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# Constrained Output Feedback Control for Discrete-Time Fuzzy Systems with Local Nonlinear Models Subject to State and Input Constraints

Anh-Tu Nguyen, *Member, IEEE*, Pedro Coutinho, Thierry-Marie Guerra, Reinaldo Palhares, and Juntao Pan

**Abstract**—This paper presents a new approach to design static output feedback (SOF) controllers for constrained Takagi-Sugeno fuzzy systems with nonlinear consequents. The proposed SOF fuzzy control framework is established via the absolute stability theory with appropriate sector-bounded properties of the local state and input nonlinearities. Moreover, both state and input constraints are explicitly taken into account in the control design using set-invariance arguments. Especially, we include the local sector-bounded nonlinearities of the fuzzy systems in the construction of both the nonlinear controller and the nonquadratic Lyapunov function. Within the considered local control context, the new class of nonquadratic Lyapunov functions provides an effective solution to estimate the closed-loop domain of attraction, which can be *nonconvex* and even *disconnected*. The convexification procedure is performed using specific congruence transformations in accordance with the special structures of the proposed SOF controllers and nonquadratic Lyapunov functions. Consequently, the fuzzy SOF control design can be reformulated as an optimization problem under strict LMI constraints with a linear search parameter. Compared to existing fuzzy SOF control schemes, the new structures of the control law and the Lyapunov function are more general and offer additional degrees of freedom for the control design. Both theoretical arguments and numerical illustrations are provided to demonstrate the effectiveness of the proposed approach in reducing the design conservatism.

**Index Terms**—Takagi-Sugeno fuzzy systems, fuzzy output feedback control, fuzzy Lyapunov functions, input constraints.

## I. INTRODUCTION

FUZZY control has been widely adopted as an effective framework to deal with complex nonlinear systems [1]. Especially, nonlinear control and observation based on Takagi-Sugeno (TS) fuzzy modeling has received considerable research investigations [2]. The success of TS fuzzy control systems comes from their ability to represent nonlinear systems in a convex structure with simple local consequents. Exploiting this convexity property of TS fuzzy systems, numerous control results have been derived using Lyapunov stability theory [3]. An outstanding feature of these results is that the design conditions are expressed in terms of linear matrix inequalities (LMIs), effectively solved with semidefinite programming [4].

Numerous results on stability analysis and control design of TS fuzzy systems have been reported in the literature, see [2] and references therein. Moreover, TS fuzzy model-based control techniques have been successfully applied to various engineering applications [5]. Despite its effectiveness for nonlinear control, TS fuzzy-model-based approaches still suffer major drawbacks on design conservatism and numerical complexity in dealing complex nonlinear systems. To reduce the conservatism caused by the use of common quadratic Lyapunov functions, many other alternative classes of Lyapunov function candidates have been proposed such as piecewise Lyapunov functions [6], [7], line-integral fuzzy Lyapunov functions [8], [9], and fuzzy Lyapunov functions [10]–[12], see [2] for a recent review. Moreover, it has been demonstrated that fuzzy Lyapunov functions are especially effective for conservatism reduction in discrete-time TS fuzzy control framework [13], [14]. Furthermore, by the sector nonlinearity approach [1], an *equivalent* TS fuzzy representation of a given nonlinear smooth system can be derived. However, the number of fuzzy rules *exponentially* increases according to the number of involved nonlinearities. To overcome this numerical difficulty, TS fuzzy systems with local nonlinear models, which will be called hereafter N-TS fuzzy systems, have been proposed [15], [16]. Although the local linearity of N-TS fuzzy systems is not preserved, exploiting judiciously the absolute stability theory [3] enables a tractable control framework when the local retained nonlinearities verify some sector-bound properties [15], [17]–[22]. Note that TS fuzzy systems with polynomial consequents have been also discussed [9], [23], [24]. Using Lyapunov stability theorem, sufficient design conditions were derived in the form of sum-of-squares constraints, see [9], [24] and references therein.

A large number of TS fuzzy control results are concerned with state feedback control schemes which are based on the concepts of parallel distributed compensation (PDC) [1], [25] or non-parallel distributed compensation (non-PDC) [11], [14], [26]. However, only *partial* state information is available for real-time control implementation in most of engineering applications [13], [27]. Output feedback control must be used in these situations [28]–[31]. Note that state feedback control approaches are not directly adapted to output feedback control designs, which are more difficult and involved [13]. Therefore, fuzzy output feedback control has received intensive research efforts [7], [27], [30], [32]–[34]. Among all output feedback schemes, static output feedback (SOF) represents the simplest control structure. Moreover, in many cases the

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designs of dynamic output feedback (DOF) controllers can be reformulated as a SOF control problem [35]. However, the design of SOF control schemes still remains one of the most challenging control topics due to its inherent nonconvex formulation [36], [37]. Based on the convexification procedure in [38], fuzzy SOF design conditions involving linear matrix *equalities* (LMEs) have been presented in [39], [40]. Other fuzzy SOF control results derived from quadratic Lyapunov functions can be found in [30], [41], [42]. To reduce the design conservatism, SOF control designs for TS fuzzy systems have been also proposed with piecewise Lyapunov functions [7] or fuzzy Lyapunov functions [13], [27], [36]. SOF control schemes have been also proposed for various classes of TS fuzzy systems such as polynomial systems [23], [24], large-scale systems [42], partial differential equations systems [28], positive systems [43], singularly perturbed systems [33], networked systems [32], semi-Markov jump systems [44]. However, existing SOF control designs usually lead to too restrictive and/or conservative results [13].

In general, both classical TS or N-TS fuzzy models are only defined within a given compact set of the state space [1]. This latter can be formulated in the form of state constraints [13]. It has been shown that taking into account these constraints into the control design is crucial to guarantee a suitable closed-loop performance of TS fuzzy systems [45]. Moreover, the control input constraints naturally arise in most of engineering applications due to the physical restrictions of the system actuators [46]. If it is not properly taken into account in the control design, then the control performance can be seriously degraded, and in the worst case the closed-loop stability may be lost [26], [47]. Due to its theoretical and practical significance, this control topic has attracted increasing research efforts in fuzzy control framework [45], [48], [49]. Model predictive control (MPC) was applied to address both state and input constraints in [50], [51]. Using the quadratic boundedness concept, constrained DOF controllers were designed in [52]. Set-invariance theory was employed with a polyhedral Lyapunov function in [49] and with a fuzzy Lyapunov function in [26] to design constrained fuzzy controllers. Fuzzy antiwindup-based DOF control schemes were proposed in [45], [48], [53] to deal with actuator saturation. Adaptive fuzzy output feedback controllers were also proposed for fuzzy systems subject to full state constraints in [54] or input constraints in [55]. Unfortunately, despite the practical and theoretical significance, the SOF control of N-TS fuzzy systems subject to both state and input constraints has not been well addressed. This motivates our new control solution for this research topic.

This paper investigates the local SOF control design for constrained N-TS fuzzy systems. The proposed control framework is established with a suitable sector-bounded characterization for both state and input nonlinearities. Based on the absolute stability theory, we provide an effective solution to estimate the closed-loop domain of attraction (DoA), which is known as an open-ended task for nonlinear systems [3]. More specifically, the main contributions can be summarized as follows.

- 1) We propose a new class of nonquadratic Lyapunov functions by taking into account the local nonlinearities of the

N-TS fuzzy systems in its construction. These new Lyapunov functions allow reducing the design conservatism and improving the estimation of the closed-loop DoA. In particular, the estimated DoA can be nonconvex and even disconnected depending on the local nonlinearity feature of the N-TS fuzzy systems.

- 2) The convexification procedure is performed via specific congruence transformations in comply with the special structures of both the proposed SOF controller and the new nonquadratic Lyapunov function. Consequently, the proposed SOF control design can be reformulated as an LMI-based optimization with a linear search parameter, effectively solved with available solvers. Especially, we demonstrate *theoretically* that the new approach provides less conservative results than those requiring LME constraints for convexification purposes. These latter constraints are hard to be satisfied for general fuzzy systems.
- 3) In contrast to [50], the proposed SOF control scheme requires neither full-state information nor heavy online optimizations to deal with input and state constraints. The new control approach can be applied to N-TS fuzzy systems with nonlinear and/or uncertain output matrix, which is not the case of many other fuzzy output feedback control approaches [16], [36], [41], [45], [56].

The paper is organized as follows. The control problem is defined in Section II and solved in Section III, where the control design algorithms are proposed. The extension to the robust SOF control design for constrained N-TS systems subject to structured uncertainties is discussed in Section IV. Several illustrative examples are given in Section V, and concluding remarks are drawn in Section VI.

*Notation.* For a vector  $x$ ,  $x_i$  denotes its  $i$ th entry. Inequalities between vectors are componentwise, *i.e.*,  $x \succeq y$  means that  $x_i \geq y_i$ . For two vectors  $x, y \in \mathbb{R}^n$ , the convex hull of these vectors is denoted as  $\text{co}\{x, y\} = \{\alpha_1 x + \alpha_2 y : \alpha_1 + \alpha_2 = 1, \alpha_1 \geq 0, \alpha_2 \geq 0\}$ . For a matrix  $X$ ,  $X^\top$  denotes its transpose,  $X \succ 0$  means  $X$  is symmetric positive definite,  $X_{(i)}$  denotes its  $i$ th row.  $\text{diag}(X_1, X_2)$  denotes a block-diagonal matrix composed of  $X_1, X_2$ . For a matrix  $P \succ 0 \in \mathbb{R}^{n_x}$ ,  $\mathcal{E}(P) = \{x \in \mathbb{R}^{n_x} : x^\top P x \leq 1\}$ .  $I$  denotes the identity matrix of appropriate dimension. For two integers  $k_1 < k_2$ ,  $\mathcal{I}[k_1, k_2] = \{k_1, k_1 + 1, \dots, k_2\}$ . The set of nonnegative integers is denoted by  $\mathbb{Z}_+$ . In symmetric block matrices, the symbol  $\star$  stands for the terms deduced by symmetry. Arguments are omitted when their meaning is straightforward.

## II. PROBLEM FORMULATION

This section provides the preliminaries and formalizes the constrained control design of N-TS fuzzy systems.

