



HAL
open science

A Unified Framework for Asymptotic Observer Design of Fuzzy Systems With Unmeasurable Premise Variables

Jun-Tao Pan, Tran Anh-Tu Nguyen, Thierry-Marie Guerra, Dalil Ichalal

► **To cite this version:**

Jun-Tao Pan, Tran Anh-Tu Nguyen, Thierry-Marie Guerra, Dalil Ichalal. A Unified Framework for Asymptotic Observer Design of Fuzzy Systems With Unmeasurable Premise Variables. IEEE Transactions on Fuzzy Systems, 2021, 29 (10), pp.2938-2948. 10.1109/TFUZZ.2020.3009737. hal-03425007

HAL Id: hal-03425007

<https://uphf.hal.science/hal-03425007>

Submitted on 25 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/343000186>

A Unified Framework for Asymptotic Observer Design of Fuzzy Systems With Unmeasurable Premise Variables

Article in IEEE Transactions on Fuzzy Systems · October 2021

DOI: 10.1109/TFUZZ.2020.3009737

CITATIONS

28

READS

398

4 authors, including:



Anh-Tu Nguyen

Université Polytechnique Hauts-de-France

140 PUBLICATIONS 2,084 CITATIONS

[SEE PROFILE](#)



Thierry-Marie Guerra

Université Polytechnique Hauts-de-France

349 PUBLICATIONS 9,873 CITATIONS

[SEE PROFILE](#)



Dalil Ichalal

Université d'Évry-Val-d'Essonne

147 PUBLICATIONS 1,926 CITATIONS

[SEE PROFILE](#)

A Unified Framework for Asymptotic Observer Design of Fuzzy Systems with Unmeasurable Premise Variables

Juntao Pan*, Anh-Tu Nguyen*†, *Member, IEEE*, Thierry-Marie Guerra, Dalil Ichalal

Abstract—This paper develops a *unified* framework to design fuzzy-model-based observers of general nonlinear systems for both discrete-time and continuous-time cases. This observer problem is known as a challenging task due to the mismatch caused by the *unmeasurable* premise variables. To deal with this major challenge, we propose to rewrite the nonlinear system as a specific fuzzy model with two types of local nonlinearities: measurable and unmeasurable. Then, a differential mean value theorem for vector-valued functions is applied to local unmeasurable nonlinearities. This allows to represent the estimation error dynamics in a special polytopic form involving measurable membership functions and unknown but bounded time-varying parameters. Using Lyapunov-based arguments, design conditions in terms of linear matrix inequalities are derived to guarantee the asymptotic convergence of the estimation error. Three illustrative examples are given to demonstrate the interests of the new fuzzy observer framework in reducing: i) the design conservatism, ii) the numerical complexity of the fuzzy observer structure for real-world applications.

Index Terms—Fuzzy systems, fuzzy observers, unmeasured premise variables, state estimation, linear matrix inequality.

I. INTRODUCTION

REAL-time knowledge of the state evolution of dynamical systems is essential for feedback control [1], [2], fault detection/diagnosis [3]–[5], monitoring and decision making [6], [7]. However, full-state information is not available for most of engineering applications due to economical and/or technical reasons. Hence, estimation of the system state has received considerable attention in the literature [8]. Moreover, this problem is particularly challenging for nonlinear systems

*The authors contributed equally to this work.

This work is supported in part by the French Ministry of Higher Education and Research, in part by the National Center for Scientific Research (CNRS), in part by the Nord-Pas-de-Calais Region under the project ELSAT 2020, in part by the Natural Science Foundation of Ningxia Hui Autonomous Region under Grant 2018AAC03107, in part by Ningxia first-class discipline and scientific research projects (electronic science and technology), NO. NXYLXK2017A07, in part by the Third Batch of Ningxia Youth Talents Supporting Program, in part by the Advanced Intelligent Perception and Control Technology Innovative Team of Ningxia, in part by Major Special project of North Minzu University, No. ZDZX201902.

A.-T. Nguyen and T.-M. Guerra are with the LAMIH laboratory UMR CNRS 8201, Université Polytechnique Hauts-de-France, F-59313 Valenciennes, France.

J. Pan is with the School of Electrical and Information Engineering, North Minzu University, Yinchuan, China 750021. This work has been done during his stay at the LAMIH laboratory UMR CNRS 8201, Université Polytechnique Hauts-de-France, F-59313 Valenciennes, France.

D. Ichalal is with the IBISC laboratory, University of Évry, Université Paris Saclay, 91000 Évry, France.

†Corresponding author (e-mail: nguyen.tranhanhtu@gmail.com).

[9]. Motivated by these crucial issues of control engineering, this paper investigates the observer design for nonlinear control systems whose dynamics is described as

$$\begin{aligned}\delta[x(t)] &= f(z(x(t)), u(t)), \\ y(t) &= g(z(x(t)), u(t)),\end{aligned}\quad (1)$$

where $x(t) \in \mathcal{D}_x \subseteq \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathcal{D}_u \subseteq \mathbb{R}^{n_u}$ is the control input and $y(t) \in \mathbb{R}^{n_y}$ is the system output. $\delta[\cdot]$ denotes the time derivative operator for continuous-time systems and the shift operator for discrete-time systems. The vector of premise variables $z(x(t)) \in \mathbb{R}^{n_z}$ is bounded in a compact set \mathcal{D}_z of the state space. The nonlinear functions $f : \mathcal{D}_x \times \mathcal{D}_u \rightarrow \mathbb{R}^{n_x}$ and $g : \mathcal{D}_x \times \mathcal{D}_u \rightarrow \mathbb{R}^{n_y}$ are differentiable with respect to the state $x(t)$ in the set \mathcal{D}_x .

The problem is to design a dynamical observer, generating a state estimate $\hat{x}(t)$ of the nonlinear system (1), such that

$$\lim_{t \rightarrow \infty} |x(t) - \hat{x}(t)| = 0.$$

Up to now, several approaches have been proposed to address this problem, including high-gain observers [10], adaptive observers [11], sliding-mode observers [6], [12], and so on. A survey on observer tools for nonlinear systems can be found in [9]. It is important to note that observer approaches existing in the literature are generally involved and only applicable to some specific classes of nonlinear systems. A promising solution for this major drawback is to exploit the Takagi-Sugeno (TS) fuzzy representation [13], [14] of nonlinear systems for observer design, see for example [8], [15]–[18] and numerous related references.

Existing approaches to design fuzzy-model-based observers are usually based on the following N -rule TS fuzzy model of the nonlinear system (1):

$$\delta[x] = \sum_{i=1}^N \nu_i(z) \mathcal{A}_i(x, u), \quad y = \sum_{i=1}^N \nu_i(z) \mathcal{C}_i(x, u), \quad (2)$$

where $\mathcal{A}_i(x, u) = A_i x + B_i u$ and $\mathcal{C}_i(x, u) = C_i x + D_i u$. The local matrices with appropriate dimensions (A_i, B_i, C_i, D_i) are known. The membership functions (MFs) satisfy the convex sum property, *i.e.*, $\nu_i(z) \geq 0$, $\sum_{i=1}^N \nu_i(z) = 1$.

In this paper, we are particularly interested in designing fuzzy observers with Luenberger-type structure [8], which proved useful in fuzzy-model-based nonlinear feedback control [1], [13], [19]–[21]. Other types of fuzzy observers have been also considered in the literature, for instance sliding mode observers [12], adaptive observers [11], [22]. The classical

fuzzy Luenberger observer of the TS fuzzy system (2) is given as follows [8]:

$$\begin{aligned}\delta[\hat{x}] &= \sum_{i=1}^N \nu_i(\hat{z})(A_i\hat{x} + B_iu) + M(\hat{z})(y - \hat{y}), \\ \hat{y} &= \sum_{i=1}^N \nu_i(\hat{z})(C_i\hat{x} + D_iu),\end{aligned}\quad (3)$$

where the designed MF-dependent matrix $M(\hat{z})$ can be of parallel distributed compensation (PDC) structure [13] or non-PDC structure [23]. Let us denote the state estimation error as $e = x - \hat{x}$. Then, the estimation error dynamics is defined from (2) and (3) as follows:

$$\delta[e] = \sum_{i=1}^N \nu_i(\hat{z})(A_i - M(\hat{z})C_i)e + \omega(z, \hat{z}), \quad (4)$$

where

$$\begin{aligned}\omega(z, \hat{z}) &= \sum_{i=1}^N \Delta_i(z, \hat{z})(\mathcal{A}_i(x, u) + M(\hat{z})C_i(x, u)), \\ \Delta_i(z, \hat{z}) &= \nu_i(z) - \nu_i(\hat{z}).\end{aligned}$$

From the error dynamics (4), fuzzy-model-based observer design can be classified into two following categories depending on the real-time availability of the premise variables.

A. Observer Design with Measurable Premise Variables

If all the premise variables are measurable, *i.e.*, $\hat{z} = z$, then the term $\omega(z, \hat{z})$ involved in (4) vanishes. The fuzzy observer design is relatively simple in this case [13]. The main design goal here is to reduce the design conservatism by exploiting different Lyapunov candidate functions and/or introducing slack variables [19], [24]–[27]. However, such design approaches are only applicable for a restrictive class of TS fuzzy systems, thus nonlinear systems. Therefore, the research mainstream has focused on the case where the premise variables cannot be measured [8].

