f(x,t)\varphi(x)dx \\ \quad\quad - \int_0^l f(x,0)\varphi(x)dx - h_c T(l,t)\varphi(l) + h_c T_0(l)\varphi(l) \\ \quad -\Theta(T(l,t))\varphi(l) + \Theta(T_0(l))\varphi(l) + \Theta(u(t))\varphi(l) - \Theta(u(0))\varphi(l). \end{array}\right.$$

Using the compatibility condition:

$$-k_h T_0'(l) = h_c\left(T_0(l) - T_a\right) + \Theta(T_0(l)) - \Theta(u(0)),$$

and $f(x,t) = \psi(T_u(x,t)) + h_{T_u,u}(x,t)$, we obtain after simplifications $\forall t \in]0, t_f[$:

$$\left\{\begin{array}{l} c_p m_g \int_0^l \frac{\partial T}{\partial t}(x,t)\varphi(x)dx + k_h \int_0^l \frac{\partial T}{\partial x}(x,t)\varphi'(x)dx - \int_0^l \psi(T_u(x,t))\varphi(x)dx \\ \quad - \int_0^l h_{T_u,u}(x,t)\varphi(x)dx - \left[\Theta(u(t)) - \Theta(T(l,t))\right]\varphi(l) \\ \quad - \left[h_c(T_a - T(l,t))\right]\varphi(l) = 0,\ \forall \varphi \in H_L^1(]0,l[) \\ \quad T(0,t) = T_a,\ 0 < t < t_f\ ,\ \text{(B.C. at } x = 0), \\ \quad T(x,0) = T_0(x),\ \forall x \in [0,l],\ \text{(I.C. at } t = 0). \end{array}\right. \tag{31}$$

By (7) and (31), we have $\forall t \in]0, t_f[$:

$$\left\{\begin{array}{l} c_p m_g \frac{d}{dt}\left\langle T_u(\cdot,t) - T(\cdot,t), \varphi\right\rangle_{H_L^1(]0,l[)^*, H_L^1(]0,l[)} + k_h \int_0^l \frac{\partial (T_u - T)}{\partial x}(x,t)\varphi'(x)dx \\ \quad + \left[\Theta(T_u(l,t)) - \Theta(T(l,t))\right]\varphi(l) \\ \quad + \left[h_c(T_u(l,t) - T(l,t))\right]\varphi(l) = 0,\ \forall \varphi \in H_L^1(]0,l[) \\ \quad (T_u - T)(0,t) = 0,\ 0 < t < t_f\ ,\ \text{(B.C. at } x = 0), \\ \quad (T_u - T)(x,0) = 0,\ \forall' x \in [0,l],\ \text{(I.C. at } t = 0). \end{array}\right. \tag{32}$$

Considering $\varphi = T_u(\cdot,t) - T(\cdot,t) \in H_L^1(]0,l[)$ for some fixed $t \in]0, t_f[$ in (32), we obtain:

$$\frac{d}{dt}\int_0^l \left(T_u(x,t) - T(x,t)\right)^2 dx \leq 0,\ \forall' t \in]0, t_f[.$$

Thus, the absolutely continuous function

$$t \mapsto \int_0^l (T_u(x,t) - T(x,t))^2 \, dx$$

is a positive decreasing function with respect to the variable t. As this function takes the value 0 for $t = 0$, this function is zero. Thus $T_u = T$ which implies $\frac{dT_u}{dt} = \frac{dT}{dt} = v \in L^2(]0, t_f[; H^1(]0, l[))$. What was to be proved. ■

Corollary 8 *If we suppose moreover that* $T_0 \in H^3(]0, l[)$, *then* $\frac{dT_u}{dt} \in H^{2,1}(Q)$.

Proof. We keep the same notations as in the proof of the preceding proposition. By using formulas (10), (3) and (4) with $t = 0$, deriving with respect to x, using the fact that $B_g^k(T_0) \in H^1(]0, l[)$ and decomposing that function as the sum of an affine function defined on the interval $]0, l[$ and a function in $\mathring{H}^1(]0, l[)$, function that we extend by 0 as a function on the whole real line and using properties of the convolution operator, one can prove that $\frac{df(.,0)}{dx} \in L^2(]0, l[)$. Consequently $\frac{1}{c_p m_g}(f(.,0) + k_h T_0'') \in H^1(]0, l[)$. As $\frac{\partial f}{\partial t} \in L^2(Q)$, by Lemma 4,

$$t \mapsto h_c v(l,t) + \Theta'(\int_0^t v(l,s)ds + T_0(l))v(l,t) - \Theta'(u(t))u'(t)$$

belongs to $L^2(]0, t_f[)$. Thus, the initial boundary value problem (12) is of the form (9) with T_a replaced by 0 in the boundary condition at $x = 0$. Thus by Proposition 3, $\frac{dT_u}{dt} = v$ solution of (30), i.e. weak solution of the initial boundary value problem (12), belongs to $H^{2,1}(Q)$. ■

3 The semi-discrete problem

In this section, we will suppose that the hypotheses of Corollary 8 are satisfied i.e. that $T_0 \in H^3(]0, l[)$, that $T_0(0) = T_a$, and that

$$-k_h T_0'(l) = h_c \left(T_0(l) - T_a\right) + \Theta(T_0(l)) - \Theta(u(0)).$$

This implies that T_u and $\frac{dT_u}{dt}$ belong to $H^{2,1}(Q)$. In particular T_u and $\frac{dT_u}{dt}$ belong to $L^2(]0, t_f[; H^2(]0, l[))$ implying that $T_u \in C([0, t_f]; H^2(]0, l[))$. Let us now define what is the semi-discrete problem corresponding to the exact problem (2). Given $N_h \in \mathbb{N}^*$, let us define the mesh $x_0 = 0 < x_1 < x_2 < \cdots < x_j < \cdots < x_{N_h} = l$ on the interval $[0, l]$. By φ_i $(i = 0, \ldots, N_h)$, let us denote the continuous function on the interval $[0, l]$, affine on each subinterval $[x_{j-1}, x_j]$ $(j = 1, \ldots, N_h)$ which takes the value 1 at the node x_i and 0 at all the other nodes x_j, $j \neq i$. Thus:

$$\varphi_0(x) := \begin{cases} \frac{x_1 - x}{x_1 - x_0} \text{ if } x \in [x_0, x_1], \\ 0 \text{ if } x \in]x_1, x_{N_h}], \end{cases} \tag{33}$$

$$\varphi_i(x) := \begin{cases} 0 \text{ if } x \in [0, x_{i-1}[, \\ \frac{x-x_{i-1}}{x_i-x_{i-1}} \text{ if } x \in [x_{i-1}, x_i[, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} \text{ if } x \in [x_i, x_{i+1}], \\ 0 \text{ if } x \in]x_{i+1}, x_{N_h}], \end{cases} \tag{34}$$

for $i = 1, \ldots, N_h - 1$, and

$$\varphi_{N_h}(x) := \begin{cases} 0 \text{ if } x \in [x_0, x_{N_h-1}], \\ \frac{x-x_{N_h-1}}{x_{N_h}-x_{N_h-1}} \text{ if } x \in [x_{N_h-1}, x_{N_h}]. \end{cases} \tag{35}$$

Every function φ_i $(i = 0, \ldots, N_h)$ belongs to the space $H^1(]0, l[)$. Let us denote by X_h the finite dimensional vector subspace of $H^1(]0, l[)$ generated by the functions φ_i $(i = 0, \ldots, N_h)$. The semi-discrete problem is the following: find

$$T_{u,h}(x_j, \cdot) \in C^1([0, t_f]; \mathbb{R}), \ j = 1, \ldots, N_h$$

satisfying the initial conditions

$$T_{u,h}(x_j, 0) = T_0(x_j), \ \forall j = 1, \ldots, N_h \tag{36}$$

such that the function

$$T_{u,h} : [0, t_f] \to X_h : t \mapsto T_{u,h}(\cdot, t) = T_a\varphi_0 + \sum_{j=1}^{N_h} T_{u,h}(x_j, t)\varphi_j \tag{37}$$

satisfies on the time interval $[0, t_f]$ the system of differential equations:

$$\begin{gathered} c_p m_g \sum_{j=1}^{N_h} \tfrac{dT_{u,h}}{dt}(x_j, t) \int_0^l \varphi_j(x)\varphi_k(x)dx + T_a k_h \int_0^l \varphi_0'(x)\varphi_k'(x)dx \\ + \sum_{j=1}^{N_h} T_{u,h}(x_j, t) k_h \int_0^l \varphi_j'(x)\varphi_k'(x)dx - \int_0^l \psi(T_a\varphi_0(x) + \sum_{j=1}^{N_h} T_{u,h}(x_j, t) \\ \int_0^l \varphi_j(x)\varphi_k(x)dx - \int_0^l h_{T_a\varphi_0 + \sum_{j=1}^{N_h} T_{u,h}(x_j,t)\varphi_j, u}(x, t)\varphi_k(x)dx \\ + [\Theta(T_{u,h}(l, t)) - \Theta(u(t))]\delta_{k,N_h} \\ + h_c \left(T_{u,h}(l, t) - T_a\right) \delta_{k,N_h} = 0, \ \forall k = 1, \ldots, N_h. \end{gathered} \tag{38}$$