### A. System Description

Consider the following class of N-TS fuzzy systems:

#### Plant Rule $i$ :

IF  $\zeta_1(k)$  is  $\mathcal{F}_{i1}$  and  $\zeta_2(k)$  is  $\mathcal{F}_{i2} \dots \zeta_p(k)$  is  $\mathcal{F}_{ip}$ ,

THEN  $x(k+1) = A_i x(k) + G_i \varphi(z(k)) + B_i \text{sat}(u(k))$

$$z(k) = L_i x(k), \quad y(k) = C_i x(k), \quad (1)$$

where  $i \in \mathcal{I}[1, N]$ , and  $N$  is the number of IF-THEN fuzzy rules.  $\zeta = [\zeta_1 \ \zeta_2 \ \dots \ \zeta_p]^\top \in \mathbb{R}^p$  denotes the vector of measurable premise variables whose measurements can be obtained from  $y(k)$  and/or  $z(k)$ . For system (1),  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control,  $z \in \mathbb{R}^{n_z}$  and  $y \in \mathbb{R}^{n_y}$  are measured outputs. The vector valued saturation function  $\text{sat}(\cdot) : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$  is defined as

$$\text{sat}(u_l) = \text{sign}(u_l) \min(|u_l|, \bar{u}_l), \quad l \in \mathcal{I}[1, n_u],$$

where  $\bar{u}_l > 0$  denotes the given bound of the  $l$ th input. The matrices of appropriate dimensions  $A_i$ ,  $B_i$ ,  $C_i$ ,  $G_i$  and  $L_i$  represent the  $i$ th local model. Assume that the nonlinearity  $\varphi(\cdot) : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_z}$  verifies the cone-bounded sector condition [3]. This means that  $\varphi(0) = 0$ , and there exists a positive definite matrix  $\Omega \in \mathbb{R}^{n_z \times n_z}$  such that

$$\begin{aligned} \mathcal{SC}(\varphi(z(k)), \Lambda(\zeta)) &= \varphi^\top(z(k))\Lambda^{-1}(\zeta(k))[\varphi(z(k)) - \Omega z(k)] \\ \mathcal{SC}(\varphi(z(k)), \Lambda(\zeta)) &\leq 0, \quad \forall z(k) \in \mathbb{R}^{n_z}, \quad \forall k \in \mathbb{Z}_+. \end{aligned} \quad (2)$$

The positive definite matrix  $\Lambda(\zeta) = \text{diag}\{\lambda_1(\zeta), \dots, \lambda_{n_z}(\zeta)\}$  can be viewed as a degree of freedom for the SOF control design as shown in (8).

Using center-average defuzzifier, product inference and singleton fuzzifier, system (1) can be rewritten as follows:

$$\begin{aligned} x(k+1) &= A(\zeta)x(k) + G(\zeta)\varphi(z(k)) + B(\zeta)\text{sat}(u(k)) \\ z(k) &= L(\zeta)x(k), \quad y(k) = C(\zeta)x(k), \end{aligned} \quad (3)$$

where the state-space matrices are defined as

$$\begin{bmatrix} A(\zeta) & B(\zeta) & C(\zeta) \\ G(\zeta) & L(\zeta) & 0 \end{bmatrix} = \sum_{i=1}^N \eta_i(\zeta) \begin{bmatrix} A_i & B_i & C_i \\ G_i & L_i & 0 \end{bmatrix}.$$

Note that the membership functions (MFs) satisfy the following convex sum property [1]:

$$\eta_i(\zeta) \geq 0, \quad \sum_{i=1}^N \eta_i(\zeta) = 1. \quad (4)$$

Let  $\Xi$  be the set of membership functions satisfying (4), i.e.,  $\eta(\zeta) = [\eta_1(\zeta) \dots \eta_N(\zeta)]^\top \in \Xi$ .

Besides the control input limitations, the system states are also bounded in engineering applications due to physical and/or safety reasons. In particular, the *equivalent* TS fuzzy representation of a general nonlinear system obtained with the sector nonlinearity approach [1] is generally valid within a specific bounded set [13], [45]. This domain of validity can be represented by a polyhedral set  $\mathcal{D}_x \subset \mathbb{R}^{n_x}$  defined as

$$\mathcal{D}_x = \{x \in \mathbb{R}^{n_x} : S_{(m)}x \leq 1, \quad m \in \mathcal{I}[1, n_s]\}, \quad (5)$$

where the matrix  $S \in \mathbb{R}^{n_s \times n_x}$  characterizes the domain  $\mathcal{D}_x$ .

**Remark 1.** If  $G_i = 0$ ,  $\forall i \in \mathcal{I}[1, N]$ , then the N-TS fuzzy system (3) reduces to  $x(k+1) = A(\zeta(k))x(k) + B(\zeta(k))\text{sat}(u(k))$ . That is, the N-TS fuzzy system (3) reduces to the classical saturated TS fuzzy system studied in literature [13], [26], [53], [55]. Hence, the form (3) is a more general TS fuzzy representation. A particular interest of this form is that it generally requires fewer number of rules to represent a nonlinear system than a classical TS fuzzy model. Hence, it

can also lead to design results with less computational burden and conservatism, see for instance [15], [16], [22], [45], [50].

**Remark 2.** Note that (2) is equivalent to

$$[\varphi^\top(z) - z^\top \Omega] \Lambda^{-1}(\zeta) \Omega z \leq 0. \quad (6)$$

Then, it follows from (2) and (6) that

$$\begin{aligned} 0 &\leq \varphi^\top(z) \Lambda^{-1}(\zeta) \varphi(z) \leq \varphi^\top(z) \Lambda^{-1}(\zeta) \Omega z \\ &\leq z^\top \Omega \Lambda^{-1}(\zeta) \Omega z, \quad \forall z \in \mathbb{R}^{n_z}. \end{aligned} \quad (7)$$

## B. Problem Definition

Let us consider a nonlinear SOF controller of the form

$$u(k) = K(\zeta)M^{-1}(\zeta)y(k) + F(\zeta)\Lambda^{-1}(\zeta)\varphi(z(k)), \quad (8)$$

where the matrices of appropriate dimensions  $K(\zeta)$ ,  $M(\zeta)$ ,  $F(\zeta)$ ,  $\Lambda(\zeta)$  are to be designed. These MF-dependent matrices are defined as

$$\begin{bmatrix} K(\zeta) & M(\zeta) \\ F(\zeta) & \Lambda(\zeta) \end{bmatrix} = \sum_{i=1}^N \eta_i(\zeta) \begin{bmatrix} K_i & M_i \\ F_i & \Lambda_i \end{bmatrix}.$$

The *nonsingularity* of matrix  $M(\zeta(k))$  in (8) is examined in Theorem 1. We define the dead-zone nonlinearity  $\psi(\cdot) : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u}$  as

$$\psi(u(k)) = u(k) - \text{sat}(u(k)). \quad (9)$$

Then, from (3) and (8), the closed-loop TS fuzzy system can be represented as follows:

$$\begin{aligned} x(k+1) &= A_{cl}(\zeta)x(k) + G_{cl}(\zeta)\varphi(z(k)) - B(\zeta)\psi(u(k)) \\ z(k) &= L(\zeta)x(k), \quad y(k) = C(\zeta)x(k), \end{aligned} \quad (10)$$

with

$$\begin{aligned} A_{cl}(\zeta) &= A(\zeta) + B(\zeta)K(\zeta)M^{-1}(\zeta)C(\zeta), \\ G_{cl}(\zeta) &= G(\zeta) + B(\zeta)F(\zeta)\Lambda^{-1}(\zeta). \end{aligned}$$

For the control design of system (10), we propose a new non-quadratic Lyapunov function associated with the TS fuzzy systems with nonlinear local models (3). This Lyapunov function is constructed by incorporating the information on the nonlinearity  $\varphi(z)$  into the Lyapunov function candidate as

$$\mathbb{V}(x, \varphi, \zeta) = x^\top Q^{-1}(\zeta)x + \varphi(x)^\top \Lambda^{-1}(\zeta)\varphi(x), \quad (11)$$

where  $Q(\zeta) = \sum_{i=1}^N \eta_i(\zeta)Q_i$ , and  $Q_i \succ 0$ , for  $\forall i \in \mathcal{I}[1, N]$ .

**Remark 3.** If the decision variables  $\lambda_i(\zeta)$ ,  $\forall i \in \mathcal{I}[1, n_z]$ , are sufficiently large, then  $\Lambda^{-1}(\zeta) \rightarrow 0$ . As a result, the following classical Lyapunov function is directly recovered from (11):

$$\mathbb{V}^*(x, \zeta) = x^\top Q^{-1}(\zeta)x. \quad (12)$$

For this reason, the non-quadratic Lyapunov function (12) largely studied in the fuzzy control literature can be viewed as a particular case of function (11). Hence, less conservative results are expected when exploiting (11) for the control design. Moreover, due to the property (7) of  $\varphi(z)$ , the new Lyapunov function candidate is bounded as

$$\underline{\mathbb{V}}(x, \zeta) \leq \mathbb{V}(x, \varphi, \zeta) \leq \overline{\mathbb{V}}(x, \zeta), \quad (13)$$

with  $\bar{\mathbb{V}}(x, \zeta) = x^\top (Q^{-1}(\zeta) + L^\top(\zeta)\Omega\Lambda^{-1}(\zeta)\Omega L(\zeta))x$  and  $\underline{\mathbb{V}}(x, \zeta) = \mathbb{V}^*(x, \zeta) = x^\top Q^{-1}(\zeta)x$ . The level set associated with the Lyapunov function  $\mathbb{V}(x, \varphi, \zeta)$  is defined as

$$\mathcal{L}_\mathbb{V} = \{x \in \mathbb{R}^{n_x} : \mathbb{V}(x, \varphi, \zeta) \leq 1, \text{ for } \forall \eta(\zeta) \in \Xi\}. \quad (14)$$

Due to (13), it is clear that

$$\mathcal{E}(\mathcal{Q}(\zeta)) \subseteq \mathcal{L}_\mathbb{V} \subseteq \mathcal{E}(Q^{-1}(\zeta)), \quad \forall \eta(\zeta) \in \Xi, \quad (15)$$

with  $\mathcal{Q}(\zeta) = Q^{-1}(\zeta) + L^\top(\zeta)\Omega\Lambda^{-1}(\zeta)\Omega L(\zeta)$ .

The concept of *invariant set* [26] is crucial to characterize the domain of attraction of the input-saturated system (10).