B. Observer Design with Unmeasurable Premise Variables

In the presence of unmeasurable premise variables, the design of fuzzy observers/filters becomes much challenging [28]–[30]. Indeed, using (2)–(3) for fuzzy observer design leads to a major difficulty in handling the nonlinear term $\omega(z, \hat{z})$, which is due to the “mismatch” term $\Delta_i(z, \hat{z})$ between $\nu_i(z)$ and $\nu_i(\hat{z})$, for $i = 1, 2, \dots, N$. To deal with this technical difficulty, various approaches have been proposed in the literature. The most simple approach is to consider the “mismatch” term $\omega(z, \hat{z})$ as a *disturbance* of the error system (4). Then, an \mathcal{L}_2 -gain performance is taken into account in the fuzzy observer design to minimize the effects of this state-dependent disturbance [31]. Note that such an \mathcal{L}_2 observer design cannot guarantee an asymptotic convergence of the estimation error. Moreover, it generally leads to high observer gains to counteract the effects of the nonlinear term $\omega(z, \hat{z})$, see [18] and related references. A common approach to achieve asymptotic error convergence is based on the well-known Lipschitz property $\|\omega(z, \hat{z})\| \leq \gamma\|e\|$, with some

Lipschitz constant $\gamma > 0$, see for instance [15], [32]–[34] and references therein. Despite its simplicity, this approach often leads to over-conservative design results, especially when the Lipschitz constant γ of nonlinear systems is large [8]. An uncertain system approach was proposed to design an \mathcal{H}_∞ fuzzy filter with estimated premise variables in [28]. However, this norm-bounded uncertainty approach may lead to conservative results for complex TS fuzzy systems. To avoid unmeasurable premise variables in the design of fuzzy observers, a dynamic-extension-based approach was proposed in [18]. Based on immersion techniques, this approach is only applicable to a restrictive class of TS fuzzy systems due to the *non-convergence* of the algorithm generating auxiliary dynamics. Recently, based on a decomposition of the premise-space into crisp regions and a new piecewise Lyapunov function, the authors in [30] proposed a new approach to design piecewise fuzzy observers. Despite its effectiveness to deal with unmeasurable premise variables, this approach can be only applied to TS fuzzy system with linear output matrix. Moreover, the proposed fuzzy observers may cause difficulties for real-time implementation, especially for complex nonlinear systems, due to the involved piecewise regions. Using the differential mean value theorem (DMVT) is another promising way to handle the mismatch caused by unmeasurable MFs [29], [35]. Despite its systematic feature to deal with general nonlinear systems, the most recent DMVT-based approach [29] has several major drawbacks. First, the design conditions strongly depend on various *predefined* bounds not only of the system state $x(t)$ but also of the control input $u(t)$. Second, the time-varying terms stemmed from the application of DMVT are treated using a norm-bounded uncertainty approach. These features lead to a complex design framework which can be over-conservative, especially when both $f(\cdot)$ and $g(\cdot)$ depend on unmeasurable premise variables.

C. Methodology and Contributions

Motivated by the above issues, a new framework is proposed for fuzzy observer design of nonlinear systems. This framework is established based on the following key ideas. First, we reformulate the general nonlinear system (1) in the form of a specific TS fuzzy model with two parts of local nonlinearities. The first part contains nonlinear functions of unmeasurable premise variables while the second one is defined by measured premise variables. For brevity, this class of fuzzy systems will be called hereafter N-TS fuzzy systems [36]. Second, a differential mean value theorem for *vector-valued* functions [37] is exclusively applied to the local unmeasurable nonlinearities of the N-TS fuzzy model to deal with the mismatch caused by the unmeasurable premise variables. Hence, the estimation error dynamics can be rewritten in a polytopic system involving measurable MFs and time-varying unknown parameters. These unknown parameters, stemmed from the application of DMVT, belong to a bounded convex set which can be characterized by the Jacobians of the local unmeasurable nonlinearities. Using Lyapunov-based arguments, sufficient conditions are derived to design a N-TS fuzzy observer for system (1) while guaranteeing an asymptotic convergence of the estimation

error dynamics. The main contributions of the paper can be summarized as follows.

- We propose a *unified* framework to design an asymptotic fuzzy observer for system (1) in both discrete-time and continuous-time settings. The results of fuzzy observer design with measurable premise variables are straightforwardly recovered from the proposed framework.
- Using a specific N-TS fuzzy formulation, not only the design conservatism but also the complexity of the fuzzy observer structure can be effectively reduced for real-world applications. The design procedure is recast as a convex optimization problem under linear matrix inequalities (LMIs), easily solved with numerical solvers.
- In contrast to the recent DMVT-based approach [29], here the norm-bounded uncertainty approach can be avoided. Moreover, in many situations the bounds on the control input $u(t)$ are not required for observer design, *i.e.*, when the Jacobians of the local unmeasurable nonlinearities are independent to $u(t)$. Hence, the new design procedure is much less complex while offering less conservative results as demonstrated by three illustrative examples.

This paper is organized as follows. The fuzzy observer problem is formulated in Section II. We present the new observer framework in Section III for both discrete-time and continuous-times cases. Three examples are provided in Section IV to demonstrate the interest of the proposed observer approach. Concluding remarks are given in Section V.

Notation. The set of nonnegative integers is denoted by \mathbb{Z}_+ and $\mathcal{I}_r = \{1, 2, \dots, r\} \subset \mathbb{Z}_+$. For $i \in \mathcal{I}_r$, we denote $\xi_r(i) = [0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots, 0]^\top \in \mathbb{R}^r$ a vector of the canonical basis of \mathbb{R}^r . For a vector x , x_i denotes its i th entry. For two vectors $x, y \in \mathbb{R}^n$, the convex hull of these vectors is denoted as $\text{co}(x, y) = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$. For a matrix X , X^\top denotes its transpose, $X \succ 0$ means X is symmetric positive definite, and $\text{He}X = X + X^\top$. $\text{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1, X_2 . I denotes the identity matrix of appropriate dimension. In block matrices, the symbol \star stands for the terms deduced by symmetry. Arguments are omitted when their meaning is clear.

II. PROBLEM FORMULATION

This section formulates the N-TS fuzzy observer design problem. Then, we present two technical lemmas which are useful to develop the new observer design framework.

A. Description of N-TS Fuzzy Systems for Observer Design

For fuzzy observer design, we decompose the set of premise variables of system (1) into two following subsets:

$$\begin{aligned} \mathcal{Z}_\alpha &= \{z_{\alpha i} \in z : z_{\alpha i} \text{ is measurable, } i \in \mathcal{I}_p\}, \\ \mathcal{Z}_\beta &= \{z_{\beta j} \in z : z_{\beta j} \text{ is unmeasurable, } j \in \mathcal{I}_{n_z - p}\}, \end{aligned} \quad (5)$$

with $p \in \mathbb{Z}_+$ and $p \leq n_z$. With the decomposition (5) and inspired by the N-TS fuzzy representation [33], [36], the nonlinear system (1) can be reformulated in the form

$$\begin{aligned} \delta[x] &= A(z_\alpha)x + f_\alpha(z_\alpha, u) + F(z_\alpha)\phi(x, u), \\ y &= C(z_\alpha)x + g_\alpha(z_\alpha, u) + G(z_\alpha)\psi(x, u), \end{aligned} \quad (6)$$

where the nonlinear functions $\phi(\cdot) \in \mathbb{R}^{n_\phi}$ and $\psi(\cdot) \in \mathbb{R}^{n_\psi}$ cannot be measured from the output, and all the elements of functions $f_\alpha(\cdot)$ and $g_\alpha(\cdot)$ are measurable. Moreover, the state-space matrices $A(\cdot)$, $C(\cdot)$, $F(\cdot)$ and $G(\cdot)$ exclusively depend on the measurable premise variables.