A more "functional analytic" equivalent definition traced on the definition of what is a weak solution (7) of the exact problem (2) is given below by formula (47). Let us set

$$\alpha^h(t) = (T_{u,h}(x_1, t), \ldots, T_{u,h}(x_{N_h}, t)). \tag{39}$$

Let us define the square matrix of order N_h, $M^h = (m^h_{j,k})_{1 \le j,k \le N_h}$ by $m^h_{j,k} = \int_0^l \varphi_j(x)\varphi_k(x)dx$, $\forall j, k = 1, \ldots, N_h$ and the related matrix $A^h = c_p m_g M^h$. We

will need also in the following, the square matrix $B^h = (b^h_{j,k})_{1\leq j,k\leq N_h}$ of order N_h by $b^h_{j,k} = k_h \int_0^l \varphi'_j(x)\varphi'_k(x)dx$, $\forall j,k = 1,\ldots,N_h$. These three square matrices of order N_h are tridiagonal, symmetric, positive definite and thus invertible. In the particular case of the equidistributed grid $x_j = jh$, $j = 0,1,\ldots,N_h$ on the interval $[0,l]$ with $h = \frac{l}{N_h}$, an easy computation shows that

$$m^h_{i,i} = \frac{2h}{3},\ m^h_{i,i+1} = m^h_{i+1,i} = \frac{h}{6},\ \forall i = 1,\ldots,N_h-1, m^h_{N_h,N_h} = \frac{h}{3}$$

and

$$b^h_{i,i} = \frac{2k_h}{h},\ b^h_{i,i+1} = b^h_{i+1,i} = \frac{-k_h}{h},\ \forall i = 1,\ldots,N_h-1, b^h_{N_h,N_h} = \frac{k_h}{h}.$$

Now, let us introduce the nonlinear function

$$\widetilde{f}^h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} : \alpha^h = (\alpha_1,\ldots,\alpha_{N_h}) \mapsto \widetilde{f}^h(\alpha^h) = (\widetilde{f}^h_1(\alpha^h),\ldots,\widetilde{f}^h_{N_h}(\alpha^h))$$

defined as follows:

$$\begin{aligned}
\widetilde{f}^h_k(\alpha^h) = \int_0^l \psi(T_a\varphi_0(x) + \sum_{j=1}^{N_h}\alpha_j\varphi_j(x))\varphi_k(x)dx \\
+\sum_{k^*=1}^{M} 2\pi\kappa_{k^*}\int_0^l [\int_0^l G_{k^*}(x,x')B_g^{k^*}(T_a\varphi_0(x') + \sum_{j=1}^{N_h}\alpha_j\varphi_j(x'))dx']\varphi_k(x)dx + \\
\sum_{k^*=1}^{M} 2\pi\kappa^2_{k^*}\int_0^l [\int_0^l \Phi_1(\kappa_{k^*}(2l-x-x'))B_g^{k^*}(T_a\varphi_0(x') + \sum_{j=1}^{N_h}\alpha_j\varphi_j(x'))dx']\varphi_k(x)dx \\
-\Theta\left(\alpha_{N_h}\right)\delta_{k,N_h} - h_c\alpha_{N_h}\delta_{k,N_h},\ \forall k = 1,\ldots,N_h.
\end{aligned} \tag{40}$$

Lemma 9 *The nonlinear mapping*

$$\widetilde{f}^h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} : \alpha^h = (\alpha_1,\ldots,\alpha_{N_h}) \mapsto \widetilde{f}^h(\alpha^h)$$

is lipschitzian.

Proof. By ([11], p.14), the first order derivative of the Planck function with respect to the absolute temperature T satisfies the bounds: $0 \leq \frac{\partial B}{\partial T}(T,\lambda) \leq \frac{2C_1}{C_2\lambda^4}$. This inequality implies that the first order derivatives $\left(B_g^{k^*}\right)'$ and Θ' are bounded which in turn implies that the first order partial derivatives $\frac{\partial \widetilde{f}^h}{\partial \alpha_j}$ $(j = 1,\ldots,N_h)$ are also bounded. From this last fact follows that $\widetilde{f}^h$ is lipschitzian. ■

We also define the nonlinear function $\widetilde{g}^h : \mathbb{R} \to \mathbb{R}^{N_h} : u \mapsto \widetilde{g}^h(u) = (\widetilde{g}_1(u), \ldots, \widetilde{g}_{N_h}(u))$ as follows:

$$\begin{aligned} \widetilde{g}_k(u) = \sum_{k^*=1}^{M} 2\pi\kappa_{k^*} \int_0^l [E_2(\kappa_{k^*}(l-x)) - \Phi_2(\kappa_{k^*}(l-x))]\varphi_k(x)dx \; B_g^{k^*}(u) \\ +\Theta(u)\,\delta_{k,N_h},\ \forall k = 1, \ldots, N_h, \end{aligned} \tag{41}$$

and the vector $C^h = (c_k^h)_{1 \le k \le N_h}$ of $\mathbb{R}^{N_h}$ by

$$\begin{aligned} c_k^h = -k_h T_a \int_0^l \varphi_0'(x))\varphi_k'(x)dx + \sum_{k^*=1}^{M} 2\pi\kappa_{k^*} B_g^{k^*}(T_a) \int_0^l E_2(\kappa_{k^*}x)\varphi_k(x)dx \\ + \sum_{k^*=1}^{M} 2\pi\kappa_{k^*} B_g^{k^*}(T_a) \int_0^l \Phi_2(\kappa_{k^*}(2l-x))\varphi_k(x)dx + \\ +h_c T_a\ \delta_{k,N_h},\ \forall k = 1, \ldots, N_h. \end{aligned} \tag{42}$$

It is easy to see that the nonlinear function $\widetilde{g}^h : \mathbb{R} \to \mathbb{R}^{N_h}$ is Lipschitz-continuous. Let us still introduce the initial condition, the vector

$$T_0^h := (T_0(x_j))_{1 \le j \le N_h} \tag{43}$$

of $\mathbb{R}^{N_h}$. Having introduced these notations, in particular (39) and (43), we may rewrite the semi-discrete problem (38) in the following matrix differential form:

$$\begin{cases} A^h \dot{\alpha}^h(t) + B^h \alpha^h(t) = \tilde{f}^h(\alpha^h(t)) + \tilde{g}^h(u(t)) + C^h,\ \forall t \in [0, t_f], \\ \alpha^h(0) = T_0^h. \end{cases} \tag{44}$$

Proposition 10 *The nonlinear Cauchy problem (44) possesses one and only one solution.*

Proof. Let us set

$$F : [0, t_f] \times \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} : (t, y) \mapsto (A^h)^{-1}[-B^h y + \tilde{f}^h(y) + \tilde{g}^h(u(t)) + C^h].$$

F is a continuous function, which by Lemma 9 is moreover lipschitzian with respect to the y variable uniformly in t. Applying the Cauchy-Lipschitz Theorem ([4], p. 65), the result follows. ■

As an immediate corollary, we have:

Corollary 11 *The semi-discrete problem defined by the conditions (36),(37) and (38) possesses one and only one solution.*

Let us set:

$$S_h = span < \varphi_1, \ldots, \varphi_{N_h} > . \tag{45}$$

It is a vectorial subspace of dimension N_h of $H^1_L(]0,l[)$. Formula (37) may be rewritten:

$$T_{u,h}(t) = T_a + \sum_{j=1}^{N_h} (T_{u,h}(x_j,t) - T_a)\varphi_j, \tag{46}$$

which shows us that $\forall t \in [0,t_f]$, $T_{u,h}(t) \in T_a + S_h$. The semi-discrete problem defined by the conditions (36),(37) and (38) may be rewritten in the following form: find $T_{u,h} \in C^1([0,t_f];T_a+S_h)$ such that $\forall t \in [0,t_f]$:

$$\left\{\begin{array}{c} c_p m_g \int_0^l \frac{\partial T_{u,h}}{\partial t}(x,t)\chi(x)dx + k_h \int_0^l \frac{\partial T_{u,h}}{\partial x}(x,t)\chi'(x)dx \\ -\int_0^l \psi(T_{u,h}(x,t))\chi(x)dx - \int_0^l h_{T_{u,h}(.,t),u}(x,t)\chi(x)dx \\ +\left[\Theta(T_{u,h}(l,t)) - \Theta(u(t))\right]\chi(l) + \left[h_c\left(T_{u,h}(l,t) - T_a\right)\right]\chi(l) = 0,\ \forall \chi \in S_h, \\ T_{u,h}(0,t) = T_a,\ \forall t \in [0,t_f], \\ T_{u,h}(.,0) = I_h(T_0). \end{array}\right. \tag{47}$$

In formula (47), I_h denotes the Lagrange interpolation operator at the N_h+1 points $x_0 = 0,\ x_1,\ x_2, \ldots,\ x_{N_h} = l$ by continuous functions, which are polynomials of degree at most 1 in each interval $]x_j, x_{j+1}[$, $\forall j = 0,\ldots,N_h-1$. Our purpose is now to establish, an a priori error estimate. In that purpose, let us firstly write the error equation that is the equation for the error $e := T_{u,h} - T_u$:

Proposition 12 *The error $e := T_{u,h} - T_u$ on $T_{u,h}$ satisfies for $\forall'\ t \in]0,t_f[$:*

$$\left\{\begin{array}{c} c_p m_g \frac{d}{dt}\left\langle e(\cdot,t),\chi\right\rangle_{H^1_L(]0,l[)^*,H^1_L(]0,l[)} + k_h \int_0^l \frac{\partial e}{\partial x}(x,t)\chi'(x)dx \\ -\int_0^l \left[\psi(T_u(x,t)+e(x,t)) - \psi(T_u(x,t))\right]\chi(x)dx \\ -\int_0^l \left[h_{T_u(.,t)+e(.,t),u}(x,t) - h_{T_u(.,t),u}(x,t)\right]\chi(x)dx \\ +\left[\Theta(T_u(l,t)+e(l,t)) - \Theta(T_u(l,t))\right]\chi(l) + h_c e(l,t)\chi(l) = 0,\ \forall \chi \in S_h, \\ e(0,t) = 0,\ \forall t \in [0,t_f], \\ e(.,0) = I_h(T_0) - T_0. \end{array}\right. \tag{48}$$

Proof. Equation $(48)_{(i)}$ follows by taking the difference between equation $(47)_{(i)}$ and equation $(7)_{(i)}$ which are both valid for every $\chi \in S_h$. We have proved in [11], that $T_u \in C(\bar{Q})$ which implies that $T_u(0,.) \in C([0,t_f])$. Thus $T_u(0,t) = T_a$ for all $t \in [0,t_f]$. Taking the difference between equation $(47)_{(ii)}$ and equation $(7)_{(ii)}$, we obtain $e(0,t) = 0,\ \forall t \in [0,t_f]$ (not only for almost

every $t \in [0, t_f]$). Finally, taking the difference between equation $(47)_{(iii)}$ and equation $(7)_{(iii)}$, we obtain $e(.,0) = I_h(T_0) - T_0$ (this equality being true at every point $x \in [0, l]$, not only almost everywhere, as both sides are continuous functions). ■

For an arbitrary t fixed in the interval $[0, t_f]$, let us introduce the elliptic projection $\tilde{T}_{u,h}(.,t)$ of $T_u(.,t)$ defined by: $\tilde{T}_{u,h}(.,t) \in T_a + S_h$, and satisfies:

$$\int_0^l \frac{\partial \tilde{T}_{u,h}}{\partial x}(x,t)\chi'(x)dx = \int_0^l \frac{\partial T_u}{\partial x}(x,t)\chi'(x)dx, \ \forall \chi \in S_h. \tag{49}$$

In particular $\tilde{T}_{u,h}(0,t) = T_a$. Also $T_u(0,t) = T_a$. Thus $T_u(.,t) - T_a$ and $\tilde{T}_{u,h}(.,t) - T_a$ both belong to $H^1_L(]0,l[)$. Equation (49) may be rewritten in the equivalent form:

$$\int_0^l \frac{\partial \left(\tilde{T}_{u,h} - T_a\right)}{\partial x}(x,t)\chi'(x)dx = \int_0^l \frac{\partial \left(T_u - T_a\right)}{\partial x}(x,t)\chi'(x)dx, \ \forall \chi \in S_h. \tag{50}$$

Equation (50), shows us that $\tilde{T}_{u,h}(.,t) - T_a$ is the orthogonal projection of $T_u(.,t) - T_a$ onto $S_h \subset H^1_L(]0,l[)$ for the scalar product of $H^1_L(]0,l[)$. As already said in the introduction, if we would have a Dirichlet boundary condition at the point $x = l$, then the technique of elliptic projection is well known ([19], p.7). But in our case, we have the nonlinear Robin boundary condition $(2)_{(ii)}$ at the point $x = l$. The key property of our elliptic projection (49), which will allow us to prove optimal a priori error estimates is that

$$\tilde{T}_{u,h}(l,t) = T_u(l,t),$$

a property that we now establish:

Proposition 13 *Equation (49) is equivalent to* $\tilde{T}_{u,h}(l,t) = T_u(l,t)$ *and*

$$\int_0^l \frac{\partial \tilde{T}_{u,h}}{\partial x}(x,t)\chi'(x)dx = \int_0^l \frac{\partial T_u}{\partial x}(x,t)\chi'(x)dx, \forall \chi \in \mathring{S}_h,$$

where $\mathring{S}_h := \{\chi \in S_h; \chi(l) = 0\}$.

Proof. Let us consider the function

$$\chi :]0,l[\to \mathbb{R} : x \mapsto \frac{x}{l}.$$

This function $\chi \in S_h$ and $\chi(l) = 1$. Also $\chi'(x) = \frac{1}{l}$, $\forall x \in]0,l[$. Equation (49) with that function χ gives us:

$$\frac{1}{l}(\tilde{T}_{u,h}(l,t) - T_a) = \frac{1}{l}(T_u(l,t) - T_a)$$

and thus $\tilde{T}_{u,h}(l,t) = T_u(l,t)$. The equivalence with equation (49) follows from the fact that the finite dimensional vectorial subspace S_h defined by formula (45) is also equal to $span\,\langle \varphi_1, \ldots, \varphi_{N_h-1}, \chi \rangle$ and also that $span\,\langle \varphi_1, \ldots, \varphi_{N_h-1} \rangle = \mathring{S}_h$. ■

Remark 14 *The above definition and properties of the elliptic projection of $T_u(.,t)$ extends to an arbitrary function $T \in H^1(]0,l[)$ by replacing T_a by $T(0)$. In particular, we will speak in the next corollary of the elliptic projection of $\frac{dT_u}{dt}(.,t)$ for almost every $t \in [0,t_f]$; in that case $\frac{dT_u}{dt}(.,t)(0) = 0$ and T_a must be replaced by 0.*

Now, we split the error $e := T_{u,h} - T_u$ on $T_{u,h}$, into two terms:

$$e = \theta + \rho \tag{51}$$

where:

$$\begin{cases} \theta = T_{u,h} - \tilde{T}_{u,h}, \\ \rho = \tilde{T}_{u,h} - T_u. \end{cases} \tag{52}$$

θ verifies the following equations:

Proposition 15 *For almost every $t \in [0,t_f]$, we have $\forall \chi \in S_h$:*

$$\begin{cases} c_p m_g \left\langle \theta_t(\cdot,t), \chi \right\rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} + k_h \displaystyle\int_0^l \frac{\partial \theta}{\partial x}(x,t)\chi'(x)dx - \int_0^l [\psi(T_{u,h}(x,t)) \\ -\psi(\tilde{T}_{u,h}(x,t))]\chi(x)dx - \displaystyle\int_0^l [h_{T_{u,h}(.,t),u}(x,t) - \\ h_{\tilde{T}_{u,h}(.,t),u}(x,t)]\chi(x)dx + \left[\Theta(T_{u,h}(l,t)) - \Theta(\tilde{T}_{u,h}(l,t))\right]\chi(l) + h_c\theta(l,t)\chi(l) \\ = -c_p m_g \left\langle \rho_t(\cdot,t), \chi \right\rangle_{H^1_L(]0,l[)^*, H^1_L(]0,l[)} + \displaystyle\int_0^l [\psi(\tilde{T}_{u,h}(x,t)) \\ -\psi(T_u(x,t))]\chi(x)dx + \displaystyle\int_0^l \left[h_{\tilde{T}_{u,h}(.,t),u}(x,t) - h_{T_u(.,t),u}(x,t)\right]\chi(x)dx, \\ \\ \theta(0,t) = 0, \ \forall t \in [0,t_f], \\ \theta(.,0) = T_{u,h}(\cdot,0) - \tilde{T}_{u,h}(\cdot,0). \end{cases} \tag{53}$$

Proof. The first equation follows by splitting the terms in the error equation (48) using the decomposition $e = \theta + \rho$ of the error e, from equation (49) which implies that $\displaystyle\int_0^l \frac{\partial \rho}{\partial x}(x,t)\chi'(x)dx = 0$, $\forall \chi \in S_h$ and $\tilde{T}_{u,h}(l,t) = T_u(l,t)$. The second one follows from the fact that $T_{u,h} - T_a \in C([0,t_f]; S_h)$ and also

$\tilde{T}_{u,h} - T_a$ due to the fact that $T_u \in C([0,t_f];H^1(]0,l[))$ and equation (50). Thus $\theta(0,t) = T_{u,h}(0,t) - \tilde{T}_{u,h}(0,t)$ is null for every $t \in [0,t_f]$. The third equation has sense due to the continuity of $T_{u,h}$ and $\tilde{T}_{u,h}$ and is trivial. ■

Corollary 16 *For almost every $t \in]0,t_f[$, $\theta(.,t)$ verifies the following inequality:*

$$\begin{gathered}\frac{c_p m_g}{2}\frac{d}{dt}\left\|\theta(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h\left\|\nabla_x\theta(\cdot,t)\right\|^2_{L^2(]0,l[)^2} + h_c\theta(l,t)^2 \leq c_p m_g \\ \cdot\left\|\rho_t(\cdot,t)\right\|_{L^2(]0,l[)}\left\|\theta(\cdot,t)\right\|_{L^2(]0,l[)} + (L_h + L_\psi)\left\|\rho(\cdot,t)\right\|_{L^2(]0,l[)}\left\|\theta(\cdot,t)\right\|_{L^2(]0,l[)} \\ + L_h\left\|\theta(\cdot,t)\right\|^2_{L^2(]0,l[)},\end{gathered} \tag{54}$$

where L_ψ and L_h denote the Lipschitz constants of ψ and of the mapping $L^2(]0,l[) \to L^2(]0,l[) : T \mapsto h_{T,u}$.