**Definition 1.** The set  $\mathcal{L}_\mathbb{V}$  defined in (14) is said to be *contractively invariant* with respect to the closed-loop system (10) if the Lyapunov difference  $\delta_k \mathbb{V}$  verifies the condition

$$\delta_k \mathbb{V} = \mathbb{V}(x(k+1), \zeta(k+1)) - \mathbb{V}(x(k), \zeta(k)) < 0, \quad (16)$$

for  $\forall k \in \mathbb{Z}_+, \forall x(k) \in \mathcal{L}_\mathbb{V} \setminus \{0\}$ , all  $\eta(\zeta(k)), \eta(\zeta(k+1)) \in \Xi$ , and all nonlinearities  $\varphi(\cdot)$  satisfying the sector condition (2). Clearly, if  $\mathcal{L}_\mathbb{V}$  is contractively invariant with respect to system (10), then for every initial condition  $x(0) \in \mathcal{L}_\mathbb{V}$ , the corresponding trajectory converges to the origin while remaining inside this set [3]. Note that  $\mathcal{L}_\mathbb{V}$  can be used as an estimate of the domain of attraction.

This paper proposes a constructive LMI-based solution for the following control problem.

**Problem 1.** Determine the nonlinear SOF controller with the MF-dependent matrices  $K(\zeta)$ ,  $M(\zeta)$ ,  $F(\zeta)$  and  $\Lambda(\zeta)$  as in (8) and a contractively invariant set  $\mathcal{L}_\mathbb{V}$  defined in (14), as large as possible inside  $\mathcal{D}_x$ , of the closed-loop system (10) with any  $\eta(\zeta) \in \Xi$  and any  $\varphi(z)$  satisfying the sector condition (2).

The following technical lemma is useful for the subsequent theoretical developments.

**Lemma 1.** Consider matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $Y_i \in \mathbb{R}^{n_u \times n_z}$ , for  $i \in \mathcal{I}[1, N]$ , and the nonlinearity  $\varphi(z)$  satisfying (2). Let us define the following set:

$$\mathcal{D}_u = \{x \in \mathbb{R}^{n_x} : |\Phi(x, z)| \preceq \bar{u}\}, \quad (17)$$

where

$$\Phi(x, z) = H(\zeta)Q^{-1}(\zeta)x + Y(\zeta)\Lambda^{-1}(\zeta)\varphi(z),$$

and  $[H(\zeta) \ Y(\zeta)] = \sum_{i=1}^N \eta_i(\zeta) [H_i \ Y_i]$ , for  $\forall \eta(\zeta) \in \Xi$ . If  $x \in \mathcal{D}_u \subset \mathbb{R}^{n_x}$ , then the following inequality is verified:

$$\mathcal{SC}(\psi(u), U(\zeta)) = \psi(u)^\top U^{-1}(\zeta) [u - \psi(u) + \Phi(x, z)] \geq 0,$$

for  $\psi(u)$  defined in (9), and for any diagonal positive definite matrix  $U(\zeta) = \sum_{i=1}^N \eta_i(\zeta) U_i$ .

*Proof.* The proof of this lemma follows the similar lines as Lemma 1 in [26]. Then, it is omitted here for brevity.  $\square$

Lemma 1 presents a new sector condition to deal with the dead-zone nonlinearity  $\psi(u)$  in the presence of the cone-bounded nonlinearity  $\varphi(z)$  satisfying (2). This technical lemma provides a regional characterization of the closed-loop properties for the nonlinear TS fuzzy system (10) by means of an extension of the absolute stability theory [47].

### III. STATIC OUTPUT FEEDBACK CONTROL DESIGN FOR CONSTRAINED N-TS FUZZY SYSTEMS

The following theorem provides a numerically tractable solution to design a nonlinear SOF controller (8) that can stabilize asymptotically the N-TS fuzzy system (3). For brevity, we denote  $\Theta(\zeta_+) = \sum_{i=1}^N \eta_i(\zeta(k+1))\Theta_i$  for any matrices  $\Theta_i$  of appropriate dimensions.

**Theorem 1.** Consider the N-TS fuzzy system (3) with the validity domain  $\mathcal{D}_x$  defined in (5). If there exist symmetric positive definite matrices  $Q_i, \bar{Q} \in \mathbb{R}^{n_x \times n_x}$ , diagonal positive definite matrices  $U_i \in \mathbb{R}^{n_u \times n_u}$ ,  $\Lambda_i \in \mathbb{R}^{n_z \times n_z}$ , matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ ,  $F_i \in \mathbb{R}^{n_u \times n_z}$ ,  $M_i \in \mathbb{R}^{n_y \times n_y}$ ,  $Y_i \in \mathbb{R}^{n_u \times n_z}$ , for  $i \in \mathcal{I}[1, N]$ , and a positive scalar  $\epsilon > 0$  such that the optimization problem (18) is feasible. Then, the SOF controller (8) solves Problem 1 with the guaranteed contractively invariant set  $\mathcal{L}_\mathbb{V}$  of the closed-loop system (10).

$$\begin{aligned} \max_{\xi_i, i \in \mathcal{I}[1, N]} \quad & \log \det(\bar{Q}) \\ \text{s.t.} \quad & \text{inequalities (19), (20), (21), (22)} \end{aligned} \quad (18)$$

where  $\xi_i = (\epsilon, Q_i, \bar{Q}, U_i, \Lambda_i, H_i, K_i, F_i, M_i, Y_i)$  and

$$\begin{bmatrix} \bar{Q} & \star & \star \\ \bar{Q} & Q_i & \star \\ \Omega C_i \bar{Q} & 0 & \Lambda_i \end{bmatrix} \succeq 0 \quad (19)$$

$$\begin{bmatrix} Q_i & \star & \star \\ 0 & \Lambda_i & \star \\ H_{i(l)} & Y_{i(l)} & \bar{u}_i^2 \end{bmatrix} \succeq 0 \quad (20)$$

$$\begin{bmatrix} Q_i & \star & \star \\ S_{(m)} Q_i & I & \star \\ 0 & 0 & \Lambda_i \end{bmatrix} \succeq 0 \quad (21)$$

$$\left. \begin{aligned} \Upsilon_{iik} &< 0, & i = j = q \\ \Upsilon_{iik} + \Upsilon_{iqk} + \Upsilon_{qik} &< 0, & i = j, j < q \\ \Upsilon_{ijqk} + \Upsilon_{iqjk} + \Upsilon_{jqik} && \\ + \Upsilon_{jqik} + \Upsilon_{qijk} + \Upsilon_{qjik} &< 0, & i < j, j < q \\ \Upsilon_{iqqk} + \Upsilon_{qiqk} + \Upsilon_{qqik} &< 0, & i < j, j = q \end{aligned} \right\} \quad (22)$$

for  $\forall i, j, q, k \in \mathcal{I}[1, N]$ ,  $\forall m \in \mathcal{I}[1, n_s]$ , and  $\forall l \in \mathcal{I}[1, n_u]$ . The quantity  $\Upsilon_{ijqk}$  in (22) is defined as

$$\Upsilon_{ijqk} = \begin{bmatrix} -Q_j & \star & \star & \star & \star & \star \\ \Upsilon_{jq}^{[21]} & -2U_j & \star & \star & \star & \star \\ 0 & \Upsilon_j^{[32]} & -\Lambda_j & \star & \star & \star \\ \Upsilon_{ijqk}^{[41]} & \Upsilon_{ijk}^{[42]} & \Upsilon_{ijk}^{[43]} & -\Lambda_k & \star & \star \\ \Upsilon_{ijq}^{[51]} & -B_i U_j & \Upsilon_{ij}^{[53]} & 0 & -Q_k & \star \\ \Upsilon_{jq}^{[61]} & \epsilon K_j^\top & 0 & \Upsilon_{ijk}^{[64]} & \epsilon K_j^\top B_i^\top & \Upsilon_j^{[66]} \end{bmatrix},$$

with

$$\begin{aligned} \Upsilon_{jq}^{[21]} &= K_j C_q + H_j, & \Upsilon_j^{[32]} &= F_j^\top + Y_j^\top, \\ \Upsilon_{ijqk}^{[41]} &= \Omega L_k (A_i Q_j + B_i K_j C_q), & \Upsilon_{ijk}^{[42]} &= -\Omega L_k B_i U_j, \\ \Upsilon_{ijk}^{[43]} &= \Omega L_k (G_i \Lambda_j + B_i F_j), & \Upsilon_{ijq}^{[51]} &= A_i Q_j + B_i K_j C_q, \\ \Upsilon_{ij}^{[53]} &= G_i \Lambda_j + B_i F_j, & \Upsilon_{jq}^{[61]} &= C_q Q_j - M_j C_q, \\ \Upsilon_{ijk}^{[64]} &= \epsilon K_j^\top B_i^\top L_k^\top \Omega, & \Upsilon_j^{[66]} &= -\epsilon (M_j + M_j^\top). \end{aligned}$$

*Proof.* Since  $\eta(\zeta(k)), \eta(\zeta(k+1)) \in \Xi$ , for  $\forall k \in \mathbb{Z}_+$ , then condition (22) implies that

$$\begin{aligned} & \sum_{i=1}^N \sum_{k=1}^N \eta_i^3(\zeta) \eta_k(\zeta_+) \Upsilon_{iik} \\ & + \sum_{i=1}^{N-1} \sum_{q>i}^N \sum_{k=1}^N \eta_i^2(\zeta) \eta_q(z) \eta_k(\zeta_+) \hat{\Upsilon}_1 \\ & + \sum_{i=1}^{N-1} \sum_{q>i}^N \sum_{k=1}^N \eta_i(\zeta) \eta_q^2(\zeta) \eta_k(\zeta_+) \hat{\Upsilon}_2 \\ & + \sum_{i=1}^{N-2} \sum_{j>i}^{N-1} \sum_{q>j}^N \sum_{k=1}^N \eta_i(\zeta) \eta_j(\zeta) \eta_q(\zeta) \eta_k(\zeta_+) \hat{\Upsilon}_3 \prec 0, \quad (23) \end{aligned}$$

with

$$\begin{aligned} \hat{\Upsilon}_1 &= \Upsilon_{iiqk} + \Upsilon_{iqik} + \Upsilon_{qii k}, \\ \hat{\Upsilon}_2 &= \Upsilon_{iiqq} + \Upsilon_{qi qk} + \Upsilon_{qqik}, \\ \hat{\Upsilon}_3 &= \Upsilon_{ijqk} + \Upsilon_{iqjk} + \Upsilon_{jiqk} + \Upsilon_{jqik} + \Upsilon_{qijk} + \Upsilon_{qjik}. \end{aligned}$$

Condition (23) can be rewritten in the following compact form:

$$\Upsilon = \sum_{i=1}^N \sum_{j=1}^N \sum_{q=1}^N \sum_{k=1}^N \eta_i(\zeta) \eta_j(\zeta) \eta_q(\zeta) \eta_k(\zeta_+) \Upsilon_{ijqk} \prec 0, \quad (24)$$