Using fuzzy modeling technique [13], the following N-TS fuzzy model of the nonlinear system (6) can be obtained:

$$\begin{aligned} \text{RULE } \mathcal{R}_i : & \text{ IF } z_{\alpha 1} \text{ is } \mathcal{M}_1^i \text{ and } \dots \text{ and } z_{\alpha p} \text{ is } \mathcal{M}_p^i \\ \text{THEN } & \begin{cases} \delta[x] = A_i x + F_i \phi(x, u) + f_\alpha(z_\alpha, u) \\ y = C_i x + G_i \psi(x, u) + g_\alpha(z_\alpha, u) \end{cases} \end{aligned} \quad (7)$$

where the constant matrices with appropriate dimensions (A_i, F_i, C_i, G_i) are known. \mathcal{R}_i denotes the i th fuzzy inference rule, and r is the number of inference rules. \mathcal{M}_j^i , with $i \in \mathcal{I}_r$ and $j \in \mathcal{I}_p$, is the fuzzy set. The fuzzy MFs are given by $h_i(z_\alpha) = \frac{\prod_{j=1}^p \mu_j^i(z_{\alpha j})}{\sum_{i=1}^r \prod_{j=1}^p \mu_j^i(z_{\alpha j})}$, $i \in \mathcal{I}_r$, where $\mu_j^i(z_{\alpha j})$ represents the membership grade of $z_{\alpha j}$ in the respective fuzzy set \mathcal{M}_j^i . Note that the MFs satisfy the following property:

$$\sum_{i=1}^r h_i(z_\alpha) = 1, \quad 0 \leq h_i(z_\alpha) \leq 1, \quad \forall i \in \mathcal{I}_r. \quad (8)$$

Let \mathcal{H} be the set of the membership functions satisfying (8), *i.e.*, $h = [h_1(z_\alpha), h_2(z_\alpha), \dots, h_r(z_\alpha)] \in \mathcal{H}$. Using the center-of-gravity method for defuzzification, the fuzzy system (7) can be represented in the following compact form [13]:

$$\begin{aligned} \delta[x] &= A(h)x + F(h)\phi(x, u) + f_\alpha(z_\alpha, u), \\ y &= C(h)x + G(h)\psi(x, u) + g_\alpha(z_\alpha, u), \end{aligned} \quad (9)$$

where

$$\begin{bmatrix} A(h) & F(h) \\ C(h) & G(h) \end{bmatrix} = \sum_{i=1}^r h_i(z_\alpha) \begin{bmatrix} A_i & F_i \\ C_i & G_i \end{bmatrix}.$$

The Jacobian matrices of the nonlinear functions $\phi(x, u)$ and $\psi(x, u)$ with respect to the state x are defined as

$$\begin{aligned} \mathcal{J}_\phi(x, u) &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1}(x, u) & \dots & \frac{\partial \phi_1}{\partial x_{n_x}}(x, u) \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{n_\phi}}{\partial x_1}(x, u) & \dots & \frac{\partial \phi_{n_\phi}}{\partial x_{n_x}}(x, u) \end{bmatrix} \in \mathbb{R}^{n_\phi \times n_x}, \\ \mathcal{J}_\psi(x, u) &= \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1}(x, u) & \dots & \frac{\partial \psi_1}{\partial x_{n_x}}(x, u) \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_{n_\psi}}{\partial x_1}(x, u) & \dots & \frac{\partial \psi_{n_\psi}}{\partial x_{n_x}}(x, u) \end{bmatrix} \in \mathbb{R}^{n_\psi \times n_x}. \end{aligned}$$

For observer design, we consider the following assumption.

Assumption 1. The elements of $\mathcal{J}_\phi(x, u)$ and $\mathcal{J}_\psi(x, u)$ satisfy the following boundedness conditions:

$$\begin{aligned} \underline{\rho}_{ij} &\leq \frac{\partial \phi_i}{\partial x_j}(x, u) \leq \bar{\rho}_{ij}, \quad x \in \mathcal{D}_x, \quad u \in \mathcal{D}_u, \\ \underline{\zeta}_{kj} &\leq \frac{\partial \psi_k}{\partial x_j}(x, u) \leq \bar{\zeta}_{kj}, \quad x \in \mathcal{D}_x, \quad u \in \mathcal{D}_u, \end{aligned} \quad (10)$$

where

$$\underline{\rho}_{ij} = \min_{\mu \in \mathcal{D}_x \times \mathcal{D}_u} \left(\frac{\partial \phi_i}{\partial x_j}(\mu) \right), \quad \bar{\rho}_{ij} = \max_{\mu \in \mathcal{D}_x \times \mathcal{D}_u} \left(\frac{\partial \phi_i}{\partial x_j}(\mu) \right),$$

$$\underline{\zeta}_{kj} = \min_{\mu \in \mathcal{D}_x \times \mathcal{D}_u} \left(\frac{\partial \psi_k}{\partial x_j}(\mu) \right), \quad \bar{\zeta}_{kj} = \max_{\mu \in \mathcal{D}_x \times \mathcal{D}_u} \left(\frac{\partial \psi_k}{\partial x_j}(\mu) \right),$$

for $\forall (i, j, k) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x} \times \mathcal{I}_{n_\psi}$.

Remark 1. The assumption on the Jacobian matrices $\mathcal{J}_\phi(\cdot)$ and $\mathcal{J}_\psi(\cdot)$ represented by the bounds in (10) is not restrictive for N-TS fuzzy observer design. First, for a nonlinear system, its N-TS fuzzy representation is generally defined within a compact set of the state space \mathcal{D}_x , especially when using the sector nonlinear approach [13, Chapter 2]. Second, the control input of engineering systems is measured and physically bounded, *i.e.*, $u \in \mathcal{D}_u$. Moreover, the bounds $\underline{\rho}_{ij}$, $\bar{\rho}_{ij}$, $\underline{\zeta}_{kj}$, $\bar{\zeta}_{kj}$, for $\forall (i, j, k) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x} \times \mathcal{I}_{n_\psi}$, can be easily computed from the mathematical expressions of $\phi(x, u)$ and $\psi(x, u)$.

B. Observer Problem Definition

For the state estimation of system (9), we consider the following N-TS fuzzy observer structure:

$$\begin{aligned} \delta[\hat{x}] &= A(h)\hat{x} + f_\alpha(z_\alpha, u) + F(h)\phi(\hat{x}, u) + \mathcal{L}(h)(y - \hat{y}), \\ \hat{y} &= C(h)\hat{x} + g_\alpha(z_\alpha, u) + G(h)\psi(\hat{x}, u), \quad \hat{x}(0) = 0, \end{aligned} \quad (11)$$

where the MF-dependent observer gain $\mathcal{L}(h)$ is defined in (20) for the discrete-time case or in (26) for the continuous-time case.

Remark 2. All the *unmeasurable* premise variables of system (1) are ‘‘isolated’’ in $\phi(x, u)$ and $\psi(x, u)$. Moreover, since the MFs $h_i(z_\alpha)$, $i \in \mathcal{I}_r$, are measurable, the fuzzy observer (11) can be constructed using the same MFs. This particular feature of the N-TS fuzzy system (9) enables an effective observer design for general nonlinear systems of the form (1).

Remark 3. In contrast to recent results [8], [23], [29], [30] using the classical TS fuzzy model (2), the proposed observer design is based on the N-TS fuzzy form (9). Due to the retained nonlinearities $f_\alpha(z_\alpha, u)$, $g_\alpha(z_\alpha, u)$, $\phi(x, u)$ and $\psi(x, u)$ in the consequents, the latter fuzzy modeling requires significantly fewer rules than the former, *i.e.*, $r < N$, especially for complex nonlinear systems. Hence, the fuzzy observer (11) can be of much simpler structure than (3), which is particularly interesting for real-time implementation. Moreover, N-TS fuzzy model-based designs can lead to less conservative results and computational burden than standard TS fuzzy approaches [33].

The dynamics of the state estimation error can be defined from (9) and (11) as

$$\delta[e] = (A(h) - \mathcal{L}(h)C_h)e + F(h)\Delta_\phi - \mathcal{L}(h)G(h)\Delta_\psi \quad (12)$$

where $\Delta_\phi = \phi(x, u) - \phi(\hat{x}, u)$ and $\Delta_\psi = \psi(x, u) - \psi(\hat{x}, u)$.

This paper presents an effective solution for the following observer design problem.

Problem 1. Consider a general nonlinear system (1) with its equivalent TS fuzzy representation given in (9). Determine the MF-dependent observer gain $\mathcal{L}(h)$ such that the error dynamics (12) converges asymptotically to the origin.

The following differential mean value theorem for vector-valued functions is useful for theoretical developments.

Lemma 1. [38] Let $f(x) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^q$ and $a, b \in \mathbb{R}^{n_x}$. If f is differentiable on $\text{co}(a, b)$, then there are constant vectors $c_i \in \text{co}(a, b)$, $c_i \neq a$, $c_i \neq b$, for $\forall i \in \mathcal{I}_q$, such that

$$f(a) - f(b) = \left(\sum_{i=1}^q \sum_{j=1}^n \xi_q(i) \xi_n^\top(j) \frac{\partial f_i}{\partial x_j}(c_i) \right) (a - b). \quad (13)$$

Relation (13) is used to deal with Δ_ϕ and Δ_ψ involved in the error dynamics (12). This allows establishing an asymptotic observer design framework as detailed in Section III.

TS fuzzy observer design problems can be reformulated in the MF-dependent inequality of the form

$$\Upsilon_{hhh_+} = \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r h_i(t) h_j(t) h_k(t+1) \Upsilon_{ijk} \prec 0, \quad (14)$$

where $h \in \mathcal{H}$, and the matrices Υ_{ijk} are linearly dependent on the unknown decision variables. The following relaxation result provides a good tradeoff between numerical complexity and design conservatism [39].

Lemma 2. Let Υ_{ijk} be symmetric matrices of appropriate dimensions where $i, j, k \in \mathcal{I}_r$. Then, inequality (14) holds if

$$\begin{aligned} \Upsilon_{iik} &\prec 0, \\ \frac{2}{r-1} \Upsilon_{iik} + \Upsilon_{ijk} + \Upsilon_{jik} &\prec 0, \quad i \neq j. \end{aligned}$$

Note that other relaxation results with different degrees of complexity and/or conservatism can be found in [40].