Proof. Firstly, let us observe that by equation (50), $\frac{d}{dt}\tilde{T}_{u,h}(.,t) = \widetilde{\left(\frac{dT_u}{dt}(.,t)\right)}$ and that being the orthogonal projection of $\frac{dT_u}{dt}(.,t)$ onto S_h, the mapping $t \mapsto \frac{d}{dt}\tilde{T}_{u,h}(.,t)$ belongs to $L^2(]0,t_f[;S_h)$. Also, $t \mapsto \frac{d}{dt}T_{u,h}(.,t)$ belongs to $L^2(]0,t_f[;S_h)$. Thus $t \mapsto \frac{d\theta}{dt}(.,t)$ belongs to $L^2(]0,t_f[;S_h)$. Also $t \mapsto \frac{d\rho}{dt}(.,t)$ belongs to $L^2(]0,t_f[;H^1(]0,l[))$. Putting $\chi = \theta(\cdot,t)$ in equality (53), we obtain:

$$\begin{gathered}\frac{c_p m_g}{2}\frac{d}{dt}\left\|\theta(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h\int_0^l \frac{\partial\theta}{\partial x}(x,t)^2dx + h_c\theta(l,t)^2 \\ -\int_0^l\left[\psi(T_{u,h}(x,t)) - \psi(\tilde{T}_{u,h}(x,t))\right]\theta(x,t)dx + \left[\Theta(T_{u,h}(l,t)) - \Theta(\tilde{T}_{u,h}(l,t))\right] \\ \cdot\theta(l,t) \leq \left|\int_0^l\left[h_{T_{u,h}(.,t),u}(x,t) - h_{\tilde{T}_{u,h}(.,t),u}(x,t)\right]\theta(x,t)dx\right| \\ + \left|\int_0^l\left[h_{\tilde{T}_{u,h}(.,t),u}(x,t) - h_{T_u(.,t),u}(x,t)\right]\theta(x,t)dx\right| \\ + c_p m_g\left\|\rho_t(\cdot,t)\right\|_{L^2(]0,l[)}\left\|\theta(\cdot,t)\right\|_{L^2(]0,l[)} \\ + \left|\int_0^l\left[\psi(\tilde{T}_{u,h}(x,t)) - \psi(T_u(x,t))\right]\theta(x,t)dx\right|.\end{gathered} \tag{55}$$

Now, $\psi(T) := -\sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T)$ and thus the mapping $T \longmapsto -\psi(T)$ is an increasing function of T. Thus

$$\begin{aligned}-\left[\psi(T_{u,h}(x,t)) - \psi(\tilde{T}_{u,h}(x,t))\right]\theta(x,t) = \left[(-\psi)(T_{u,h}(x,t)) - (-\psi)(\tilde{T}_{u,h}(x,t))\right] \\ \cdot(T_{u,h}(x,t) - \tilde{T}_{u,h}(x,t))\end{aligned}$$

is always positive. In the same way, as $\Theta(T) := \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T,\lambda) d\lambda$, it is also an increasing function of T, which implies the positivity of the term

$$\left[\Theta(T_{u,h}(l,t)) - \Theta(\tilde{T}_{u,h}(l,t))\right]\theta(l,t) = \left[\Theta(T_{u,h}(l,t)) - \Theta(\tilde{T}_{u,h}(l,t))\right] \cdot (T_{u,h}(l,t) - \tilde{T}_{u,h}(l,t)).$$

We are thus allowed to drop the last two terms in the left hand side of inequality (55). Now by Corollary 3.7 page 14 and Lemma 3.18 page 22 of [11] and formula (4), it follows that the mapping $L^2(]0,l[) \to L^2(]0,l[) : T \mapsto h_{T,u}$ is lipschitzian. Calling its Lipschitz constant L_h, it follows that:

$$\begin{aligned} &\left| \int_0^l \left[h_{T_{u,h}(.,t),u}(x,t) - h_{\tilde{T}_{u,h}(.,t),u}(x,t)\right]\theta(x,t)dx \right| \leq \\ &\qquad \left\| h_{T_{u,h}(.,t),u}(\cdot,t) - h_{\tilde{T}_{u,h}(.,t),u}(\cdot,t) \right\|_{L^2(]0,l[)} \|\theta(\cdot,t)\|_{L^2(]0,l[)} \\ &\leq L_h \left\| T_{u,h}(\cdot,t) - \tilde{T}_{u,h}(\cdot,t) \right\|_{L^2(]0,l[)} \|\theta(\cdot,t)\|_{L^2(]0,l[)} = L_h \|\theta(\cdot,t)\|^2_{L^2(]0,l[)}. \end{aligned} \tag{56}$$

Proceeding analogously, we also have the bound:

$$\begin{aligned} &\left| \int_0^l \left[h_{\tilde{T}_{u,h}(.,t),u}(x,t) - h_{T_u(.,t),u}(x,t)\right]\theta(x,t)dx \right| \\ &\qquad \leq L_h \|\rho(\cdot,t)\|_{L^2(]0,l[)} \|\theta(\cdot,t)\|_{L^2(]0,l[)}. \end{aligned} \tag{57}$$

By Lemma 3.6 page 13 of [11] and formula (3), it follows that the mapping $\mathbb{R} \to \mathbb{R} : T \longmapsto \psi(T)$ is Lipschitzian. Calling its Lipschitz constant L_ψ, it follows that:

$$\left| \int_0^l \left[\psi(\tilde{T}_{u,h}(x,t)) - \psi(T_u(x,t))\right]\theta(x,t)dx \right| \leq L_\psi \|\rho(\cdot,t)\|_{L^2(]0,l[)} \|\theta(\cdot,t)\|_{L^2(]0,l[)}. \tag{58}$$

From (55) and inequalities (56) to (58), follows inequality (54). What was to be proved. ■

We can not a priori absorb the term $L_h \|\theta(\cdot,t)\|^2_{L^2(]0,l[)}$ in the right-hand side of inequality (54) of our previous corollary, by a fraction of a term in the left-hand side. To remedy to that, we multiply both sides of inequality (54), by $\exp(2\lambda t)$ for $\lambda = -\frac{L_h}{c_p m_g}$. Let us set:

$$\tilde{\theta}(\cdot,t) := \exp(\lambda t)\theta(\cdot,t) \text{ and } \tilde{\rho}(\cdot,t) := \exp(\lambda t)\rho(\cdot,t). \tag{59}$$

Proposition 17 *The following bound holds:*

$$\begin{aligned} c_p m_g \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + h_c \int_0^t \tilde{\theta}(l,s)^2 ds \\ \leq c_p m_g \left\|\tilde{\theta}(\cdot,0)\right\|^2_{L^2(]0,l[)} + 2\frac{(c_p m_g C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds \\ +2\frac{(L_b C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds, \end{aligned} \tag{60}$$

where $L_b := 2L_h + L_\psi$.

Proof. Multiplying both sides of inequality (54) by $\exp(2\lambda t)$, and using

$$\exp(\lambda t)\theta_t(\cdot,t) = \partial_t(\exp(\lambda t)\theta(\cdot,t)) - \lambda \exp(\lambda t)\theta(\cdot,t),$$

as well as

$$\exp(\lambda t)\rho_t(\cdot,t) = \partial_t(\exp(\lambda t)\rho(\cdot,t)) - \lambda \exp(\lambda t)\rho(\cdot,t),$$

we obtain:

$$\begin{aligned} \tfrac{c_p m_g}{2} \tfrac{d}{dt} \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h \left\|\nabla_x \tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)^2} + h_c \tilde{\theta}(l,t)^2 \\ \leq c_p m_g \|\tilde{\rho}_t(\cdot,t)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,t)\right\|_{L^2(]0,l[)} + (L_h + L_\psi + |\lambda|\, c_p m_g) \|\tilde{\rho}(\cdot,t)\|_{L^2(]0,l[)} \\ \cdot \left\|\tilde{\theta}(\cdot,t)\right\|_{L^2(]0,l[)} + (L_h + \lambda c_p m_g) \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)}. \end{aligned}$$

Taking $\lambda = -\frac{L_h}{c_p m_g}$, we have thus $L_h + \lambda c_p m_g = 0$. The previous inequality becomes:

$$\begin{aligned} \tfrac{c_p m_g}{2} \tfrac{d}{dt} \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h \left\|\nabla_x \tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)^2} + h_c \tilde{\theta}(l,t)^2 \\ \leq c_p m_g \|\tilde{\rho}_t(\cdot,t)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,t)\right\|_{L^2(]0,l[)} + L_b \|\tilde{\rho}(\cdot,t)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,t)\right\|_{L^2(]0,l[)}. \end{aligned} \tag{61}$$