Note that inequality (24) can be, in turn, represented as

$$\begin{bmatrix} -Q(\zeta) & \star & \star & \star & \star & \star \\ \Upsilon_{21} & -2U(\zeta) & \star & \star & \star & \star \\ 0 & \Upsilon_{32} & -\Lambda(\zeta) & \star & \star & \star \\ \Upsilon_{41} & \Upsilon_{42} & \Upsilon_{43} & \Upsilon_{44} & \star & \star \\ \Upsilon_{51} & \Upsilon_{52} & \Upsilon_{53} & 0 & \Upsilon_{55} & \star \\ \Upsilon_{61} & \Upsilon_{62} & 0 & \Upsilon_{64} & \Upsilon_{65} & \Upsilon_{66} \end{bmatrix} \prec 0, \quad (25)$$

with

$$\begin{aligned} \Upsilon_{21} &= K(\zeta)C(\zeta) + H(\zeta), & \Upsilon_{32} &= F^\top(\zeta) + Y^\top(\zeta), \\ \Upsilon_{41} &= \Omega L(\zeta_+) (A(\zeta)Q(\zeta) + B(\zeta)K(\zeta)C(\zeta)), \\ \Upsilon_{42} &= -\Omega L(\zeta_+) B(\zeta)U(\zeta), & \Upsilon_{44} &= -\Lambda(\zeta_+), \\ \Upsilon_{43} &= \Omega L(\zeta_+) (G(\zeta)\Lambda(\zeta) + B(\zeta)F(\zeta)), \\ \Upsilon_{51} &= A(\zeta)Q(\zeta) + B(\zeta)K(\zeta)C(\zeta), & \Upsilon_{52} &= -B(\zeta)U(\zeta), \\ \Upsilon_{53} &= G(\zeta)\Lambda(\zeta) + B(\zeta)F(\zeta), & \Upsilon_{55} &= -Q(\zeta_+), \\ \Upsilon_{61} &= C(\zeta)Q(\zeta) - M(\zeta)C(\zeta), & \Upsilon_{62} &= \epsilon K^\top(\zeta), \\ \Upsilon_{64} &= \epsilon K^\top(\zeta)B^\top(\zeta)L^\top(\zeta_+)\Omega, & \Upsilon_{65} &= \epsilon K^\top(\zeta)B^\top(\zeta), \\ \Upsilon_{66} &= -\epsilon (M(\zeta) + M^\top(\zeta)). \end{aligned}$$

Note that inequality (25) implies  $M(\zeta) + M^\top(\zeta) \succ 0$ . This guarantees that  $M(\zeta)$  is *nonsingular*, thus the existence of  $M^{-1}(\zeta)$ . Multiplying inequality (25) by

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & K(\zeta)M^{-1}(\zeta) \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & \Omega L(\zeta_+)B(\zeta)K(\zeta)M^{-1}(\zeta) \\ 0 & 0 & 0 & 0 & I & B(\zeta)K(\zeta)M^{-1}(\zeta) \end{bmatrix},$$

on the left and its transpose on the right produces

$$\begin{bmatrix} -Q(\zeta) & \star & \star & \star & \star \\ \Gamma_{21} & -2U(\zeta) & \star & \star & \star \\ 0 & \Gamma_{32} & -\Lambda(\zeta) & \star & \star \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & -\Lambda(\zeta_+) & \star \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & 0 & -Q(\zeta_+) \end{bmatrix} \prec 0, \quad (26)$$

where

$$\begin{aligned} \Gamma_{21} &= K(\zeta)M^{-1}(\zeta)C(\zeta) + H(\zeta), \\ \Gamma_{32} &= \Upsilon_{32} = F^\top(\zeta) + Y^\top(\zeta), & \Gamma_{42} &= -\Omega L(\zeta_+)B(\zeta)U(\zeta), \\ \Gamma_{41} &= \Omega L(\zeta_+) (A(\zeta)Q(\zeta) + B(\zeta)K(\zeta)M^{-1}(\zeta)C(\zeta)Q(\zeta)), \\ \Gamma_{43} &= \Omega L(\zeta_+) (G(\zeta)\Lambda(\zeta) + B(\zeta)F(\zeta)), \\ \Gamma_{51} &= A(\zeta)Q(\zeta) + B(\zeta)K(\zeta)M^{-1}(\zeta)C(\zeta)Q(\zeta), \\ \Gamma_{52} &= B(\zeta)U(\zeta), & \Gamma_{53} &= G(\zeta)\Lambda(\zeta) + B(\zeta)F(\zeta). \end{aligned}$$

Pre- and postmultiplying inequality (26) by the block-diagonal matrix  $\text{diag}(Q^{-1}(\zeta), U^{-1}(\zeta), \Lambda^{-1}(\zeta), \Lambda^{-1}(\zeta_+), I)$  yields

$$\begin{bmatrix} Q^{-1}(\zeta) & \star & \star & \star & \star \\ \Sigma_{21} & 2U^{-1}(\zeta) & \star & \star & \star \\ 0 & \Sigma_{32} & \Lambda^{-1}(\zeta) & \star & \star \\ -\Sigma_{41} & -\Sigma_{42} & -\Sigma_{43} & \Lambda^{-1}(\zeta_+) & \star \\ \Sigma_{51} & B(\zeta) & \Sigma_{53} & 0 & Q(\zeta_+) \end{bmatrix} \succeq 0, \quad (27)$$

where

$$\begin{aligned} \Sigma_{21} &= -U^{-1}(\zeta) (K(\zeta)M^{-1}(\zeta)C(\zeta) + H(\zeta)Q^{-1}(\zeta)), \\ \Sigma_{32} &= -\Lambda^{-1}(\zeta) (F^\top(\zeta) + Y^\top(\zeta)) U^{-1}(\zeta), \\ \Sigma_{41} &= \Lambda^{-1}(\zeta_+) \Omega L(\zeta_+) A_{cl}(\zeta), \\ \Sigma_{42} &= -\Lambda^{-1}(\zeta_+) \Omega L(\zeta_+) B(\zeta), \\ \Sigma_{43} &= \Lambda^{-1}(\zeta_+) \Omega L(\zeta_+) G_{cl}(\zeta), \\ \Sigma_{51} &= -A(\zeta) - B(\zeta)K(\zeta)M^{-1}(\zeta)C(\zeta), \\ \Sigma_{53} &= -G(\zeta) - B(\zeta)F(\zeta)\Lambda^{-1}(\zeta). \end{aligned}$$

By Schur complement lemma [4], we prove that inequality (27) is equivalent to

$$\begin{bmatrix} Q^{-1}(\zeta) & \star & \star & \star \\ \Sigma_{21} & 2U^{-1}(\zeta) & \star & \star \\ 0 & \Sigma_{32} & \Lambda^{-1}(\zeta) & \star \\ -\Sigma_{41} & -\Sigma_{42} & -\Sigma_{43} & \Lambda^{-1}(\zeta_+) \end{bmatrix} - \mathcal{P} \succeq 0, \quad (28)$$

with  $\mathcal{P} = \mathcal{W}^\top Q^{-1}(\zeta_+) \mathcal{W}$  and  $\mathcal{W} = [\Sigma_{51} \ B(\zeta) \ \Sigma_{53} \ 0]$ . Let us denote  $\varphi(z_+) = \varphi(z(k+1))$ . Pre- and postmultiplying (28) with  $[x^\top \ \psi(u)^\top \ \varphi(z)^\top \ \varphi(z_+)^\top]$  and its transpose together with the use of expressions (8) and (10), we obtain the following inequality after simple manipulations:

$$\begin{aligned} & \delta_k \nabla + 2\psi(u)^\top U^{-1}(\zeta) [u - \psi(u) + \Phi(x, z)] \\ & - 2\varphi(z_+)^\top \Lambda^{-1}(\zeta_+) [\varphi(z_+) - \Omega L(\zeta_+) z_+] \prec 0, \quad (29) \end{aligned}$$

where  $\delta_k \nabla$  is defined in (16).

Since  $\eta(\zeta) \in \Xi$ , then condition (20) implies that

$$\begin{bmatrix} Q(\zeta) & \star & \star \\ 0 & \Lambda(\zeta) & \star \\ H(\zeta)_{(l)} & Y(\zeta)_{(l)} & \bar{u}_l^2 \end{bmatrix} \succeq 0, \quad l \in \mathcal{I}[1, n_u]. \quad (30)$$

Pre- and postmultiplying inequality (30) by the block-diagonal matrix  $\text{diag}(Q^{-1}(\zeta), \Lambda^{-1}(\zeta), I)$  leads to

$$\begin{bmatrix} Q^{-1}(\zeta) & \star & \star \\ 0 & \Lambda^{-1}(\zeta) & \star \\ H(\zeta)_{(l)}Q^{-1}(\zeta) & Y(\zeta)_{(l)}\Lambda^{-1}(\zeta) & \bar{u}_l^2 \end{bmatrix} \succeq 0, \quad (31)$$

for  $l \in \mathcal{I}[1, n_u]$ . Applying Schur complement lemma to (31), it is easy to show that

$$\begin{bmatrix} Q^{-1}(\zeta) & \star \\ 0 & \Lambda^{-1}(\zeta) \end{bmatrix} - \frac{1}{\bar{u}_l^2} \mathcal{R}^\top \mathcal{R} \succeq 0, \quad (32)$$

with  $\mathcal{R} = [H(\zeta)_{(l)}Q^{-1}(\zeta) \ Y(\zeta)_{(l)}\Lambda^{-1}(\zeta)]$ ,  $l \in \mathcal{I}[1, n_u]$ . Pre- and postmultiplying (32) with  $[x^\top \ \varphi(z)^\top]$  and its transpose, we obtain the following condition:

$$\mathbb{V}(x, \varphi(z), \zeta) \geq \frac{1}{\bar{u}_l^2} \begin{bmatrix} x^\top \\ \varphi(z)^\top \end{bmatrix} \mathcal{R}^\top \mathcal{R} \begin{bmatrix} x \\ \varphi(z) \end{bmatrix}, \quad (33)$$

for  $\forall l \in \mathcal{I}[1, n_u]$ ,  $\forall x \in \mathbb{R}^{n_x}$ , and  $\forall \varphi(z)$  satisfying (2). For  $\forall x \in \mathcal{L}_V$ , it follows from (33) that  $x \in \mathcal{D}_u$  defined in (17), namely  $\mathcal{L}_V \subseteq \mathcal{D}_u$ , since  $\mathbb{V}(x, \zeta) \leq 1$ , for  $\forall \eta(\zeta) \in \Xi$ . Following the same line, we can prove that (21) implies the inclusion  $\mathcal{L}_V \subseteq \mathcal{D}_x$  with  $\mathcal{D}_x$  defined in (5).