III. LMI-BASED FRAMEWORK FOR FUZZY ASYMPTOTIC OBSERVER DESIGN

This section provides a general framework to design a fuzzy model-based asymptotic observer for the general nonlinear system (1) in both discrete-time and continuous-time cases.

For observer design, the terms $\Delta_\phi = \phi(x, u) - \phi(\hat{x}, u)$ and $\Delta_\psi = \psi(x, u) - \psi(\hat{x}, u)$ in (12) are first reformulated as functions of the estimation error e . To this end, applying Lemma 1 to the nonlinear functions $\phi(x, u)$ and $\psi(x, u)$, it follows that there exist $v_i(t) \in \text{co}(x(t), \hat{x}(t))$ and $w_k(t) \in \text{co}(x(t), \hat{x}(t))$, for $\forall (i, k) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_\psi}$, such that

$$\begin{aligned} \Delta_\phi &= \left(\sum_{i=1}^{n_\phi} \sum_{j=1}^{n_x} \xi_{n_\phi}(i) \xi_{n_x}^\top(j) \frac{\partial \phi_i}{\partial x_j}(v_i, u) \right) (x - \hat{x}), \\ \Delta_\psi &= \left(\sum_{k=1}^{n_\psi} \sum_{j=1}^{n_x} \xi_{n_\psi}(k) \xi_{n_x}^\top(j) \frac{\partial \psi_k}{\partial x_j}(w_k, u) \right) (x - \hat{x}). \end{aligned} \quad (15)$$

For brevity, we denote $\rho_{ij}(\cdot) = \frac{\partial \phi_i}{\partial x_j}(v_i(t), u(t))$ and $\zeta_{kj}(\cdot) = \frac{\partial \psi_k}{\partial x_j}(w_k(t), u(t))$, for $\forall (i, j, k) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x} \times \mathcal{I}_{n_\psi}$, and

$$\begin{aligned} \rho &= [\rho_{11}, \dots, \rho_{1n_x}, \dots, \rho_{n_\phi n_x}], \\ \zeta &= [\zeta_{11}, \dots, \zeta_{1n_x}, \dots, \zeta_{n_\psi n_x}]. \end{aligned} \quad (16)$$

Due to the boundedness condition (10) on the Jacobian matrix \mathcal{J}_ϕ , the parameter vector ρ belongs to a bounded convex set \mathcal{S}_ϕ , whose the set of $2^{n_\phi n_x}$ vertices is given by

$$\mathcal{V}_\phi = \{\rho = [\rho_{11}, \dots, \rho_{1n_x}, \dots, \rho_{n_\phi n_x}] : \rho_{ij} \in \{\underline{\rho}_{ij}, \bar{\rho}_{ij}\}\}.$$

Similarly, ζ belongs to a bounded convex set \mathcal{S}_ψ , whose the set of $2^{n_\psi n_x}$ vertices is given by

$$\mathcal{V}_\psi = \{\zeta = [\zeta_{11}, \dots, \zeta_{1n_x}, \dots, \zeta_{n_\psi n_x}] : \zeta_{ij} \in \{\underline{\zeta}_{ij}, \bar{\zeta}_{ij}\}\}.$$

From (12) and (15), the estimation error dynamics can be rewritten in the form

$$\delta[e] = (\mathcal{A}(h, \rho) - \mathcal{L}(h)\mathcal{C}(h, \zeta))e, \quad (17)$$

where

$$\mathcal{A}(h, \rho) = \sum_{i=1}^r h_i(z_\alpha) \mathcal{A}_i(\rho), \quad \mathcal{C}(h, \zeta) = \sum_{i=1}^r h_i(z_\alpha) \mathcal{C}_i(\zeta),$$

for $h \in \mathcal{H}$, $\rho \in \mathcal{S}_\phi$ and $\zeta \in \mathcal{S}_\psi$, with

$$\begin{aligned} \mathcal{A}_i(\rho) &= A_i + F_i \sum_{l=1}^{n_\phi} \sum_{j=1}^{n_x} \xi_{n_\phi}(l) \xi_{n_x}^\top(j) \rho_{lj}, \\ \mathcal{C}_i(\zeta) &= C_i + G_i \sum_{k=1}^{n_\psi} \sum_{j=1}^{n_x} \xi_{n_\psi}(k) \xi_{n_x}^\top(j) \zeta_{kj}. \end{aligned} \quad (18)$$

We now distinguish the observer design for two cases: discrete-time and continuous-time N-TS fuzzy systems.

A. Discrete-Time N-TS Fuzzy Observer Design

The following theorem provides sufficient conditions to design a N-TS fuzzy observer (11) in the discrete-time case.

Theorem 1. Consider the discrete-time N-TS fuzzy system (9) and its associated observer structure (11). If there exist positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices $Q_i \in \mathbb{R}^{n_x \times n_x}$ and matrices $L_i \in \mathbb{R}^{n_x \times n_y}$, for $i \in \mathcal{I}_r$, such that the following linear matrix inequalities hold:

$$\begin{aligned} \Gamma_{iik}(\alpha_l, \beta_m) &< 0, \\ \frac{2}{r-1} \Gamma_{iik}(\alpha_l, \beta_m) + \Gamma_{ijk}(\alpha_l, \beta_m) + \Gamma_{jik}(\alpha_l, \beta_m) &< 0, \end{aligned} \quad (19)$$

for $i, j, k \in \mathcal{I}_r$, $i \neq j$. The quantity $\Gamma_{ijk}(\alpha_l, \beta_m)$ is given by

$$\Gamma_{ijk}(\alpha_l, \beta_m) = \begin{bmatrix} -P_j & \star \\ Q_j \mathcal{A}_i(\alpha_l) - L_j \mathcal{C}_i(\beta_m) & P_k - Q_j - Q_j^\top \end{bmatrix},$$

where $\mathcal{A}_i(\alpha_l)$ and $\mathcal{C}_i(\beta_m)$ are defined as in (18), with $\alpha_l \in \mathcal{V}_\phi$, for $\forall l \in \mathcal{I}_{2^{n_\phi n_x}}$, and $\beta_m \in \mathcal{V}_\psi$, for $\forall m \in \mathcal{I}_{2^{n_\psi n_x}}$.

Then, the origin of the error system (17) is asymptotically stable. Moreover, the observer gain is expressed as

$$\mathcal{L}(h) = Q(h)^{-1} L(h), \quad (20)$$

with $[Q(h) \quad L(h)] = \sum_{i=1}^r h_i(z_\alpha) [Q_i \quad L_i]$.

Proof. Consider the fuzzy Lyapunov candidate function

$$V(e(t)) = e(t)^\top P(h) e(t), \quad (21)$$

where $P(h) = \sum_{i=1}^r h_i(z_\alpha) P_i$ and $P_i \succ 0$, for $i \in \mathcal{I}_r$. The goal is to prove that condition (19) guarantees the inequality

$$\Delta V(t) = V(e(t+1)) - V(e(t)) < 0 \quad (22)$$

along the trajectory of system (17).

By Lemma 2 and the convexity property of the bounded sets \mathcal{S}_ϕ and \mathcal{S}_ψ , it follows from (19) that

$$\begin{bmatrix} -P(h) & \star \\ \mathcal{X}(h, \alpha, \beta) & P(h_+) - Q(h) - Q(h)^\top \end{bmatrix} \prec 0, \quad (23)$$

with $\mathcal{X}(h, \alpha, \beta) = Q(h)\mathcal{A}(h, \alpha) - L(h)\mathcal{C}(h, \beta)$ and $P(h_+) = \sum_{i=1}^r h_i(z_\alpha(t+1)) P_i$, for $h \in \mathcal{H}$, $\alpha \in \mathcal{S}_\phi$ and $\beta \in \mathcal{S}_\psi$. Inequality (23) guarantees that $Q(h) + Q(h)^\top \succ 0$. This, in turn, guarantees the existence of $Q(h)^{-1}$, thus the validity of the gain expression (20).

Multiplying (23) with $[I \quad -\mathcal{Y}(h, \alpha, \beta)^\top]$ on the left and its transpose on the right, we obtain

$$\mathcal{Y}(h, \alpha, \beta)^\top P(h_+) \mathcal{Y}(h, \alpha, \beta) - P(h) \prec 0, \quad (24)$$

with $\mathcal{Y}(h, \alpha, \beta) = \mathcal{A}(h, \alpha) - Q(h)^{-1} L(h) \mathcal{C}(h, \beta)$, for $h \in \mathcal{H}$, $\alpha \in \mathcal{S}_\phi$ and $\beta \in \mathcal{S}_\psi$. Pre- and postmultiplying (24) with $e(t)$ and its transpose, we obtain $\Delta V(t) < 0$ after some manipulations, where $\Delta V(t)$ is defined in (22). This concludes the proof. \square

Remark 4. Using a ‘‘basic’’ design formulation, this paper aims to provide new insights on the application of DMVT for fuzzy observer design with unmeasured premise variables. For the discrete-time case, other LMI-based formulations using different nonquadratic Lyapunov functions [24], [27], [41] or various techniques to introduce slack variables [26], [32] can be directly applied to reduce further the design conservatism at the expense of increasing numerical complexity.