Integrating both sides of the previous inequality from 0 to t, we obtain:

$$\begin{aligned} \tfrac{c_p m_g}{2} \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + h_c \int_0^t \tilde{\theta}(l,s)^2 ds \\ \leq \tfrac{c_p m_g}{2} \left\|\tilde{\theta}(\cdot,0)\right\|^2_{L^2(]0,l[)} + c_p m_g \int_0^t \|\tilde{\rho}_t(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds \\ +L_b \int_0^t \|\tilde{\rho}(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds. \end{aligned} \tag{62}$$

Now, let us apply "Cauchy's inequality with δ": $\forall a \in \mathbb{R}_+$, $\forall b \in \mathbb{R}_+$, $\forall \delta \in \mathbb{R}_+^*$: $ab \leq \delta a^2 + \frac{b^2}{4\delta}$ to bound the products in the right-hand side of the previous inequality. Firstly, let us denote by C_{PO} the Poincaré constant i.e. $\forall \varphi \in H_L^1(]0,l[) : \|\varphi\|_{L^2(]0,l[)} \leq C_{PO} \|\nabla_x \varphi\|_{L^2(]0,l[)^2}$ (it is easy to see that $C_{PO} = \frac{2}{\pi} l$).
1°) $\forall \varepsilon > 0$:

$$
\begin{aligned}
c_p m_g \int_0^t \|\tilde{\rho}_t(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds & \\
&\leq c_p m_g \left(\int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds \right)^{1/2} \left(\int_0^t \left\|\tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)} ds \right)^{1/2} \\
&\leq \varepsilon \int_0^t \left\|\tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)} ds + \tfrac{(c_p m_g)^2}{4\varepsilon} \int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds \\
&\leq \varepsilon C_{PO}^2 \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + \tfrac{(c_p m_g)^2}{4\varepsilon} \int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds.
\end{aligned}
$$

Let us take $\varepsilon = \frac{k_h}{4C_{PO}^2}$. Thus:

$$
\begin{aligned}
c_p m_g \int_0^t \|\tilde{\rho}_t(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds & \\
&\leq \tfrac{k_h}{4} \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + \tfrac{(c_p m_g C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds.
\end{aligned}
\tag{63}
$$

2°) $\forall \tilde{\varepsilon} > 0$:

$$
\begin{aligned}
L_b \int_0^t \|\tilde{\rho}(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds & \\
&\leq L_b \left(\int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds \right)^{1/2} \left(\int_0^t \left\|\tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)} ds \right)^{1/2} \\
&\leq \tilde{\varepsilon} \int_0^t \left\|\tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)} ds + \tfrac{L_b^2}{4\tilde{\varepsilon}} \int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds \\
&\leq \tilde{\varepsilon} C_{PO}^2 \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + \tfrac{L_b^2}{4\tilde{\varepsilon}} \int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds.
\end{aligned}
$$

Let us take $\tilde{\varepsilon} = \frac{k_h}{4C_{PO}^2}$. Thus:

$$L_b \int_0^t \|\tilde{\rho}(\cdot,s)\|_{L^2(]0,l[)} \left\|\tilde{\theta}(\cdot,s)\right\|_{L^2(]0,l[)} ds$$
$$\leq \tfrac{k_h}{4} \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + \tfrac{(L_b C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds. \tag{64}$$

Multiplying both sides of inequality (62) by 2 and using inequalities (63) to (64), we obtain:

$$c_p m_g \left\|\tilde{\theta}(\cdot,t)\right\|^2_{L^2(]0,l[)} + k_h \int_0^t \left\|\nabla_x \tilde{\theta}(\cdot,s)\right\|^2_{L^2(]0,l[)^2} ds + h_c \int_0^t \tilde{\theta}(l,s)^2 ds$$
$$\leq c_p m_g \left\|\tilde{\theta}(\cdot,0)\right\|^2_{L^2(]0,l[)} + 2\tfrac{(c_p m_g C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds$$
$$+2\tfrac{(L_b C_{PO})^2}{k_h} \int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds.$$

What was to be proved. ■

In view of the previous proposition, we must bound $\left\|\tilde{\theta}(\cdot,0)\right\|^2_{L^2(]0,l[)}$, $\int_0^t \|\tilde{\rho}_t(\cdot,s)\|^2_{L^2(]0,l[)} ds$ and $\int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{L^2(]0,l[)} ds$. Firstly:

$$\|\theta(\cdot,0)\|_{L^2(]0,l[)} = \left\|T_{u,h}(\cdot,0) - \tilde{T}_{u,h}(\cdot,0)\right\|_{L^2(]0,l[)}$$
$$\leq \|T_{u,h}(\cdot,0) - T_u(\cdot,0)\|_{L^2(]0,l[)} + \left\|T_u(\cdot,0) - \tilde{T}_{u,h}(\cdot,0)\right\|_{L^2(]0,l[)}. \tag{65}$$

Lemma 18 *$\forall t \in [0,t_f]$:*

$$\left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{H^1_L(]0,l[)} \lesssim h \|T_u(\cdot,t)\|_{H^2(]0,l[)}.$$

Proof. Equation (50), shows us that $\tilde{T}_{u,h}(.,t) - T_a$ is the orthogonal projection of $T_u(.,t) - T_a$ onto $S_h \subset H^1_L(]0,l[)$ for the scalar product of $H^1_L(]0,l[)$. Thus:

$$\left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{H^1_L(]0,l[)} = \left\|(T_u(\cdot,t) - T_a) - \left(\tilde{T}_{u,h}(\cdot,t) - T_a\right)\right\|_{H^1_L(]0,l[)}$$
$$= \inf_{\chi \in S_h} \|(T_u(\cdot,t) - T_a) - \chi\|_{H^1_L(]0,l[)} = \inf_{\chi \in T_a + S_h} \|T_u(\cdot,t) - \chi\|_{H^1_L(]0,l[)}$$
$$\leq \|T_u(\cdot,t) - I_h T_u(\cdot,t)\|_{H^1_L(]0,l[)} \lesssim h \|T_u(\cdot,t)\|_{H^2(]0,l[)}.$$

■

But, in view of (65), what we need (at least for $t = 0$) is a bound in the $L^2(]0,l[)$-norm of $\tilde{T}_{u,h}(\cdot,0) - T_u(\cdot,0)$. We will thus apply a duality argument (like in [19], p.5):

Proposition 19 *The following bound holds* $\forall t \in [0, t_f]$*:*

$$\|\rho(\cdot,t)\|_{L^2(]0,l[)} = \left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{L^2(]0,l[)} \lesssim h^2 \|T_u(\cdot,t)\|_{H^2(]0,l[)}.$$

In the right hand side of the above inequality, h *means* $\max_{j=1,\ldots,N_h}(x_j - x_{j-1})$*.*

Proof. Let us consider the two-points boundary value problem in the interval $]0,l[$: given $\varphi \in L^2(]0,l[)$, find $\psi \in H^2(]0,l[)$ such that

$$\begin{cases} -\psi''(x) = \varphi(x), \ \forall x \in]0,l[, \\ \psi(0) = 0, \\ \psi(l) = 0. \end{cases}$$

By the closed graph theorem, there exists a constant $C > 0$, such that

$$\|\psi\|_{H^2(]0,l[)} \leq C \|\varphi\|_{L^2(]0,l[)}.$$

Using $\tilde{T}_{u,h}(0,t) = T_u(0,t) = T_a$ and Proposition 13 which tells us that $\tilde{T}_{u,h}(l,t) = T_u(l,t)$, we have:

$$(\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)|\varphi)_{L^2(]0,l[)} = -(\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)|\psi'')_{L^2(]0,l[)}$$

$$= -(\tilde{T}_{u,h}(l,t) - T_u(l,t))\psi'(l) + (\tilde{T}_{u,h}(0,t) - T_u(0,t))\psi'(0) + \int_0^l (\tfrac{\partial \tilde{T}_{u,h}}{\partial x}(x,t)$$

$$- \tfrac{\partial T_u}{\partial x}(x,t))\psi'(x)dx = \int_0^l (\tfrac{\partial \tilde{T}_{u,h}}{\partial x}(x,t) - \tfrac{\partial T_u}{\partial x}(x,t))\psi'(x)dx$$

$$= \int_0^l (\tfrac{\partial \tilde{T}_{u,h}}{\partial x}(x,t) - \tfrac{\partial T_u}{\partial x}(x,t))(\psi'(x) - \chi'(x))dx, \ \forall \chi \in S_h,$$

by equation (49). Thus using the preceding lemma:

$$\left|(\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)|\varphi)_{L^2(]0,l[)}\right| \leq \left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{H^1(]0,l[)}$$

$$\cdot \inf_{\chi \in S_h} \|\psi - \chi\|_{H^1(]0,l[)} \leq h \|T_u(\cdot,t)\|_{H^2(]0,l[)} \inf_{\chi \in S_h} \|\psi - \chi\|_{H^1(]0,l[)}$$

$$\leq h \|T_u(\cdot,t)\|_{H^2(]0,l[)} \|\psi - I_h\psi\|_{H^1(]0,l[)}$$

$$\lesssim h^2 \|T_u(\cdot,t)\|_{H^2(]0,l[)} \|\psi\|_{H^2(]0,l[)}$$

$$\lesssim h^2 \|T_u(\cdot,t)\|_{H^2(]0,l[)} \|\varphi\|_{L^2(]0,l[)}.$$

Taking the supremum over all $\varphi \in L^2(]0,l[)$ such that $\|\varphi\|_{L^2(]0,l[)} \leq 1$, we obtain:

$$\left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{L^2(]0,l[)} \lesssim h^2 \|T_u(\cdot,t)\|_{H^2(]0,l[)}.$$

What was to be proved. ■

Corollary 20 *Assuming that* $T_u(\cdot,0) \in H^2(]0,l[)$*:* $\left\|\tilde{\theta}(\cdot,0)\right\|_{L^2(]0,l[)} \lesssim h^2 \left\|T_u(\cdot,0)\right\|_{H^2(]0,l[)}$.