Since  $x \in \mathcal{D}_u$ , by Lemma 1 and the sector property (2), it follows from (29) that  $\delta_k \mathbb{V} < 0$ , for  $\forall x \in \mathcal{L}_V \setminus \{0\}$ , and  $\forall \eta(\zeta) \in \Xi$ . Thus, we can conclude that inequalities (20), (21) and (22) guarantee that  $\mathcal{L}_V$  is a contractively invariant set with respect to the closed-loop system (10) and  $\mathcal{L}_V \subseteq \mathcal{D}_x$ . Furthermore, inequality (19) implies that

$$\begin{bmatrix} \bar{Q} & \star & \star \\ \bar{Q} & Q(\zeta) & \star \\ \Omega C(\zeta)\bar{Q} & 0 & \Lambda(\zeta) \end{bmatrix} \succeq 0. \quad (34)$$

By Schur complement lemma, followed by a congruence transformation with  $\text{diag}(\bar{Q}^{-1}, \Lambda^{-1}(\zeta))$ , we can prove that (34) is equivalent to

$$\begin{bmatrix} \bar{Q}^{-1} - Q^{-1}(\zeta) & \star \\ \Lambda^{-1}(\zeta)\Omega C(\zeta) & \Lambda^{-1}(\zeta) \end{bmatrix} \succeq 0. \quad (35)$$

Pre- and postmultiplying (35) with  $[x^\top \ \varphi(z)^\top]$ , the following inequality can be obtained after some manipulations:

$$x^\top \bar{Q}^{-1} x + 2\mathcal{S}C(\varphi(z), \Lambda(\zeta)) \geq \mathbb{V}(x, \varphi, \zeta), \quad (36)$$

where  $\mathcal{S}C(\varphi(z), \Lambda(\zeta))$  and  $\mathbb{V}(x, \varphi, \zeta)$  are respectively defined in (2) and (11). Since  $\mathcal{S}C(\varphi(z), \Lambda(\zeta)) \leq 0$ , it follows from (36) that  $x^\top \bar{Q}^{-1} x \geq \mathbb{V}(x, \varphi, \zeta)$ . This, in turn, guarantees that  $\mathcal{E}(\bar{Q}^{-1}) \subseteq \mathcal{L}_V$ . Note that the quantity  $\log \det(\bar{Q})$  represents the volume of the ellipsoid  $\mathcal{E}(\bar{Q}^{-1})$  [4]. Then, the optimization problem (18) implicitly maximizes the volume of the set  $\mathcal{L}_V$  included inside  $\mathcal{D}_x$ . This concludes the proof.  $\square$

**Remark 4.** Several relaxation results with different degrees of design conservatism and/or numerical complexity [57] can be directly applied to convert the MF-based inequality (24) into a finite set of LMI-based constraints.

**Remark 5.** The nonlinear SOF control law (8) is constructed using the slack variable  $M(\zeta)$ , which is *independent* to the Lyapunov matrix  $Q(\zeta)$ . Together with the specific congruence

transformation to obtain (26), this feature enables an LMI-based formulation to design a SOF controller (8) for the N-TS fuzzy system (3) where all the matrix decision variables can be MF-dependent to reduce the conservatism.

**Remark 6.** Theorem 1 is a *generalization* of the SOF control result using linear matrix equalities to convexify the design conditions [40]. This result is formulated in Corollary 1.

**Corollary 1.** Consider the N-TS fuzzy system (3) with the validity domain  $\mathcal{D}_x$  defined in (5), and  $C_i = C$ ,  $\forall i \in \mathcal{I}[1, N]$ , where  $C$  is a matrix of full row rank. If there exist symmetric positive definite matrices  $Q_i, \bar{Q} \in \mathbb{R}^{n_x \times n_x}$ , diagonal positive definite matrices  $U_i \in \mathbb{R}^{n_u \times n_u}$ ,  $\Lambda_i \in \mathbb{R}^{n_z \times n_z}$ , and matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ ,  $F_i \in \mathbb{R}^{n_u \times n_z}$ ,  $M_i \in \mathbb{R}^{n_y \times n_y}$ ,  $Y_i \in \mathbb{R}^{n_u \times n_z}$ , for  $i \in \mathcal{I}[1, N]$ , such that the optimization problem (1) is feasible. Then, the SOF controller (8) solves Problem 1 with the guaranteed contractively invariant set  $\mathcal{L}_V$  of the closed-loop fuzzy system (10).

$$\begin{aligned} \max_{\chi_i, i \in \mathcal{I}[1, N]} \quad & \log \det(\bar{Q}) \\ \text{s.t.} \quad & \text{inequalities (19), (20), (21), (37), (38)} \end{aligned}$$

where  $\chi_i = (Q_i, \bar{Q}, U_i, \Lambda_i, H_i, K_i, F_i, M_i, Y_i)$  and

$$\begin{aligned} CQ_i &= M_i C & (37) \\ \left. \begin{aligned} \Upsilon_{iik}^* &< 0, & i = j = q \\ \Upsilon_{iik}^* + \Upsilon_{iik}^* + \Upsilon_{qik}^* &< 0, & i = j, j < q \\ \Upsilon_{ijqk}^* + \Upsilon_{ijqk}^* + \Upsilon_{jqk}^* \\ + \Upsilon_{jqik}^* + \Upsilon_{qijk}^* + \Upsilon_{qijk}^* &< 0, & i < j, j < q \\ \Upsilon_{iqk}^* + \Upsilon_{qik}^* + \Upsilon_{qik}^* &< 0, & i < j, j = q \end{aligned} \right\} & (38) \end{aligned}$$

for  $\forall i, j, q, k \in \mathcal{I}[1, N]$ , and  $\forall l \in \mathcal{I}[1, n_u]$ . The quantity  $\Upsilon_{ijqk}^*$  in (38) is defined as

$$\Upsilon_{ijqk}^* = \begin{bmatrix} -Q_j & \star & \star & \star & \star \\ \Upsilon_{jq}^{[21]} & -2U_j & \star & \star & \star \\ 0 & \Upsilon_j^{[32]} & -\Lambda_j & \star & \star \\ \Upsilon_{ijqk}^{[41]} & \Upsilon_{ijk}^{[42]} & \Upsilon_{ijk}^{[43]} & -\Lambda_k & \star \\ \Upsilon_{ijq}^{[51]} & -B_i U_j & G_i \Lambda_j + B_i F_j & 0 & -Q_k \end{bmatrix}.$$

**Remark 7.** In contrast to Theorem 1, Corollary 1 requires two additional restrictions: (i)  $C_i = C$ ,  $\forall i \in \mathcal{I}[1, N]$ , and (ii)  $C$  must be of full row rank. Restriction (i) is imposed since in general  $C(\zeta)Q(\zeta) = M(\zeta)C(\zeta)$  cannot be verified when  $C_i \neq C_j$ , for  $i, j \in \mathcal{I}[1, N]$  and  $i \neq j$ , see [37]. Note that equality (37) implies that

$$CQ(\zeta) = M(\zeta)C. \quad (39)$$

Then, with  $Q(\zeta) \succ 0$ , restriction (ii) aims to guarantee the nonsingularity of  $M(\zeta)$ . The design conditions in Theorem 1 allow avoiding not only these restrictions but also LME condition (37), which may induce numerical difficulties. Remark that the SOF control approaches in [36], [38]–[41], [56] for classical TS fuzzy systems lead to the same drawbacks.

**Remark 8.** Since  $\eta(\zeta(k)), \eta(\zeta(k+1)) \in \Xi$ , for  $\forall k \in \mathbb{Z}_+$ , then condition (38) implies that

$$\Upsilon^* = \begin{bmatrix} -Q(\zeta) & \star & \star & \star & \star \\ \Upsilon_{21} & -2U(\zeta) & \star & \star & \star \\ 0 & \Upsilon_{32} & -\Lambda(\zeta) & \star & \star \\ \Upsilon_{41} & \Upsilon_{42} & \Upsilon_{43} & \Upsilon_{44} & \star \\ \Upsilon_{51} & \Upsilon_{52} & \Upsilon_{53} & 0 & \Upsilon_{55} \end{bmatrix} \prec 0. \quad (40)$$

Applying Schur complement lemma to (25) in the proof of Theorem 1 while considering equality (39) yields

$$\Upsilon^* + \epsilon \mathcal{Z}^\top (M(\zeta) + M^\top(\zeta))^{-1} \mathcal{Z} \prec 0, \quad (41)$$

where  $\mathcal{Z} = [\Upsilon_{61} \ \Upsilon_{62} \ 0 \ \Upsilon_{64} \ \Upsilon_{65}]$ . It is clear that (41) is equivalent to (40) for a sufficiently small scalar  $\epsilon > 0$ . Then, for this case, it is expected that Theorem 1 provides less conservative results than those of Corollary 1.

**Remark 9.** The design of *structured* controllers can be reformulated as a SOF control problem. Indeed, the expression  $\mathcal{H}(\zeta) = K(\zeta)M^{-1}(\zeta)$  enables us to design a structured gain  $\mathcal{H}(\zeta)$  by imposing  $M(\zeta)$  as a block-diagonal matrix and  $K(\zeta)$  with the same structure desired for  $\mathcal{H}(\zeta)$ . For example, if we are interested in obtaining  $\mathcal{H}(\zeta)$  with

$$\mathcal{H}(\zeta) = \begin{bmatrix} \mathcal{H}_{11}(\zeta) & \mathcal{H}_{12}(\zeta) & 0 \\ 0 & \mathcal{H}_{22}(\zeta) & 0 \\ \mathcal{H}_{31}(\zeta) & 0 & \mathcal{H}_{33}(\zeta) \end{bmatrix}.$$

Then, it suffices to include the additional constraints

$$M(\zeta) = \text{diag}(M_{11}(\zeta), M_{22}(\zeta), M_{33}(\zeta)),$$

$$K(\zeta) = \begin{bmatrix} K_{11}(\zeta) & K_{12}(\zeta) & 0 \\ 0 & K_{22}(\zeta) & 0 \\ K_{31}(\zeta) & 0 & K_{33}(\zeta) \end{bmatrix},$$

into the design conditions of Theorem 1 and Corollary 1. A similar remark can be reported for the control gain  $\mathcal{F}(\zeta) = F(\zeta)\Lambda^{-1}(\zeta)$ , where  $\Lambda(\zeta)$  is already a block-diagonal matrix.