B. Continuous-Time N-TS Fuzzy Observer Design

We now provide sufficient LMI-based conditions to design N-TS fuzzy observers for continuous-time nonlinear systems.

Theorem 2. Consider the continuous-time N-TS fuzzy system (9) and its associated observer structure (11). If there exist a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$, and matrices $L_i \in \mathbb{R}^{n_x \times n_y}$, for $i \in \mathcal{I}_r$, such that

$$\begin{aligned} \Xi_{ii}(\alpha_l, \beta_m) &< 0, \\ \frac{2}{r-1} \Xi_{ii}(\alpha_l, \beta_m) + \Xi_{ij}(\alpha_l, \beta_m) + \Xi_{ji}(\alpha_l, \beta_m) &< 0, \end{aligned} \quad (25)$$

for $i, j \in \mathcal{I}_r$, $i \neq j$. The quantity $\Xi_{ij}(\alpha_l, \beta_m)$ is given by

$$\Xi_{ij}(\alpha_l, \beta_m) = \text{He}(P \mathcal{A}_i(\alpha_l) - L_j \mathcal{C}_i(\beta_m)),$$

where $\mathcal{A}_i(\alpha_l)$ and $\mathcal{C}_i(\beta_m)$ are defined as in (18), with $\alpha_l \in \mathcal{V}_\phi$, for $\forall l \in \mathcal{I}_{2^{n_\phi n_x}}$, and $\beta_m \in \mathcal{V}_\psi$, for $\forall m \in \mathcal{I}_{2^{n_\psi n_x}}$.

Then, the origin of the error system (17) is asymptotically stable. Moreover, the observer gain is expressed as

$$\mathcal{L}(h) = P^{-1} L(h). \quad (26)$$

Proof. The state estimation error (17) converges asymptotically to the origin if there exists $P \succ 0$ such that

$$\text{He}(P\mathcal{A}(h, \alpha) - P\mathcal{L}(h)\mathcal{C}(h, \beta)) \prec 0, \quad (27)$$

for $h \in \mathcal{H}$, $\alpha \in \mathcal{S}_\phi$ and $\beta \in \mathcal{S}_\psi$. Following the same arguments as in the proof of Theorem 1 and using the gain expression (26), we can prove that condition (25) implies (27). This concludes the proof. \square

Remark 5. For the continuous-time case, a common Lyapunov matrix P is considered for the design of N-TS fuzzy observers. Note that fuzzy Lyapunov functions as (21) can be also exploited to design fuzzy observers using the results in [23]. However, in contrast to the discrete-time case, significant technical difficulties arise in dealing with the *unknown* time-derivatives of the membership functions, which may lead to conservative and restrictive results [14]. This issue is not the focus of this paper.

Remark 6. Theorems 1 and 2 offer a unified framework to design fuzzy asymptotic observers for general nonlinear systems with *partially* or *fully* measurable premise variables. Indeed, if the nonlinear function(s) $f(\cdot)$ and/or $g(\cdot)$ of system (1) can be measured, then it suffices to replace, in (19) or (25), $\mathcal{A}_i(\alpha_i) = A_i$ and/or $\mathcal{C}_i(\beta_m) = C_i$, for $\forall(i, l, m) \in \mathcal{I}_r \times \mathcal{I}_{2^{n_\phi n_x}} \times \mathcal{I}_{2^{n_\psi n_x}}$. Therefore, the proposed framework is more general and flexible than most of recent results on fuzzy observer design [8], [23], [23], [26], [29], [30].

Remark 7. The design conditions in Theorems 1 and 2 are expressed in terms of LMIs, which are easily solved with numerical solvers. All optimization problems in this paper are performed with MATLAB R2016a platform using YALMIP toolbox and SDPT3 solver [42].

IV. ILLUSTRATIVE EXAMPLES

Numerical examples are presented hereafter to demonstrate the effectiveness of the proposed N-TS fuzzy observer design for both discrete-time and continuous-time cases.

Example 1 (Comparative study: Discrete-time case). Consider the following nonlinear system borrowed from [33]:

$$\begin{aligned} x_1(t+1) &= (1+T)x_1(t) + Tx_2(t) + \frac{\sin(b)}{b}Tx_3(t) \\ &\quad - 0.1Tx_4(t) + T(1+x_1^2(t))u(t) + T\eta(x(t)) \\ x_2(t+1) &= Tx_1(t) + (1-2T)x_2(t) \\ x_3(t+1) &= Tx_1(t) + (1-0.3T)x_3(t) + Tx_1^2(t)x_2(t) \\ x_4(t+1) &= (1-T)x_4(t) + \frac{\sin(b)}{b}Tx_3(t) + T\eta(x(t)) \\ y_1(t) &= x_2(t) + (1+x_1^2(t))x_4(t) \\ y_2(t) &= x_1(t) \end{aligned} \quad (28)$$

where $\eta(x) = \sin(x_3) - \frac{\sin(b)}{b}x_3$, $x_1 \in [-a, a]$, $x_3 \in [-b, b]$ and $b = \frac{\pi}{2}$. The fixed step of discretization is $T = 0.5$. The

premise variables are defined as $z_\alpha = x_1^2$ and $z_\beta = \sin(x_3)$. Then, system (28) can be rewritten in the form (6) with

$$\begin{aligned} A(z_\alpha) &= \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & Tz_\alpha & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix}, \\ f_\alpha(z_\alpha, u) &= \begin{bmatrix} T(1+z_\alpha)u \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad F(z_\alpha) = F = \begin{bmatrix} T \\ 0 \\ 0 \\ T \end{bmatrix}, \\ C(z_\alpha) &= \begin{bmatrix} 0 & 1 & 0 & 1+z_\alpha \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad g_\alpha(z_\alpha, u) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \phi(x) &= \sin(x_3), \quad \psi(x) = 0. \end{aligned}$$

Using the sector nonlinearity approach [13, Chapter 2] with the measurable premise variable $z_\alpha = x_1^2$, the following 2-rule N-TS fuzzy model can be obtained:

$$\begin{aligned} x(t+1) &= \sum_{i=1}^2 h_i(z_\alpha)A_i x(t) + f_\alpha(z_\alpha(t), u(t)) + F\phi(x(t)), \\ y(t) &= \sum_{i=1}^2 h_i(z_\alpha)C_i x(t), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & Ta^2 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & 0 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 0 & 1 & 0 & 1+a^2 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The corresponding membership functions are given by

$$h_1(z_\alpha) = \frac{x_1^2}{a^2}, \quad h_2(z_\alpha) = \frac{a^2 - x_1^2}{a^2}.$$

Applying the proposed approach, we define the vector $\rho(t)$ as in (16) with $\rho_{1i} = 0$, for $i \in \{1, 2, 4\}$, and $\rho_{13} = \cos(x_3)$. Since $x_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then the bounded convex set \mathcal{S}_ϕ has two vertices, *i.e.*, $\mathcal{V}_\phi = \{\rho_{13} : \rho_{13} \in \{0, 1\}\}$. Theorem 1 is readily applied to design a N-TS fuzzy observer for system (28).

For comparison purposes, we consider three following approaches for fuzzy observer design.

- Proposed approach with Theorem 1.
- Lipschitz-property-based approach in [33, Theorem 1] with $\eta(x)$ as the nonlinear consequent. Using Cauchy mean value theorem, the Lipschitz constant of function $\eta(x)$ can be computed ($\gamma = 0.6366$) for observer design.
- DMVT-based approach in [35]. Note that, for a fair comparison, the design conditions in [35] have been extended using a fuzzy Lyapunov function as in (21).

To illustrate the design conservatism, we find the largest value of parameter a , denoted by a^* , for which a fuzzy observer can be found for system (28). Table I summarizes

the numerical comparisons between three considered observer designs. We can see that the proposed approach provides a significant improvement in reducing the conservatism over existing results. Note that using N-TS fuzzy modeling in Theorem 1 and [33], the number of fuzzy rules is decreased from four to two, leading to a fuzzy observer structure with less numerical complexity as indicated in Remark 3. This also allows the Lipschitz-property-based approach [33, Theorem 1] to outperform the DMVT-based approach [35] in terms of conservatism reduction. Moreover, the computational burden of the new approach, represented by the number of scalar decision variables \mathcal{N}_{var} and the number of rows \mathcal{N}_{row} of all involved LMIs, is reasonable compared to two other ones.

TABLE I
NUMERICAL COMPARISON BETWEEN DIFFERENT FUZZY OBSERVER DESIGN APPROACHES.

Approach	Lipschitz-based [33]	DMVT-based [35]	Theorem 1
a^*	1399.84	996.49	2084.90
Nb. of rules	2	4	2
\mathcal{N}_{var}	53	72	68
\mathcal{N}_{row}	80	2064	136

For illustrations, we consider system (28) with $a = 2080$, for which no existing approach can provide a feasible solution for fuzzy observer design. Fig. 1 depicts the asymptotic estimation performance of the proposed N-TS fuzzy observer with respect to the initial condition $x(0) = [0.5 \ 1 \ 0 \ -3]^T$.