Proof. By inequality (65) and the previous proposition:

$$\begin{aligned}\left\|\tilde{\theta}(\cdot,0)\right\|_{L^2(]0,l[)} &\leq \left\|T_{u,h}(\cdot,0) - T_u(\cdot,0)\right\|_{L^2(]0,l[)} + \left\|T_u(\cdot,0) - \tilde{T}_{u,h}(\cdot,0)\right\|_{L^2(]0,l[)} \\ &\lesssim \left\|T_{u,h}(\cdot,0) - T_u(\cdot,0)\right\|_{L^2(]0,l[)} + h^2 \left\|T_u(\cdot,0)\right\|_{H^2(]0,l[)} \\ &= \left\|I_h(T_u(\cdot,0)) - T_u(\cdot,0)\right\|_{L^2(]0,l[)} + h^2 \left\|T_u(\cdot,0)\right\|_{H^2(]0,l[)} \\ &\lesssim h^2 \left\|T_u(\cdot,0)\right\|_{H^2(]0,l[)}.\end{aligned}$$

■

Proposition 21 $\int_0^t \left\|\tilde{\rho}(\cdot,s)\right\|^2_{L^2(]0,l[)}\ ds \lesssim h^4 \int_0^t \left\|T_u(\cdot,s)\right\|^2_{H^2(]0,l[)}\ ds.$

Proof. $\tilde{\rho}(\cdot,s) := \exp(\lambda s)\rho(\cdot,s)$ and λ is negative. Thus: $\int_0^t \left\|\tilde{\rho}(\cdot,s)\right\|^2_{L^2(]0,l[)} ds \leq \int_0^t \left\|\rho(\cdot,s)\right\|^2_{L^2(]0,l[)} ds$. Thus it suffices to bound $\int_0^t \left\|\rho(\cdot,s)\right\|^2_{L^2(]0,l[)} ds$. $\rho(\cdot,s) := \tilde{T}_{u,h}(\cdot,s) - T_u(\cdot,s)$. By the preceding proposition

$$\left\|\rho(\cdot,s)\right\|_{L^2(]0,l[)} \lesssim h^2 \left\|T_u(\cdot,s)\right\|_{H^2(]0,l[)}.$$

Thus:

$$\int_0^t \left\|\rho(\cdot,s)\right\|^2_{L^2(]0,l[)} ds \lesssim h^4 \int_0^t \left\|T_u(\cdot,s)\right\|^2_{H^2(]0,l[)}\ ds.$$

What was to be proved. ■

It remains thus to bound $\int_0^t \left\|\tilde{\rho}_t(\cdot,s)\right\|^2_{L^2(]0,l[)} ds$.

Proposition 22 $\int_0^t \left\|\tilde{\rho}_t(\cdot,s)\right\|^2_{L^2(]0,l[)} ds \lesssim h^4 \int_0^t \left\|\frac{dT_u}{dt}(\cdot,s)\right\|^2_{H^2(]0,l[)} ds + h^4 \int_0^t \left\|T_u(\cdot,s)\right\|^2_{H^2(]0,l[)} ds.$

Proof. $\partial_s(\exp(\lambda s)\rho)(.,s) = \exp(\lambda s)\left(\partial_s \rho\right)(.,s) + \lambda \exp(\lambda s)\rho(.,s)$. Thus:

$$\begin{aligned}\tilde{\rho}_t(\cdot,s) &= \exp(\lambda s)\rho_t(.,s) + \lambda \exp(\lambda s)\rho(.,s), \\ &= \exp(\lambda s)\rho_t(.,s) + \lambda\tilde{\rho}(.,s),\end{aligned}$$

which implies:

$$\begin{aligned}\left\|\tilde{\rho}_t(\cdot,s)\right\|_{L^2(]0,l[)} &\leq \exp(\lambda s)\left\|\rho_t(\cdot,s)\right\|_{L^2(]0,l[)} + |\lambda|\left\|\tilde{\rho}(\cdot,s)\right\|_{L^2(]0,l[)} \\ &\leq \left\|\rho_t(\cdot,s)\right\|_{L^2(]0,l[)} + |\lambda|\left\|\tilde{\rho}(\cdot,s)\right\|_{L^2(]0,l[)},\end{aligned}$$

as λ is negative (see (59). By Proposition 21: $\forall s \in [0, t_f]$:

$$\int_0^t \|\tilde{\rho}(\cdot, s)\|^2_{L^2(]0,l[)}\, ds \lesssim h^4 \int_0^t \|T_u(\cdot, s)\|^2_{H^2(]0,l[)}\, ds.$$

It suffices thus to bound $\int_0^t \|\rho_t(\cdot, s)\|^2_{L^2(]0,l[)}\, ds$. $\rho_t(\cdot, s) = \frac{d\tilde{T}_{u,h}}{dt}(\cdot, s) - \frac{dT_u}{dt}(\cdot, s)$.

As, we have already observed, by equation (50): $\frac{d}{dt}\tilde{T}_{u,h}(., t) = \widetilde{\left(\frac{dT_u}{dt}(., t)\right)}$. In particular, $\frac{d}{dt}\tilde{T}_{u,h}(., t)$ is the orthogonal projection of $\frac{dT_u}{dt}(., t)$ onto $S_h \subset H^1_L(]0, l[)$ for the scalar product of $H^1_L(]0, l[)$. Also by Proposition 13, $\widetilde{\frac{d\tilde{T}_{u,h}}{dt}}(l, t) = \widetilde{\frac{dT_u}{dt}}(l, t)$. Replacing $T_u(\cdot, t)$ by $\frac{dT_u}{dt}(., t)$ and $\tilde{T}_{u,h}(\cdot, t)$ by $\frac{d}{dt}\tilde{T}_{u,h}(., t) = \widetilde{\left(\frac{dT_u}{dt}(., t)\right)}$, in the estimate stated in Proposition 19, we obtain $\forall' t \in [0, t_f]$

$$\left\| \frac{d}{dt}\tilde{T}_{u,h}(\cdot, t) - \frac{dT_u}{dt}(., t) \right\|_{L^2(]0,l[)} \lesssim h^2 \left\| \frac{dT_u}{dt}(., t) \right\|_{H^2(]0,l[)}.$$

This estimate implies that:

$$\begin{aligned} \int_0^t \|\rho_t(\cdot, s)\|^2_{L^2(]0,l[)}\, ds &= \int_0^t \left\| \tfrac{d}{dt}\tilde{T}_{u,h}(\cdot, s) - \tfrac{dT_u}{dt}(., s) \right\|^2_{L^2(]0,l[)} ds \\ &\leq h^4 \int_0^t \left\| \tfrac{dT_u}{dt}(\cdot, s) \right\|^2_{H^2(]0,l[)} ds, \end{aligned} \tag{66}$$

from which the result now follows. ■

Collecting the above results, we obtain the following estimate on $\theta = T_{u,h} - \tilde{T}_{u,h}$:

Proposition 23 *The following estimate holds for $\theta = T_{u,h} - \tilde{T}_{u,h}$:*

$$\begin{aligned} &c_p m_g \|\theta(\cdot, t)\|^2_{L^2(]0,l[)} + k_h \int_0^t \|\nabla_x \theta(\cdot, s)\|^2_{L^2(]0,l[)^2}\, ds + h_c \int_0^t \theta(l, s)^2 ds \\ &\lesssim h^4 \|T_u(\cdot, 0)\|^2_{H^2(]0,l[)} + h^4 \int_0^t \left\| \tfrac{dT_u}{dt}(\cdot, s) \right\|^2_{H^2(]0,l[)} ds + h^4 \int_0^t \|T_u(\cdot, s)\|^2_{H^2(]0,l[)}\, ds. \end{aligned} \tag{67}$$

Proof. This result follows from Proposition 17, Corollary 20, Proposition 22 and Proposition 21. ■