#### IV. EXTENSION TO ROBUST SOF CONTROL DESIGN FOR CONSTRAINED UNCERTAIN N-TS FUZZY SYSTEMS

This section extends the proposed control approach to N-TS fuzzy systems (3) including structured uncertainties. To this end, let us consider the following uncertain system:

$$x(k+1) = \sum_{i=1}^N \eta_i(\zeta) \left( \hat{A}_i x(k) + \hat{G}_i \varphi(z(k)) + \hat{B}_i \text{sat}(u(k)) \right)$$

$$z(k) = \sum_{i=1}^N \eta_i(\zeta) L_i x(k), \quad y(k) = \sum_{i=1}^N \eta_i(\zeta) C_i x(k), \quad (42)$$

where  $\hat{A}_i = A_i + \Delta A_i(k)$ ,  $\hat{B}_i = B_i + \Delta B_i(k)$  and  $\hat{G}_i = G_i + \Delta G_i(k)$ , for  $i \in \mathcal{I}[1, N]$ . The structural uncertainties are described as

$$[\Delta A_i(k) \ \Delta B_i(k) \ \Delta G_i(k)] = D\Delta(k) [E_{A_i} \ E_{B_i} \ E_{G_i}],$$

where the *known* constant matrices of appropriate dimensions  $D$ ,  $E_{A_i}$ ,  $E_{B_i}$  and  $E_{G_i}$ , with  $i \in \mathcal{I}[1, N]$ , characterize how the nominal state-space matrices are affected by the uncertainty. The *unknown* matrix-function  $\Delta(\cdot)$  is with Lebesgue measurable elements and satisfies  $\Delta(k)^\top \Delta(k) \preceq I$ , for  $\forall k \in \mathbb{Z}_+$ .

Now we state the following result on nonlinear robust SOF control design for saturated TS fuzzy system (42).

**Theorem 2.** Consider the N-TS fuzzy system (42) with the validity domain  $\mathfrak{D}_x$  defined in (5). If there exist symmetric positive definite matrices  $Q_i, \bar{Q} \in \mathbb{R}^{n_x \times n_x}$ , diagonal positive definite matrices  $U_i \in \mathbb{R}^{n_u \times n_u}$ ,  $\Lambda_i \in \mathbb{R}^{n_z \times n_z}$ , matrices  $H_i \in \mathbb{R}^{n_u \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_y}$ ,  $F_i \in \mathbb{R}^{n_u \times n_z}$ ,  $M_i \in \mathbb{R}^{n_y \times n_y}$ ,  $Y_i \in \mathbb{R}^{n_u \times n_z}$ , for  $i \in \mathcal{I}[1, N]$ , and positive scalars  $\epsilon, \mu_i, \nu_i$  such that the optimization problem (43) is feasible. Then, the SOF controller (8) solves Problem 1 with the guaranteed contractively invariant set  $\mathcal{L}_V$  of the closed-loop system (42).

$$\max_{\sigma_i, i \in \mathcal{I}[1, N]} \log \det(\bar{Q}) \quad (43)$$

s.t. inequalities (19), (20), (21), (44)

where  $\sigma_i = (\epsilon, \mu_i, \nu_i, Q_i, \bar{Q}, U_i, \Lambda_i, H_i, K_i, F_i, M_i, Y_i)$  and

$$\left. \begin{aligned} \Psi_{iik} &\prec 0, & i = j = q \\ \Psi_{iiq} + \Psi_{iqi} + \Psi_{qii} &\prec 0, & i = j, j < q \\ \Psi_{ijq} + \Psi_{iqj} + \Psi_{jqi} & & \\ + \Psi_{jqk} + \Psi_{qjk} + \Psi_{qjk} &\prec 0, & i < j, j < q \\ \Psi_{iqk} + \Psi_{qik} + \Psi_{qqk} &\prec 0, & i < j, j = q \end{aligned} \right\} \quad (44)$$

for  $\forall i, j, q, k \in \mathcal{I}[1, N]$ , and  $\forall l \in \mathcal{I}[1, n_u]$ . The quantity  $\Psi_{ijqk}$  in (44) is defined as

$$\Psi_{ijqk} = \begin{bmatrix} \Upsilon_{ijqk} & \star & \star \\ \mathcal{G}_i \mathcal{Q}_k^\top & -\mathcal{G}_i & \star \\ \mathcal{W}_{ijqk} & 0 & -\mathcal{G}_i \end{bmatrix}, \quad (45)$$

with  $\Upsilon_{ijqk}$  defined in Theorem 1,  $\mathcal{G}_i = \mu_i I$ , and

$$\mathcal{Q}_k = [0 \ 0 \ 0 \ D^\top L_k^\top \Omega^\top \ D^\top \ 0]^\top,$$

$$\mathcal{W}_{ijqk} = [\mathcal{L}_{ijq} \ -E_{B_i} U_j \ \mathcal{N}_{ijk} \ 0 \ 0 \ \tau E_{B_i} K_j],$$

$$\mathcal{L}_{ijq} = E_{A_i} Q_j + E_{B_i} K_j C_q, \quad \mathcal{N}_{ijk} = E_{G_i} \Lambda_k + E_{B_i} F_j.$$

*Proof.* As in Theorem 1, it follows from (44) that

$$\Psi = \sum_{i=1}^N \sum_{j=1}^N \sum_{q=1}^N \sum_{k=1}^N \eta_i(\zeta) \eta_j(\zeta) \eta_q(\zeta) \eta_k(\zeta_+) \Psi_{ijqk} \prec 0, \quad (46)$$

with  $\Psi_{ijqk}$  defined in (45). Applying successively the Schur complement lemma, we can prove that (46) is equivalent to

$$\Upsilon + \mathcal{Q} \mathcal{G} \mathcal{Q}^\top + \mathcal{W}^\top \mathcal{G}^{-1} \mathcal{W} \prec 0, \quad (47)$$

where  $\Upsilon$  defined in (24) and

$$\mathcal{Q} = \sum_{k=1}^N \eta_k(\zeta_+) \mathcal{Q}_k, \quad \mathcal{G} = \sum_{i=1}^N \eta_i(\zeta) \mathcal{G}_i,$$

$$\mathcal{W} = \sum_{i=1}^N \sum_{j=1}^N \sum_{q=1}^N \sum_{k=1}^N \eta_i(\zeta) \eta_j(\zeta) \eta_q(\zeta) \eta_k(\zeta_+) \mathcal{W}_{ijqk}.$$

Denote  $\Delta(k) = \text{diag}(\Delta(k), \Delta(k))$ . Since  $\Delta^\top(k) \Delta(k) \preceq I$ , using the following well-known matrix property [50]:

$$\mathcal{X}^\top \mathcal{Y} + \mathcal{Y}^\top \mathcal{X} \preceq \mathcal{X}^\top \mathcal{G} \mathcal{X} + \mathcal{Y}^\top \mathcal{G}^{-1} \mathcal{Y},$$

with  $\mathcal{X} = \mathcal{Q}^\top$  and  $\mathcal{Y} = \Delta(k) \cdot \mathcal{W}$ , it follows from (47) that

$$\Upsilon + \mathcal{Q} \Delta(k) \mathcal{W} + \mathcal{W}^\top \Delta^\top(k) \mathcal{Q}^\top \prec 0. \quad (48)$$

Following similar arguments as in the proof of Theorem 1, we show that (48) implies (29) with respect to system (42). Then, by the results of Theorem 1, the proof can be concluded.  $\square$

**Remark 10.** Note that the fuzzy output feedback control problem with *unmeasurable* premise variables is very challenging [58], especially for SOF control schemes [13], [37]. The robust control result in Theorem 2 may be useful to deal with not only the norm-bounded uncertainty but also partially measurable premise vector  $\zeta$ . To this end, the set of premise variables can be divided into two following subsets:

$$\begin{aligned} \mathcal{M} &= \{\zeta_i \in \zeta : \zeta_i \text{ is measurable, } i \in \mathcal{I}[1, p_m]\}, \\ \mathcal{U} &= \{\zeta_j \in \zeta : \zeta_j \text{ is unmeasurable, } j \in \mathcal{I}[p_m + 1, p]\}, \end{aligned}$$

with  $p_m \in \mathbb{Z}_+$  and  $p_m \leq p$ . Then, only premise variables belonging to  $\mathcal{M}$  are used to construct the nonlinear control law (8). Each unmeasurable premise variable belonging to  $\mathcal{U}$  can be rewritten in the form

$$\zeta_j(k) = \zeta_{jm} + \delta_j(k)\zeta_{jr}, \quad \forall j \in \mathcal{I}[p_m + 1, p], \quad (49)$$

where  $\delta_j(k) \in [-1, 1]$ , for  $\forall k \in \mathbb{Z}_+$ , and

$$\zeta_{jm} = \frac{\zeta_{j \max} + \zeta_{j \min}}{2}, \quad \zeta_{jr} = \frac{\zeta_{j \max} - \zeta_{j \min}}{2}.$$

Since  $x \in \mathcal{D}_x$  defined in (5), the upper and lower bounds of each premise variable, *i.e.*,  $\zeta_j(k) \in [\zeta_{j \min}, \zeta_{j \max}]$ , for  $\forall j \in \mathcal{I}[p_m + 1, p]$ ,  $\forall k \in \mathbb{Z}_+$ , can be easily computed. Substituting (49) into the original nonlinear system, we can derive the uncertain N-TS fuzzy system (42) with only measurable premise variables and Theorem 2 can be then applied.

**Remark 11.** The optimization problems in Theorems 1 and 2 are expressed in terms of LMIs with a linear search parameter  $\epsilon > 0$ . A gridding method can be performed to search for  $\epsilon$ . To this end, we define a new parameter  $\tau = \frac{\epsilon}{\epsilon+1}$ , thus  $\epsilon = \frac{\tau}{1-\tau}$ . Note that  $\epsilon > 0$  if and only if  $\tau \in (0, 1)$ . Then, the optimization problem (18) or (43) can be effectively solved with YALMIP toolbox and SDPT3 solver [59] performing a linear search for the new parameter scalar  $\tau \in (0, 1)$ .

## V. ILLUSTRATIVE EXAMPLES

Numerical simulations are presented hereafter to demonstrate the effectiveness of the proposed SOF control design for constrained N-TS fuzzy systems in the form (1).