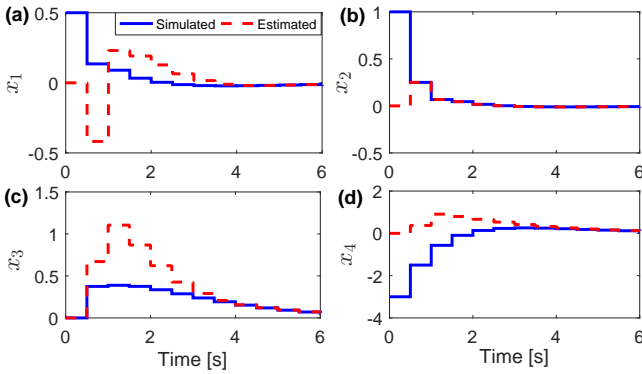


Fig. 1. Asymptotic estimation performance in Example 1 with $a = 2080$.

Example 2 (Physically motivated system). Estimating the internal human variables is crucial to understand the sitting control of persons living with spine cord injury. To this end, we consider the observer design for the following Head-Two-Arms-Trunk (H2AT) system [29], which is open-loop unstable:

$$\begin{aligned} 0 &= m_1 \ddot{x}_l - m_1 l_0 \ddot{\theta} - m_1 x_l \dot{\theta}^2 + m_1 g \sin(\theta) - F(t - \tau(t)) \\ 0 &= J \ddot{\theta} - m_1 l_0 \ddot{x}_l + 2m_1 x_l \dot{x}_l \dot{\theta} + m_1 g x_l \cos(\theta) \\ &\quad - (m_1 l_0 + m_2 l_c) g \sin(\theta), \end{aligned} \quad (29)$$

where $x_l(t)$ is the position of the trunk with respect to the mass center of the upper segment, $\theta(t)$ is the angular position of the trunk, $F(t - \tau(t))$ is the controlling force with a time-varying delay $\tau(t)$. Fig. 2 depicts the schematic of the studied H2AT system, whose parameters corresponding to a 80 kg male subject are given in Table II.

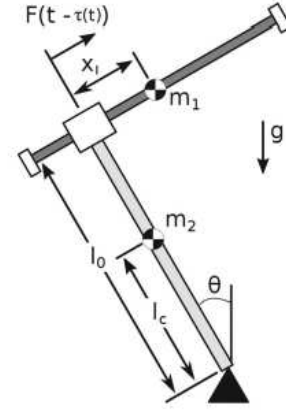


Fig. 2. Schematic of the studied H2AT system.

TABLE II
PARAMETER VALUES OF H2AT SYSTEM.

Parameter	Description	Value
m_1	Mass of the upper segment	16.1 [kg]
m_2	Mass of the trunk	26.64 [kg]
l_0	Length of the trunk	477 [mm]
l_c	Length of the mass center of the trunk	276.66 [mm]
J	Average moment of inertia of the subject	5.79 [kgm ²]
g	Gravitational constant	9.81 [m/s ²]

Let us denote $x_1(t) = x_l(t)$, $x_2(t) = \dot{x}_l(t)$, $x_3(t) = \theta(t)$, $x_4(t) = \dot{\theta}(t)$ and $u(t) = F(t - \tau(t))$. The dynamics of system (29) can be then rewritten as

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \left(2.7x_1x_4^2 - 7.2x_1x_2x_4 + 0.17u \right) \\ x_4 \\ \left(3.6x_1x_4^2 - 15x_1x_2x_4 + 0.22u \right) \\ \left(+34 \sin(x_3) - 74x_1 \cos(x_3) \right) \end{bmatrix}. \quad (30)$$

To take into account a full neck flexion with both stretched arms and an extension of the arms and neck, we consider a compact set of the state space defined as $x_1 \in [-1, 1]$ [m], $x_2 \in [-1, 1]$ [m/s], $x_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ [rad], $x_4 \in [-1, 1]$ [rad/s]. Note that the variation ranges of x_1 and x_3 considered here are much larger than that in [29] to fully take into account the subject's vertical movement.

The four nonlinearities of system (30) can be easily identified as $\cos(x_3)$, $\sin(x_3)$, $x_1x_4^2$ and $x_1x_2x_4$. In practice, only the positions x_l and θ can be measured for this real-world H2AT system, the premise variables $x_1x_4^2$ and $x_1x_2x_4$ are thus unmeasurable. To deal with this difficulty, we parameterize the nonlinear system (30) in the form

$$\dot{x} = Ax + f_\alpha(y, u) + F(z_\alpha)\phi(x), \quad y = Cx, \quad (31)$$

where $z_\alpha = x_1$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F(z_\alpha) = \begin{bmatrix} 0 & 0 \\ 2.7z_\alpha & -7.2z_\alpha \\ 0 & 0 \\ 3.6z_\alpha & -15z_\alpha \end{bmatrix},$$

$$f_\alpha(y, u) = \begin{bmatrix} 0 \\ 0.17u + 6.4 \sin(x_3) - 35x_1 \cos(x_3) \\ 0 \\ 0.22u + 34 \sin(x_3) - 74x_1 \cos(x_3) \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} x_4^2 \\ x_2 x_4 \end{bmatrix}.$$

By the sector nonlinearity approach [13], system (31) can be exactly represented by the following N-TS fuzzy system:

$$\dot{x} = Ax + f_\alpha(y, u) + \sum_{i=1}^2 h_i(z_\alpha) F_i \phi(x), \quad (32)$$

with

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0.284 & -0.756 \\ 0 & 0 \\ 0.378 & -1.575 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ -0.203 & 0.540 \\ 0 & 0 \\ -0.270 & 1.125 \end{bmatrix}.$$

The MFs of the N-TS fuzzy system (32) are given by

$$h_1(z_\alpha) = \frac{x_1 + 0.075}{0.18}, \quad h_2(z_\alpha) = \frac{0.105 - x_1}{0.18}.$$

The vector $\rho(t)$ in (16) can be now defined for system (32) with $\rho_{1i} = \rho_{2j} = 0$, for $i \in \{1, 2, 3\}$, $j \in \{1, 3\}$, and $\rho_{14} = 2\rho_{22}$, $\rho_{22} = x_4$, $\rho_{24} = x_2$. Since $x_2 \in [-1, 1]$ and $x_4 \in [-1, 1]$, then the bounded convex set \mathcal{S}_ρ has four vertices, *i.e.*, $\mathcal{V}_\rho = \{[\rho_{22}, \rho_{24}] : \rho_{22} \in \{-1, 1\}, \rho_{24} \in \{-1, 1\}\}$. Solving the design condition (25) in Theorem 2, the following N-TS fuzzy observer solution is obtained for the H2AT system (29):

$$P = \begin{bmatrix} 13.423 & -4.735 & -0.097 & 1.322 \\ -4.735 & 8.240 & 1.653 & -3.894 \\ -0.097 & 1.653 & 13.603 & -2.965 \\ 1.322 & -3.894 & -2.965 & 2.425 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 5.724 & 0.224 \\ 13.337 & -0.227 \\ 0.224 & 6.059 \\ -0.217 & 13.293 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 5.869 & 0.444 \\ 13.239 & -0.346 \\ 0.444 & 6.523 \\ -0.304 & 13.046 \end{bmatrix}.$$

Fig. 3 illustrates the estimation performance with respect to the initial condition $x(0) = [0 \ 0.2 \ 0 \ 0.3]^\top$. Note that the estimated states asymptotically converge to the states of the H2AT system (29).

To emphasize the interests of the proposed approach, a comparison is performed with the following approaches.

- Applying the Lipschitz-property-based approach [15], a 16-rule TS fuzzy observer can be found the maximum admissible Lipschitz constant $\gamma^* = 3.465$. However, the real Lipschitz constant of $\gamma = 18.600$ can be found for this example by solving a simple numerical optimization problem in [8, Chapter 4]. Since $\gamma \gg \gamma^*$, the classical approach using the Lipschitz property fails to provide a feasible estimation solution for the H2AT system.

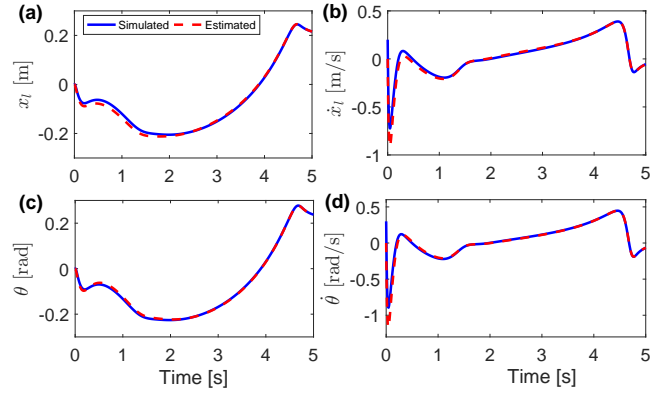


Fig. 3. Asymptotic estimation performance for H2AT system.