Corollary 24 *For the value of the solution $T_{u,h}$ ot the semi-discrete problem at the point $x = l$, we have the following a priori error estimate:*

$$\|T_{u,h}(l,.)-T_u(l,.)\|_{L^2(]0,t_f[)} \lesssim h^2 \left(\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[)} \right). \tag{68}$$

Proof. As $\tilde{T}_{u,h}(l,.) = T_u(l,.)$ by Proposition 13, it follows that:

$$T_{u,h}(l,.) - T_u(l,.) = T_{u,h}(l,.) - \tilde{T}_{u,h}(l,.) = \theta(l,.).$$

Thus by (67), it follows

$$\begin{aligned}\|T_{u,h}(l,.)-T_u(l,.)\|_{L^2(]0,t_f[)} \lesssim h^2(&\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \|T_u\|_{L^2(0,t_f;H^2(]0,l[)} \\ &+ \left\|\tfrac{dT_u}{dt}\right\|_{L^2(0,t_f;H^2(]0,l[)}).\end{aligned} \tag{69}$$

But for every $t \in [0,t_f]$:

$$\|T_u(\cdot,t)\|_{H^2(]0,l[)} \lesssim \|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))} \tag{70}$$

which implies

$$\|T_u\|_{L^2(0,t_f;H^2(]0,l[))} \lesssim \|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))}. \tag{71}$$

By inequalities (69) and (71), the result follows. ■

We are now ready to state our a priori error estimate for the solution $T_{u,h}$ ot the semi-discrete problem:

Theorem 25 *For the solution $T_{u,h}$ of the semi-discrete problem, we have the following a priori error estimate:*

$$\|T_{u,h}-T_u\|_{C([0,t_f];L^2(]0,l[))} \lesssim h^2 \left(\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))} \right). \tag{72}$$

Proof. As we have explained at the begining of this section, $T_u \in C([0,t_f];H^2(]0,l[))$. As $\tilde{T}_{u,h}(\cdot,t) - T_a$ is the orthogonal projection of $T_u(\cdot,t) - T_a$ onto S_h in $H^1_L(]0,l[)$, $\tilde{T}_{u,h} \in C([0,t_f];H^1(]0,l[))$. A fortiori $\tilde{T}_{u,h} \in C([0,t_f];L^2(]0,l[))$. Using the previous proposition, we obtain:

$$\begin{aligned}\left\|T_{u,h}-\tilde{T}_{u,h}\right\|_{C([0,t_f];L^2(]0,l[))} \lesssim h^2(&\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \|T_u\|_{L^2(0,t_f;H^2(]0,l[))} \\ &+ \left\|\tfrac{dT_u}{dt}\right\|_{L^2(0,t_f;H^2(]0,l[))}).\end{aligned} \tag{73}$$

By Proposition 19:

$$\left\|\tilde{T}_{u,h}(\cdot,t) - T_u(\cdot,t)\right\|_{L^2(]0,l[)} \lesssim h^2 \left\|T_u(\cdot,t)\right\|_{H^2(]0,l[)}, \ \forall t \in [0,t_f]. \tag{74}$$

By inequality (74) and (70) follows:

$$\left\|\tilde{T}_{u,h} - T_u\right\|_{C([0,t_f];L^2(]0,l[))} \lesssim h^2 \left(\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))} \right). \tag{75}$$

By inequalities (75), (73), and the triangular inequality follows:

$$\begin{aligned} \|T_u - T_{u,h}\|_{C([0,t_f];L^2(]0,l[))} \lesssim h^2 (&\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \|T_u\|_{L^2(0,t_f;H^2(]0,l[))} \\ &+ \left\| \tfrac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))}). \end{aligned} \tag{76}$$

Using inequality (71), the result follows. ■

Now, we want also to give some a priori error estimate for the gradient $\nabla_x T_{u,h}$. Firstly a lemma:

Lemma 26 $\displaystyle\int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{H^1(]0,l[)}\, ds \lesssim h^2 \int_0^t \|T_u(\cdot,s)\|^2_{H^2(]0,l[)}\ ds.$

Proof. Equation (50), shows us that $\tilde{T}_{u,h}(.,t) - T_a$ is the orthogonal projection of $T_u(.,t) - T_a$ onto $S_h \subset H^1_L(]0,l[)$ for the scalar product of $H^1_L(]0,l[)$. Thus:

$$\begin{aligned} \|\rho(\cdot,s)\|_{H^1(]0,l[)} &\lesssim \left\| T_u(\cdot,s) - \tilde{T}_{u,h}(\cdot,s) \right\|_{H^1_L(]0,l[)} = \inf_{\chi \in T_a + S_h} \|T_u(\cdot,s) - \chi\|_{H^1_L(]0,l[)} \\ &\leq \|T_u(\cdot,s) - I_h T_u(\cdot,s)\|_{H^1_L(]0,l[)} \lesssim h \left\|T_u(\cdot,s)''\right\|_{L^2(]0,l[)} \lesssim h \left\|T_u(\cdot,s)\right\|_{H^2(]0,l[)}. \end{aligned}$$

As λ is negative, it follows from the previous inequality:

$$\begin{aligned} &\int_0^t \|\tilde{\rho}(\cdot,s)\|^2_{H^1(]0,l[)}\, ds = \int_0^t \|\exp(\lambda s)\rho(\cdot,s)\|^2_{H^1(]0,l[)}\, ds = \int_0^t \exp(2\lambda s) \\ &\cdot \|\rho(\cdot,s)\|^2_{H^1(]0,l[)}\, ds \leq \int_0^t \|\rho(\cdot,s)\|^2_{H^1(]0,l[)}\, ds \lesssim h^2 \int_0^t \|T_u(\cdot,s)\|^2_{H^2(]0,l[)}\, ds. \end{aligned}$$

■

Proposition 27 *For the gradient $\nabla_x T_{u,h}$, we have the following a priori error estimate:*

$$\|\nabla_x (T_{u,h} - T_u)\|_{L^2(0,t_f;L^2(]0,l[)^2)} \lesssim h \left(\|T_u(\cdot,0)\|_{H^2(]0,l[)} + \left\| \frac{dT_u}{dt} \right\|_{L^2(0,t_f;H^2(]0,l[))} \right). \tag{77}$$

Proof. From inequality (67) follows that:

$$\left\|\nabla_x\left(T_{u,h}-\tilde{T}_{u,h}\right)\right\|_{L^2(0,t_f;L^2(]0,l[)^2)}\lesssim h^2(\|T_u(\cdot,0)\|_{H^2(]0,l[)}+\|T_u\|_{L^2(0,t_f;H^2(]0,l[))}+\left\|\tfrac{dT_u}{dt}\right\|_{L^2(0,t_f;H^2(]0,l[))}). \tag{78}$$

On the other hand, from Lemma 26, it follows that:

$$\left\|\nabla_x\left(\check{T}_{u,h}-T_u\right)\right\|_{L^2(0,t_f;L^2(]0,l[)^2)}\lesssim h\left\|T_u\right\|_{L^2(0,t_f;H^2(]0,l[))}. \tag{79}$$

By inequalities (78), (79) and the triangle inequality follows that:

$$\|\nabla_x(T_{u,h}-T_u)\|_{L^2(0,t_f;L^2(]0,l[)^2)}\lesssim h(\|T_u(\cdot,0)\|_{H^2(]0,l[)}+\|T_u\|_{L^2(0,t_f;H^2(]0,l[))}+\left\|\tfrac{dT_u}{dt}\right\|_{L^2(0,t_f;H^2(]0,l[))}).$$

Using inequality (71) the result follows. ■

4 Numerical Tests

Using the P_1-finite element in space with a regular grid on the interval $[0,l]$ and the implicit Euler scheme in time, with prediction-correction on the nonlinear terms in (40), we have computed the temperature along the tickness in the glass plate. Moreover, to avoid the computation of double integrals with logarithmic singularities at each time iteration step in (40) very costly in computational time, we have turned back to the origin of these terms coming from the integration of the radiative intensities $I^{k^*}(x,t,\mu)$ $(k^*=1,\ldots,M)$ with respect to the cosine direction μ on the interval $[-1,1]$ (see formula (2.18) p.9 of [11]), by performing rather a composite midpoint numerical integration in μ like in N. Siedow's program based on their publication [15]. M has been choosen to be 30 like in that program also. The numerical results shown in Figure 0 below, have been obtained by decomposing the interval $[0,l]$ (the thickness l of the plate is taken equal to 6.1 millimeters like in [15]) into $nss=64$ subintervals of equal length and by taking as time step $\tau=0.1$ second. The temperature u (denoted T_S in [11] but u in [12]) of the infinite black radiative source placed above the glass plate has been choosen equal to 1500 $^\circ C$. The ambient air temperature has been taken equal to $25^\circ C$ and also the initial temperature T_0 of the glass plate[1]. The heat transfer coefficient h_c at the upper face of the glass plate has been chosen equal to 5.0 W/(m$^2\times{}^\circ$K) (in [15], it is 280.5 W/(m$^2\times{}^\circ$K) but it concerns tempering[2]). The heat conductivity of the glass plate k_h has been chosen equal to 0.8 W/(m $\times{}^\circ$K) in conformity with ([17], p.449). The graph of the temperature along the thickness of the glass plate with respect to time is the following:

[1] for convenience the temperatures are cited in $^\circ C$; the corresponding absolute temperatures in $^\circ K$ are obtained by adding 273.15 to the temperatures cited in $^\circ C$.