**Example 1.** Consider the following nonlinear system:

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 1 - 0.2\zeta_1(k) & -0.35 \\ 0.4 & 0.6 + 0.1\zeta_3(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 1 - 1.2\zeta_1(k) \end{bmatrix} \varphi(z(k)) + \begin{bmatrix} 0.5 + \zeta_2(k) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{sat}(u_1(k)) \\ \text{sat}(u_2(k)) \end{bmatrix} \\ z(k) &= x_1(k), \quad y(k) = [x_1(k) \quad \zeta_2(k)x_2(k)]^\top, \quad (50) \end{aligned}$$

with  $|x_1(k)| \leq \frac{\pi}{2}$ ,  $|x_2(k)| \leq 2$ ,  $\bar{u}_1 = \bar{u}_2 = 1$ ,  $\zeta_1(k) = \sin^2(x_1(k)) \in [0, 1]$ ,  $\zeta_2(k) = x_1^2(k) \in [0, \frac{\pi^2}{4}]$ , and  $\zeta_3(k) = x_2(k) \in [-2, 2]$ . The nonlinearity  $\varphi(z(k))$  is defined as  $\varphi(z(k)) = \Omega \frac{z(k)(1 + \cos(50z(k)))}{2} \in \text{co}\{0, \Omega z(k)\}$ ,  $\Omega = 2$ . Note that the premise variable  $\zeta_3(k)$  is not directly available for measurement. Hence, the procedure in Remark 10 can be used

together with the sector nonlinearity approach [1] to construct the following uncertain N-TS fuzzy model (42) of system (50):

$$\begin{aligned} A_1 = A_2 &= \begin{bmatrix} 1 & -0.35 \\ 0.4 & 0.6 \end{bmatrix}, & A_3 = A_4 &= \begin{bmatrix} 0.8 & -0.35 \\ 0.4 & 0.6 \end{bmatrix}, \\ B_1 = B_3 &= \text{diag}(0.5, 1), & B_2 = B_4 &= \text{diag}(2.967, 1), \\ C_1 = C_3 &= \text{diag}(1, 0), & C_2 = C_4 &= \text{diag}(1, 2.467), \\ G_1 = G_2 &= D = [0 \quad 1]^\top, & G_3 = G_4 &= [0 \quad -0.2]^\top, \\ L_i &= [1 \quad 0], & E_{A_i} &= [0 \quad 0.2], \quad i \in \mathcal{I}[1, 4]. \end{aligned}$$

The corresponding fuzzy MFs are given by

$$\begin{aligned} \mathcal{F}_{11}(\zeta_1(k)) &= 1 - \sin^2 x_1(k), & \mathcal{F}_{21}(\zeta_1(k)) &= \sin^2 x_1(k), \\ \mathcal{F}_{12}(\zeta_2(k)) &= 1 - \frac{4x_1(k)^2}{\pi^2}, & \mathcal{F}_{22}(\zeta_2(k)) &= \frac{4x_1(k)^2}{\pi^2}. \end{aligned}$$

Note that for the case of  $C(\zeta(k)) = C_1 = C_3$ , the second sensor cannot be used to measure the output variables. Since  $C_1$  and  $C_3$  are not of full row rank, the nonlinear output matrix  $C(\zeta(k)) = \sum_{i=1}^4 \eta_i(\zeta(k))C_i$  is not always of full row rank. Hence, Corollary 1 and the SOF control approaches in [36], [39], [41], [56] cannot be applied to system (50). However, by Theorem 2, a SOF control solution can be found with  $\epsilon = 0.0421$ . This shows that the proposed SOF control approach can be applied to a larger class of nonlinear systems than numerous existing results.

Fig. 1 depicts the estimate of the DoA and the phase portrait for the nonlinear system (50) in feedback with the SOF controller (8) designed with Theorem 2. Observe that the DoA estimation given by the set  $\mathcal{L}_V$  (solid blue line) is *nonconvex* and *disconnected*, which is included inside  $\mathcal{D}_x$  and  $\mathcal{D}_u$ . To illustrate the inclusion in (15), the lower and upper bounding ellipsoids respectively denoted by  $\mathcal{E}(\mathcal{Q}(\zeta))$  and  $\mathcal{E}(\mathcal{Q}^{-1}(\zeta))$ , are also depicted (dotted-dashed line). Moreover, the phase portrait points out the stabilizing property of the designed SOF controller with respect to the closed-loop system.

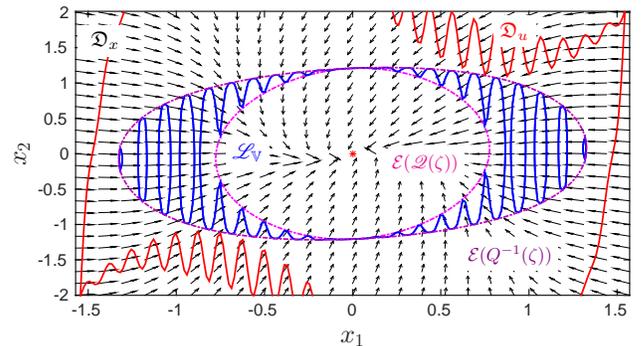


Fig. 1. Estimation of the DoA and phase portrait for the nonlinear system (50) in feedback with the control law (8). The DoA estimation given by the set  $\mathcal{L}_V$  is in solid line and the lower and upper bounding ellipsoids  $\mathcal{E}(\mathcal{Q}(\zeta))$  and  $\mathcal{E}(\mathcal{Q}^{-1}(\zeta))$  are in dotted-dashed lines.

The proposed SOF control results enable designing feedback gains with particular structures as stated in Remark 9. To illustrate this feature, assume that we search for decentralized feedback gains with diagonal structure constraints on the

matrices  $K(\zeta)$  and  $M(\zeta)$ . Then, solving the optimization problem in Theorem 2, the control gains are obtained as

$$\begin{aligned} K_1 &= -\text{diag}(1.494, 0.312), & K_2 &= -\text{diag}(0.680, 0.347), \\ K_3 &= -\text{diag}(1.232, 0.727), & K_4 &= -\text{diag}(0.507, 0.390), \\ M_1 &= \text{diag}(1.239, 1.414), & M_2 &= \text{diag}(2.654, 1.400), \\ M_3 &= \text{diag}(1.410, 1.433), & M_4 &= \text{diag}(2.659, 1.393), \end{aligned}$$

with  $\epsilon = 0.042$ , and the following Lyapunov matrices:

$$\begin{aligned} Q_1 &= \begin{bmatrix} 1.253 & 0.03 \\ 0.03 & 1.395 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 2.467 & 0.08 \\ 0.08 & 1.4 \end{bmatrix}, \\ Q_3 &= \begin{bmatrix} 1.448 & 0.03 \\ 0.03 & 1.4 \end{bmatrix}, & Q_4 &= \begin{bmatrix} 2.467 & 0.06 \\ 0.06 & 1.396 \end{bmatrix}, \\ \Lambda_1 &= 3.404, & \Lambda_2 &= 2.609, & \Lambda_3 &= 3.498, & \Lambda_4 &= 3.514. \end{aligned}$$

Note that  $\Lambda_i$ ,  $i \in \mathcal{I}[1, 4]$ , are different, and  $\Lambda^{-1}(\zeta) \nrightarrow 0$ . This numerically confirms that compared to the classical nonquadratic Lyapunov function (12), the new function (11) provides more degrees of freedom to reduce the conservatism.

**Example 2.** Consider the following nonlinear system [16]:

$$\begin{aligned} x_1(k+1) &= (1+T)x_1(k) + Tx_2(k) + \frac{\sin b}{b}Tx_3(k) \\ &\quad - 0.1Tx_4(k) + T(1+x_1^2(k))\text{sat}(u(k)) + T\varphi(z(k)) \\ x_2(k+1) &= Tx_1(k) + (1-2T)x_2(k) \\ x_3(k+1) &= Tx_1(k) + (1-0.3T)x_3(k) + Tx_1^2(k)x_2(k) \\ x_4(k+1) &= (1-T)x_4(k) + \frac{\sin b}{b}Tx_3(k) + T\varphi(z(k)) \\ y_1(k) &= x_2(k) + (1+x_1^2(k))x_4(k), & y_2(k) &= x_1(k) \\ z(k) &= x_3(k), \end{aligned} \quad (51)$$

where  $x_1 \in [-a, a]$ ,  $x_3 \in [-b, b]$ ,  $b = \frac{\pi}{2}$  and  $\varphi(z(k)) = \sin(x_3(k)) - \frac{\sin b}{b}x_3(k)$ , with  $\varphi(z(k)) \in \text{co}\{0, 1 - \frac{\sin b}{b}x_3(k)\}$ . The control input is constrained by  $\bar{u} = 1.2$  and the sampling time is  $T = 0.5$ s. Selecting  $\zeta(k) = x_1^2(k)$ , the nonlinear system (51) can be represented as a N-TS fuzzy system (1) with the following local matrices and membership functions:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1+T & T & \frac{\sin b}{b}T & -0.1T \\ T & 1-2T & 0 & 0 \\ T & a^2T & 1-0.3T & 0 \\ 0 & 0 & \frac{\sin b}{b} & 1-T \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1+T & T & \frac{\sin b}{b}T & -0.1T \\ T & 1-2T & 0 & 0 \\ T & 0 & 1-0.3T & 0 \\ 0 & 0 & \frac{\sin b}{b} & 1-T \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} B_1 &= [(1+a^2)T \ 0 \ 0 \ 0]^\top, & B_2 &= [T \ 0 \ 0 \ 0]^\top, \\ G_1 &= G_2 = [T \ 0 \ 0 \ T]^\top, & L_1 &= L_2 = [0 \ 0 \ 1 \ 0], \\ C_1 &= \begin{bmatrix} 0 & 1 & 0 & 1+a^2 \\ 1 & 0 & 0 & 0 \end{bmatrix}, & C_2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\ \eta_1(\zeta(k)) &= \frac{x_1(k)^2}{a^2}, & \eta_2(\zeta(k)) &= 1 - \eta_1(x(k)). \end{aligned}$$

As in [16], we find the largest value of  $a$ , denoted by  $a^*$ , for which a SOF controller can be found for system (51). Note that the state constraints  $|x_1(k)| \leq a$  and  $|x_3(k)| \leq b$ , guaranteeing

the validity of the considered TS fuzzy nonlinear model, and the control input saturation are not take into account in the design performed in [16]. Note also that the numerical complexity of LMI-based optimizations can be evaluated by the number of scalar decision variables  $\mathcal{N}_{\text{var}}$  and the number of rows of involved LMI conditions  $\mathcal{N}_{\text{row}}$ . The results concerning the design conservatism and the numerical complexity are reported in Table I. Remark that since  $C_1 \neq C_2$ , Corollary 1 cannot provide a control solution for any value of  $a$ . This numerically confirms the statements in Remark 7.