- The approach in [18] cannot be applied to this real-world system since the involved nonlinearities lead to a non-convergence of the dynamic-extension algorithm.
- The DMVT-based approach in [29] can be applied to design a 8-rule TS fuzzy observer. However, requiring only two fuzzy rules, the proposed N-TS fuzzy observer is of much less numerical complexity for real-time implementation. Especially, the new approach does not require any bounds on the control input u and the angular position x_3 for fuzzy observer design.

Example 3 (Comparative study: continuous-time case). Let us consider the following academic nonlinear system [29]:

$$\begin{aligned} \dot{x}_1 &= \frac{a}{4}x_1^2 + \frac{a}{4}x_1x_2 + (b-3)x_2 \\ \dot{x}_2 &= \frac{3}{4}x_1^2 + \frac{7}{4}x_1x_2 + x_2^2 + u \\ y &= x_1, \end{aligned} \quad (33)$$

where $x_1 \in [-2, 2]$ and $x_2 \in [-2, 2]$. The parameter ranges of system (33) are defined as $a \in [-5, 5]$ and $b \in [-1, 3]$. Since x_2 is unmeasurable, applying directly the sector nonlinearity approach [13] leads to an exact TS fuzzy model (2) of system (33) with unmeasurable MFs, thus technical difficulty for fuzzy observer design. With the proposed design approach, we rewrite the nonlinear system (33) in the form

$$\dot{x} = A(z_\alpha)x + f_\alpha(u) + F\phi(x), \quad (34)$$

with $z_\alpha = x_1$, $\phi(x) = x_2^2$ and

$$A(z_\alpha) = \begin{bmatrix} \frac{a}{4}x_1 & \frac{a}{4}x_1 + b - 3 \\ \frac{3}{4}x_1 & \frac{7}{4}x_1 \end{bmatrix}, \quad f_\alpha(u) = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Applying the sector nonlinearity, the following 2-rule N-TS fuzzy model can be obtained for system (34):

$$\dot{x} = \sum_{i=1}^2 h_i(z_\alpha) A_i x + f_\alpha(u) + F\phi(x),$$

where

$$A_1 = \begin{bmatrix} \frac{a}{2} & \frac{a}{2} + b - 3 \\ \frac{3}{2} & \frac{7}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{a}{2} & -\frac{a}{2} + b - 3 \\ -\frac{3}{2} & -\frac{7}{2} \end{bmatrix}.$$

The corresponding MFs are given by

$$h_1(z_\alpha) = \frac{x_1 + 2}{4}, \quad h_2(z_\alpha) = \frac{2 - x_1}{4}.$$

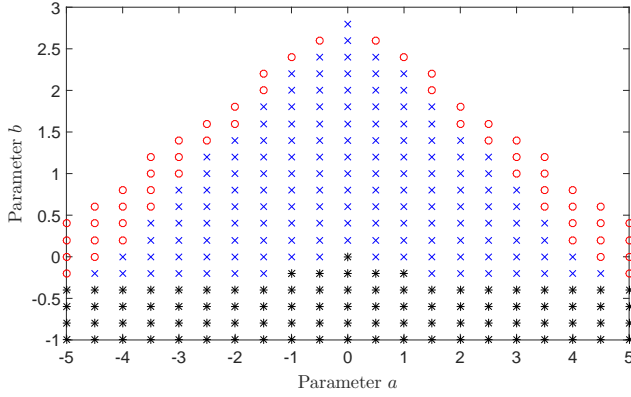


Fig. 4. Feasibility regions obtained with Theorem 3 in [15] (“*”), Theorem 1 in [29] (“*”, “x”), condition (25) of Theorem 2 (“*”, “x” and “o”).

With $\phi(x) = x_2^2$, we easily define $\rho = 2x_2$. Since $|x_2| \leq 2$, the bounded convex set \mathcal{S}_ϕ has two vertices, i.e., $\mathcal{V}_\phi = \{\rho : \rho \in \{-4, 4\}\}$. Theorem 2 can be now applied to design a N-TS fuzzy observer for the nonlinear system (33).

For comparison purposes, we examine the design conservatism between three approaches: i) the proposed approach (Theorem 2), ii) the classical Lipschitz-property-based approach in [15], and the recent DMVT-based approach in [29]. Fig. 4 depicts the feasibility regions obtained with the three considered approaches. Note that for the design approach [15], the Lipschitz constant γ is computed for each pair of parameter values (a, b) . Despite the simplicity of the benchmark system (33) with *constant* input and output matrices, we can observe that the proposed approach provides less conservative result than that of the most recent DMVT-based approach [29]. As also expected, the Lipschitz-property-based approach [15] leads to an over-conservative design result.

V. CONCLUDING REMARKS

A unified framework has been proposed to design fuzzy observers with unmeasurable premise variables for both continuous-time and discrete-time nonlinear systems. For observer design, the nonlinear systems are exactly represented by N-TS fuzzy models. The differential mean value theorem for vector-valued functions is applied to the unmeasurable nonlinear consequents of the N-TS fuzzy systems to deal with the mismatch caused by unmeasurable premise variables. Using Lyapunov-based arguments, sufficient conditions are derived in terms of LMIs to design asymptotic N-TS fuzzy observers. Three examples are given to demonstrate the interest of the new fuzzy observer design for both continuous-time and discrete-time cases. Several future research directions are related to the new observer design framework. First, we can exploit this framework for observer-based control design of fuzzy systems with unmeasurable premise variables [1], [21]. Second, an extension of the proposed results to type-2 TS fuzzy systems will be useful to deal with parametric uncertainties [34], [43]. Another promising topic consists in designing fuzzy observers for systems with unmeasured *hard* nonlinearities such as stiction, hysteresis or saturation.

REFERENCES

- [1] H. Li, C. Wu, S. Yin, and H. Lam, “Observer-based fuzzy control for nonlinear networked systems under unmeasurable premise variables,” *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 5, pp. 1233–1245, Oct. 2016.
- [2] H. Ma, Q. Zhou, L. Bai, and H. Liang, “Observer-based adaptive fuzzy fault-tolerant control for stochastic nonstrict-feedback nonlinear systems with input quantization,” *IEEE Trans. Syst., Man, Cybern.: Syst.*, vol. 49, no. 2, pp. 287–298, Feb. 2019.
- [3] J. Dong, Y. Wu, and G. Yang, “A new sensor fault isolation method for T–S fuzzy systems,” *IEEE Trans. Cybern.*, vol. 47, no. 9, pp. 2437–2447, Sep. 2017.
- [4] D. Du, S. Xu, and V. Cocquempot, “Fault detection for nonlinear discrete-time switched systems with persistent dwell time,” *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 4, pp. 2466–2474, Aug. 2018.
- [5] L. Cosme, W. Caminhas, M. D’Angelo, and R. Palhares, “A novel fault-prognostic approach based on interacting multiple model filters and fuzzy systems,” *IEEE Trans. Indus. Electron.*, vol. 66, no. 1, pp. 519–528, Jan. 2019.
- [6] C. Edwards, S. K. Spurgeon, and R. J. Patton, “Sliding mode observers for fault detection and isolation,” *Automatica*, vol. 36, no. 4, pp. 541–553, Apr. 2000.
- [7] C. Li, J. de Oliveira, M. Cerrada, D. Cabrera, R. Sanchez, and G. Zurita, “A systematic review of fuzzy formalisms for bearing fault diagnosis,” *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 7, pp. 1362–1382, July 2019.
- [8] Z. Lendek, T.-M. Guerra, R. Babuska, and B. De Schutter, *Stability Analysis and Nonlinear Observer Design Using Takagi-Sugeno Fuzzy Models*. Springer, Berlin, Heidelberg, Oct. 2011, vol. 262.
- [9] G. Besançon, Ed., *Nonlinear Observers and Applications*, ser. Lecture Notes Control Inf. Sci. Springer-Verlag, 2007, vol. 363.
- [10] H. Khalil and L. Praly, “High-gain observers in nonlinear feedback control,” *Int. J. Robust Nonlinear Control*, vol. 24, no. 6, pp. 993–1015, July 2014.
- [11] R. Li and X. Zhang, “Adaptive sliding mode observer design for a class of T-S fuzzy descriptor fractional order systems,” *IEEE Trans. Fuzzy Syst.*, pp. 1–1, 2019.
- [12] B. Jiang, H. Karimi, Y. Kao, and C. Gao, “Adaptive control of nonlinear semi-Markovian jump T–S fuzzy systems with immeasurable premise variables via sliding mode observer,” *IEEE Trans. Cybern.*, pp. 1–11, 2018.
- [13] K. Tanaka and H. Wang, *Fuzzy Control Systems Design and Analysis: a Linear Matrix Inequality Approach*. John Wiley & Sons, 2004.
- [14] A.-T. Nguyen, T. Taniguchi, L. Eciolaza, V. Campos, R. Palhares, and M. Sugeno, “Fuzzy control systems: Past, present and future,” *IEEE Comput. Intell. Mag.*, vol. 14, no. 1, pp. 56–68, Feb. 2019.
- [15] P. Bergsten, R. Palm, and D. Driankov, “Observers for Takagi-Sugeno fuzzy systems,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 32, no. 1, pp. 114–121, Feb. 2002.
- [16] M. Fadali, “Fuzzy functional observers for dynamic TSK systems,” in *The 14th IEEE Int. Conf. Fuzzy Syst.*, Reno, NV, USA, May 2005, pp. 389–394.
- [17] J. Li, Q. Zhang, X. Yan, and S. Spurgeon, “Observer-based fuzzy integral sliding mode control for nonlinear descriptor systems,” *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 5, pp. 2818–2832, Oct. 2018.
- [18] D. Ichalal, B. Marx, S. Mammar, D. Maquin, and J. Ragot, “How to cope with unmeasurable premise variables in Takagi-Sugeno observer design: dynamic extension approach,” *Eng. Appl. Artif. Intell.*, vol. 67, pp. 430–435, Jan. 2018.
- [19] M. Teixeira, E. Assuncao, and R. Avellar, “On relaxed LMI-based designs for fuzzy regulators and fuzzy observers,” *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 5, pp. 613–623, Oct. 2003.
- [20] C. Peng, S. Ma, and X. Xie, “Observer-based non-PDC control for networked T–S fuzzy systems with an event-triggered communication,” *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 2279–2287, Aug. 2017.
- [21] W.-B. Xie, Y.-L. Wang, J. Zhang, and M.-Y. Fu, “Novel separation principle based \mathcal{H}_∞ observer-controller design for a class of T–S fuzzy systems,” *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 6, pp. 3206–3221, 2018.
- [22] A. Boulkroune, M. Tadjine, M. M’Saad, and M. Farza, “Design of a unified adaptive fuzzy observer for uncertain nonlinear systems,” *Inf. Sci.*, vol. 265, pp. 139–153, May 2014.
- [23] L. Wang, H. Zhang, and X. Liu, “ \mathcal{H}_∞ observer design for continuous-time Takagi-Sugeno fuzzy model with unknown premise variables via nonquadratic Lyapunov function,” *IEEE Trans. Cybern.*, vol. 46, no. 9, pp. 1986–1996, Sep. 2016.
- [24] T.-M. Guerra, H. Kerkeni, J. Lauber, and L. Vermeiren, “An efficient Lyapunov function for discrete T–S models: Observer design,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 1, pp. 187–192, Feb. 2011.