[2] I take here the opportunity to thank my colleague F. Béchet from the Tempo laboratory in our university, who has made me that remark.

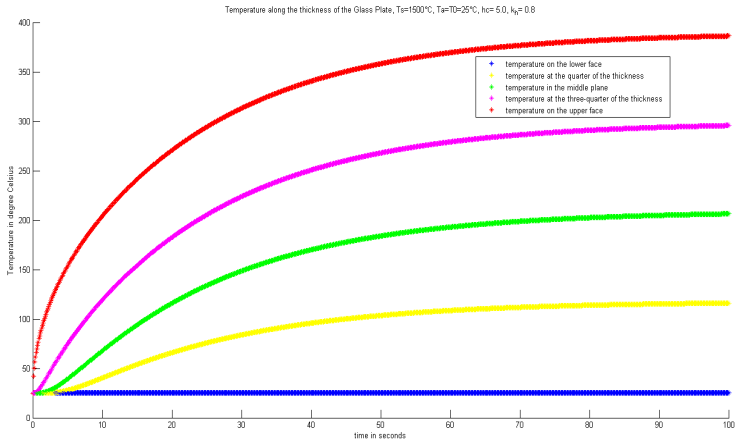

Figure 0: Temperatures in function of time

If we do not take into account the terms $\psi(T(x,t)) + h_{T,u}(x,t)$ modelizing the participating medium character of the glass[3] into equation $(2)_{(i)}$, instead of $T(l,100) = 386.29$ °C we obtain $T(l,100) = 393.58$ °C, instead of $T(\frac{3l}{4},100) = 295.71$ °C we obtain $T(\frac{3l}{4},100) = 300.50$ °C, instead of $T(\frac{l}{2},100) = 206.38$ °C we obtain $T(\frac{l}{2},100) = 208.17$ °C, and instead of $T(\frac{l}{4},100) = 116.12$ °C we obtain $T(\frac{l}{4},100) = 116.42$ °C.

To confront our error estimate (72) with a numerical experiment, we have computed the temperature along the tickness of the glass plate at the fixed time $t = 10$ seconds choosing a small fixed time step $\tau = 0.01$ second. The numerical results have been compared after one thousand of time steps for more and more refined meshes on the space interval $[0,l]$: $h = \frac{l}{8}, \frac{l}{16}, \frac{l}{32}, \frac{l}{64}, \frac{l}{128}, \frac{l}{256}$, taking as reference the solution computed with $h = \frac{l}{1024}$ (its graph is plotted in green in Figure 1):

We observe quadratic behaviour of the relative error measured in the $L^2(]0,l[)$ norm with respect to $h = \frac{l}{nss}$ in accordance with our a priori error estimate (72) in Theorem 25 (nss denotes the number of subintervals $[(i-1)h, ih]$, $i = 1, \ldots, nss$, into which the interval $[0,l]$ has been decomposed):

[3] semi-transparent medium

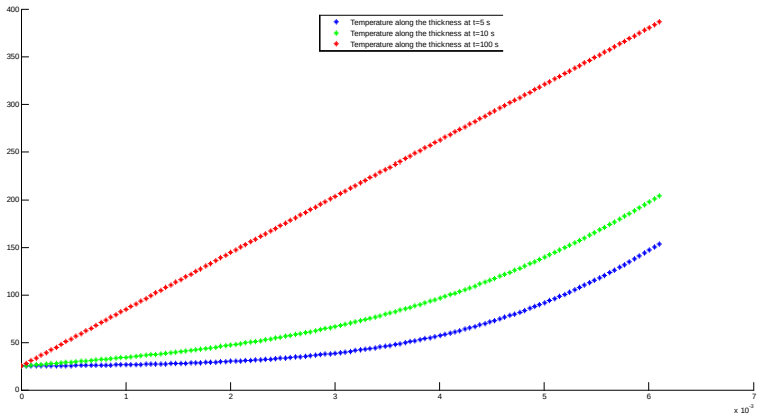

Figure 1: Temperatures in °C at 5, 10, and 100 s with respect to x.

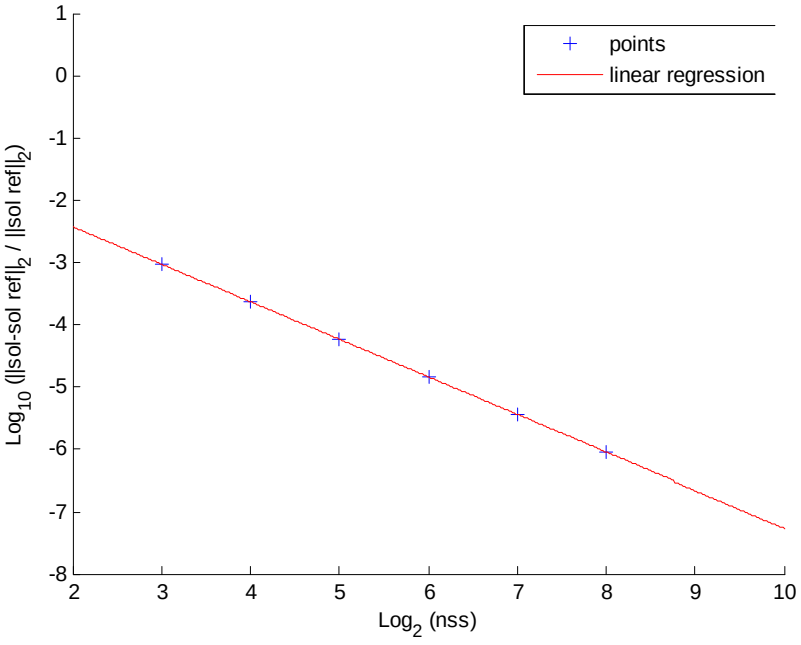

Figure 2: Relative error at time t=10 s in function of nss.

In blue in Figure 2, we have plotted the five points, results of our numerical computations:

nss	$Log_{10}\left(\frac{\|sol-sol\_ref\|_2}{\|sol\_ref\|_2}\right)$
$8 = 2^3$	-3.0402
$16 = 2^4$	-3.6421
$32 = 2^5$	-4.2443
$64 = 2^6$	-4.8470
$128 = 2^7$	-5.4515
$256 = 2^8$	-6.0637

In red, on the same figure, we have plotted the regression line fitting at best these five points in the sense of mean-squares. The equation of that straight line is the following:

$$Log_{10}(\frac{\|sol - sol\_ref\|_2}{\|sol\_ref\|_2}) = -1.2248 - 0.60424 * Log_2(nss)$$

which implies that:

$$\frac{\|sol - sol\_ref\|_2}{\|sol\_ref\|_2} = 0.059592 \times nss^{-2.00772} \ .$$

Remark 28 *For the temperature on the upper face of the glass plate at the fixed time $t = 10$ seconds, we have also observed quadratic convergence with respect to h in accordance with our a priori error estimate in Corollary 24:*

The equation of the regression line (see Fig. 3) for the relative error is:

$$Log_{10}(relative\_error) = -1.34 - 0.609 * Log_2(nss)$$

which implies that:

$$relative\_error = 0.0457 \times nss^{-2.02}.$$

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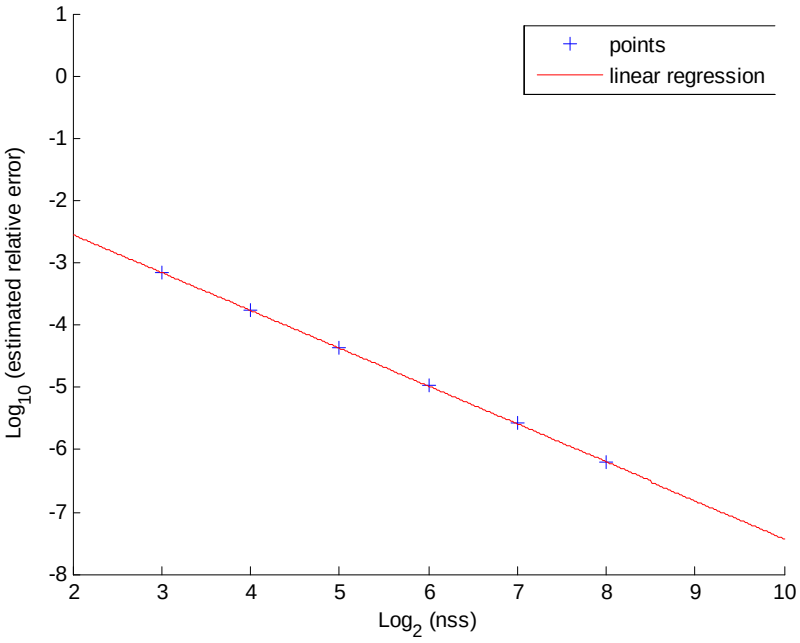

Figure 3: Relative error on the temperature of the upper face.

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