TABLE I  
LARGEST VALUE OF  $a$  WITH THE RELATED NUMERICAL COMPLEXITY.

Control design	Theorem 1	Corollary 1
$a^*$	1.1	Infeasible
Decision variables ( $\mathcal{N}_{\text{var}}$ )	58	-
LMI rows ( $\mathcal{N}_{\text{row}}$ )	186	-
Numerical complexity ( $\mathcal{N}_{\text{var}}^3 \mathcal{N}_{\text{row}}$ )	36290832	-

For illustration purposes, we consider system (51) with  $a = a^* = 1.1$ . Solving the optimization problem in Theorem 1, a nonlinear SOF controller (8) can be found with  $\epsilon = 0.1379$ . The closed-loop trajectories and the control input signal corresponding to the initial condition  $x(0) = [0.5 \ 1 \ 0 \ -0.3]^\top$  are depicted in Fig. 2. Observe that the closed-loop system is asymptotically stabilized despite the input saturation at the beginning of the simulation.

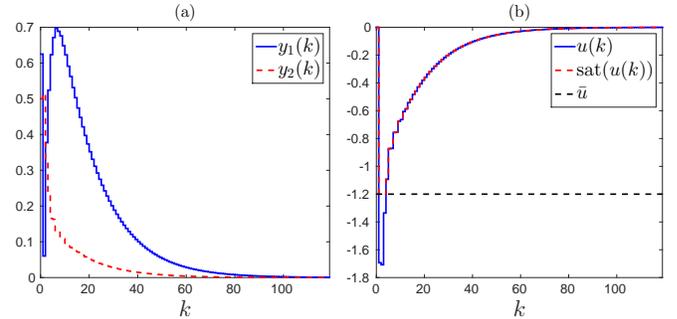


Fig. 2. Closed-loop behaviors of the nonlinear system in Example 2 with  $a = 1.1$ . (a) Output. (b) Control input.

**Example 3.** Consider the following physically motivated two-tank system borrowed from [50]:

$$\begin{aligned} \dot{h}_1(t) &= \frac{1}{A} \left( \bar{k}v(t) - a_1 \sqrt{2g(h_1(t) - h_2(t))} \right), \\ \dot{h}_2(t) &= \frac{1}{A} \left( a_1 \sqrt{2g(h_1(t) - h_2(t))} - a_2 \sqrt{2gh_2(t)} \right), \end{aligned} \quad (52)$$

where  $h_1(t)$  and  $h_2(t)$  denote the water level of the two tanks,  $v(t)$  is the flow rate of the pump,  $A = 100\text{cm}^2$  is the horizontal section,  $\bar{k} = 0.01$  is a constant,  $a_1 = 1\text{cm}^2$  is the section of the valve connecting the tanks,  $a_2 = 0.7\text{cm}^2$  is the section of the outlet valve,  $g = 981\text{cm/s}^2$  is the gravitational acceleration constant. The control goal is to keep the water level of the two tanks in the operating point  $[h_{1e} \ h_{2e}]^\top = [9 \ 6]^\top$  cm.

It follows from the equilibrium equations of system (52) that

$$\begin{aligned} 0 &= \frac{1}{A} \left( \bar{k}v_e - a_1 \sqrt{2g(h_{1e} - h_{2e})} \right), \\ 0 &= \frac{1}{A} \left( a_1 \sqrt{2g(h_{1e} - h_{2e})} - a_2 \sqrt{2gh_{2e}} \right). \end{aligned}$$

By defining  $x_1 = h_1 - h_{1e}$ ,  $x_2 = h_2 - h_{2e}$  and  $\Delta v = v - v_e$ , the following incremental model is obtained [50]:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{a_1}{A} \frac{2g}{f(x_1, x_2)} & \frac{a_1}{A} \frac{2g}{f(x_1, x_2)} \\ \frac{a_1}{A} \frac{2g}{f(x_1, x_2)} & -\frac{a_1}{A} \frac{2g}{f(x_1, x_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{k} \\ 0 \end{bmatrix} \Delta v \\ &+ \begin{bmatrix} 0 \\ -\frac{2ga_2}{A} \end{bmatrix} \bar{\varphi}(x_2), \end{aligned} \quad (53)$$

where  $f(x_1, x_2) = \sqrt{2g(x_1 - x_2 + h_{1e} - h_{2e})} + \sqrt{2g(h_{1e} - h_{2e})}$ , and  $\bar{\varphi}(x_2) = \frac{x_2}{t(x_2)}$ , with  $t(x_2) = \sqrt{2g(x_2 + h_{2e})} + \sqrt{2gh_{2e}}$ .

Assume that  $|x_1| \leq 8$  and  $|x_2| \leq 5$ , then  $\zeta_1 = \frac{1}{f(x_1, x_2)} \in [0.0054, 0.0130]$  and  $\bar{\varphi}(x_2) \in \text{co}\{0.0039x_2(k), 0.0065x_2(k)\}$ . After a loop-transformation, it follows that  $\varphi(x_2) = \bar{\varphi}(x_2) - 0.0039x_2$ . Note that  $\varphi(x_2) \in \text{co}\{0, \Omega x_2\}$  with  $\Omega = 0.0026$ . We consider  $|\Delta v(t)| \equiv |\text{sat}(u(t))| \leq \bar{u}$  with  $\bar{u} = 50$ . Hence, system (53) can be rewritten in the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{a_1}{A} \frac{2g}{f(x_1, x_2)} & \frac{a_1}{A} \frac{2g}{f(x_1, x_2)} \\ \frac{a_1}{A} \frac{2g}{f(x_1, x_2)} & -\frac{a_1}{A} \frac{2g}{f(x_1, x_2)} - \frac{2ga_2\Omega}{A} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ \begin{bmatrix} \bar{k} \\ 0 \end{bmatrix} \text{sat}(u) + \begin{bmatrix} 0 \\ -\frac{2ga_2}{A} \end{bmatrix} \varphi(x_2). \end{aligned} \quad (54)$$

A continuous-time two-rule N-TS fuzzy model can be easily derived from the nonlinear system (54). Then, performing a discretization with the sampling period  $T = 0.5\text{s}$  and a zero-order hold with no delay, its discrete-time counterpart (1) can be obtained with the following local matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.7343 & 0.2308 \\ 0.2308 & 0.5738 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.426 \\ 0.067 \end{bmatrix} \times 10^{-3}, \\ A_2 &= \begin{bmatrix} 0.5692 & 0.3671 \\ 0.3671 & 0.4634 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.364 \\ 0.123 \end{bmatrix} \times 10^{-3}, \\ G_1 &= [-13.2833 \quad -74.3406]^\top, & C_1 &= C_2 = [1 \quad -1], \\ G_2 &= [-24.2390 \quad -64.4727]^\top, & L_1 &= L_2 = [0 \quad 1]. \end{aligned}$$

Since  $C_1 = C_2$ , both Theorem 1 and Corollary 1 can be applied to this system. However, only the design conditions in Theorem 1 are feasible with  $\epsilon = 0.01$ . This numerically reconfirms that Theorem 1 provides less conservative results than Corollary 1, see Remark 8.

Furthermore, assume now that the state-state matrices are affected by the following structural uncertainties:

$$\begin{aligned} E_{A1} &= 0.001A_1, & E_{A2} &= 0.001A_2, & D &= I, \\ E_{B1} &= 0.1B_1, & E_{B2} &= 0.1B_2. \end{aligned}$$

A robust SOF controller (8) can be found with the design conditions in Theorem 2 and  $\epsilon = 0.01$ . The closed-loop behavior of the two tank system obtained with this robust SOF controller and the initial condition  $x(0) = [-3.5 \quad -1.5]^\top$  is depicted in Fig. 3. Note that  $h_1(k) = x_1(k) + h_{1e}$ ,  $h_2(k) = x_2(k) + h_{2e}$  and  $v(k) = \Delta v(k) + v_e$  with  $v_e = \frac{a_1}{\bar{k}} \sqrt{2g(h_{1e} - h_{2e})} = 7672\text{cm}^3/\text{s}$ . Then, the two tank system

is stabilized in the desired setpoint using only the output signal in the SOF control law (8) designed with Theorem 2. Note that the fuzzy dynamic output feedback control approaches proposed in [34], [45], [52], [53] cannot be applied to this uncertain N-TS fuzzy system.

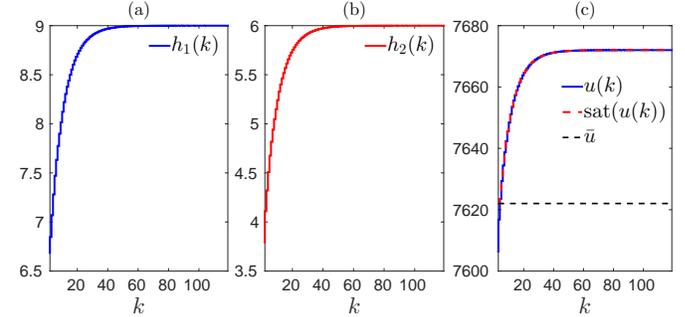


Fig. 3. Closed-loop behaviors of the nonlinear system in Example 3. (a) and (b) States. (c) Control input.

## VI. CONCLUDING REMARKS

A new solution for SOF control of constrained N-TS fuzzy systems has been proposed. The regional control design is established through an effective treatment of the nonlinear effects introduced by the input saturation and the nonlinear consequents of the fuzzy systems in the closed loop. To reduce further the design conservatism, we propose a new type of non-quadratic Lyapunov functions including the information on the nonlinear consequents in their construction. Using Lyapunov-based arguments, the SOF control design is reformulated as an LMI-based optimization with a linear search parameter. The convexification procedure is based on some special congruence transformations which are specific to the new structures of the proposed SOF controller and nonquadratic Lyapunov function. Therefore, the proposed control approach requires neither explicit matrix-rank constraints nor LMEs, inducing numerical difficulties. Especially, in terms of design conservatism, we can theoretically prove that the new approach includes the LME-based SOF control results. Three numerical examples are provided to demonstrate the effectiveness of the new control approach. For future works, we will focus on observer-based control and dynamic output feedback control for fuzzy systems under multiple state and input constraints [54], [60], especially in the presence of unmeasurable premise variables.

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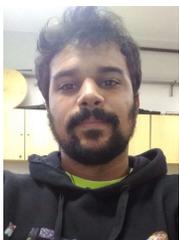
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