- [25] A.-T. Nguyen, T.-M. Guerra, and V. Campos, "Simultaneous estimation of state and unknown input with ℓ_∞ guarantee on error-bounds for fuzzy descriptor systems," *IEEE Control Syst. Lett.*, vol. 3, no. 4, pp. 1020–1025, Oct. 2019.
- [26] X. Xie, D. Yue, J. Park, and H. Li, "Relaxed fuzzy observer design of discrete-time nonlinear systems via two effective technical measures," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 5, pp. 2833–2845, Oct. 2018.
- [27] L. Li, S. Ding, J. Qiu, Y. Yang, and Y. Zhang, "Weighted fuzzy observer-based fault detection approach for discrete-time nonlinear systems via piecewise-fuzzy Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 6, pp. 1320–1333, Dec. 2016.
- [28] J. Yoneyama, " \mathcal{H}_∞ filtering for fuzzy systems with immeasurable premise variables: An uncertain system approach," *Fuzzy Sets Syst.*, vol. 160, no. 12, pp. 1738–1748, June 2009.
- [29] T.-M. Guerra, R. Márquez, A. Kruszewski, and M. Bernal, " \mathcal{H}_∞ LMI-based observer design for nonlinear systems via Takagi-Sugeno models with unmeasured premise variables," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1498–1509, June 2018.
- [30] X. Wang and G. Yang, " \mathcal{H}_∞ observer design for fuzzy system with immeasurable state variables via a new Lyapunov function," *IEEE Trans. Fuzzy Syst.*, pp. 1–1, 2019.
- [31] A. Pérez-Estrada, G. Osorio-Gordillo, M. Darouach, M. Alma, and V. Olivares-Peregrino, "Generalized dynamic observers for quasi-LPV systems with unmeasurable scheduling functions," *Int. J. Robust Nonlinear Control*, vol. 28, no. 17, pp. 5262–5278, 2018.
- [32] M. Chadli and H. Karimi, "Robust observer design for unknown inputs Takagi-Sugeno models," *IEEE Trans. Fuzzy Syst.*, vol. 1, no. 21, pp. 158–164, Feb. 2013.
- [33] J. Dong, Y. Wang, and G. Yang, "Output feedback fuzzy controller design with local nonlinear feedback laws for discrete-time nonlinear systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 6, pp. 1447–1459, Dec. 2010.
- [34] H. Hassani, J. Zarei, M. Chadli, and J. Qiu, "Unknown input observer design for interval type-2 T–S fuzzy systems with immeasurable premise variables," *IEEE Trans. Cybern.*, vol. 47, no. 9, pp. 2639–2650, Sep. 2017.
- [35] D. Ichalal, H. Arioui, and S. Mammar, "Observer design for two-wheeled vehicle: A Takagi-Sugeno approach with unmeasurable premise variables," in *19th Mediterranean Conference Control Automation*, Corfu, Greece, June 2011, pp. 934–939.
- [36] P. Coutinho, R. Araújo, A.-T. Nguyen, and R. Palhares, "A multiple parameterization approach for local stabilization of constrained Takagi-Sugeno fuzzy systems with nonlinear consequents," *Inf. Sci.*, vol. 506, pp. 295–307, Jan. 2020.
- [37] R. McLeod, "Mean value theorems for vector valued functions," *Proc. Edinburgh Math. Soc.*, vol. 14, no. 3, pp. 197–209, 1965.
- [38] G. Phanomchoeng and R. Rajamani, "The bounded Jacobian approach to nonlinear observer design," in *Proc. American Control Conference*, Baltimore, USA, July 2010, pp. 6083–6088.
- [39] H. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 324–332, Apr. 2001.
- [40] A. Sala and C. Arino, "Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem," *Fuzzy Sets Syst.*, vol. 158, no. 24, pp. 2671–2686, Dec. 2007.
- [41] A. Nasiri, S. Nguang, A. Swain, and D. Almkhles, "Reducing conservatism in an \mathcal{H}_∞ robust state-feedback control design of T–S fuzzy systems: A nonmonotonic approach," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 1, pp. 386–390, Feb. 2018.
- [42] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in Matlab," in *Proc. IEEE Int. Symp. Comput. Aided Control Syst. Des.*, Taipei, 2004, pp. 284–289.
- [43] S. Fadali and S. Jafarzadeh, "TSK observers for discrete type-1 and type-2 fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 2, pp. 451–458, Apr. 2013.



Juntao Pan received the B.S. degree in communication engineering and M.Sc. degree in control theory and control engineering from Three Gorges University, Yichang, China, in 2006 and 2008, respectively, and the Ph.D. degree in control theory and control engineering from Southeast University, Nanjing, China, in 2012.

He is currently an Associate Professor with North Minzu University, Yinchuan, China. His current research interests include fuzzy control, robust control, nonlinear control and their applications.



Anh-Tu Nguyen is an Associate Professor at the Université Polytechnique des Hauts-de-France, Valenciennes, France. He received the degree in engineering and the M.Sc. degree in automatic control from Grenoble Institute of Technology, France, in 2009, and the Ph.D. degree in automatic control from the Université Polytechnique Hauts-de-France, Valenciennes, France, in 2013.

After working a short period in 2010 at the French Institute of Petroleum, Rueil-Malmaison, France, Dr. Nguyen began his doctoral program at the University of Valenciennes, France, in collaboration with the VALEO Group. From February 2014 to August 2018, Dr. Nguyen was a postdoctoral researcher at the laboratory LAMIH UMR CNRS 8201, Valenciennes, France, and the laboratory LS2N UMR CNRS 6004, Nantes, France. Dr. Nguyen's research interests include robust control, constrained control systems, human-machine shared control for intelligent vehicles.



Thierry-Marie Guerra is a Full Professor at the Polytechnic University Hauts-de-France, France. He received his PhD degree in automatic control in 1991 and the HDR in 1999. From 2009 to 2019, he was head of the Laboratory of Industrial and Human Automation, Mechanics and Computer Science (LAMIH CNRS UMR 8201) (148 researchers and staff, 110 PhD students and post-docs) <http://www.univ-valenciennes.fr/LAMIH/>. He is chair of the Technical Committee 3.2 "Computational Intelligence in Control" for IFAC, member of the IFAC TC 7.1 "Automotive Control", Area Editor of the international journals: *Fuzzy Sets & Systems* and *IEEE Transactions on Fuzzy Systems*. His research fields and topics of interest are wine, hard rock, stamps, nonlinear control, LPV, Takagi-Sugeno models control and observation, LMI constraints, nonquadratic Lyapunov functions and applications to mobility, soft robotics and disabled persons.



Dalil Ichalal received his M.Sc. degree from Paul Cezanne Aix Marseille 3 University, France, in 2006, and a Ph.D. degree from the National Polytechnic Institute of Lorraine, France, in 2009. In 2010, he joined the Department of Electrical Engineering of the University of Évry Val d'Essonne and the laboratory IBISC (Informatique, Biologie Intégrative et Systèmes Complexes), France. His research interests include state estimation and observer design, as well as fault diagnosis and fault tolerant control of nonlinear systems with application to ground and

aerial vehicles.