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Finite-Time Stability and Stabilization Results for Switched Impulsive Dynamical Systems on Time Scales

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Abstract: In this article, we study the finite-time stability (FTS) and finite time stabilization problems for a class of switched impulsive systems evolving on an arbitrary time domain. This problem is formulated using time scale theory where the time domain can be continuous, discrete, union of disjoint intervals with variable gaps and variable lengths or any combination of these. Using common Lyapunov-quadratic and Lyapunov-like functions, we establish sufficient conditions to ensure the FTS results. Further, to solve the stabilization problem, we design state feedback controllers. We have illustrated the effectiveness of the obtained analytical results through numerical examples.

Keywords: Time scales; Switched systems; Finite-time stability; Stabilization;

1 Introduction

Most of the existing literature on stability analysis of dynamical systems has focused on exponential stability and Lyapunov asymptotic stability which is defined over an infinite time interval and does not usually involve a short period of information. However, in numerous practical applications of engineering and science such as heat transfer systems, multi-agent systems, etc., it is required that for a finite time interval, the state value of the practical system does not exceed a given bound. In this case, Lyapunov stability is not sufficient, because the system can be Lyapunov stable, however it may contain undesirable transient performances in some time intervals. Therefore, the problem of FTS is meaningful.

The concept of FTS was introduced in the Russian literature [1, 2]. Later, this concept appeared in the western journals [3, 4]. Roughly speaking, a system is called FTS if, for a fixed time interval, its state does not exceed some bound for a given bound on the initial condition. It is not difficult to observe that FTS and Lyapunov asymptotic stability are two independent properties. Indeed a system could be FTS however not Lyapunov asymptotically stable and vice-versa [5]. In reality, numerous practical issues are related to the problem of FTS. For example, the issue of guaranteeing that a space vehicle will stay in a predetermined orbit for a given time allotment in order to finish a set of experiments; in a compound procedure, the weight, temperature or some different parameters should be kept within a predetermined bound in a prescribed time interval. In these cases, FTS used since it is important to study the transient behavior over a finite time interval instead of the asymptotic behavior of the system response. Over the most recent couple of years, many authors investigated

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FTS for continuous-time and discrete-time systems [6, 7, 8, 9, 10, 11], but they are studied separately. However, there is no result concerning FTS for dynamical systems evolving on arbitrary time domains.

Switched systems consist of a group of continuous or discrete subsystems and a rule controlling the switching between them. This class of systems has been extensively studied since they model many physical or artificial systems displaying switching features such as electrical engineering, computer science, network control systems, etc. One important studied problem for this class of systems is the stability and stabilization problems. In the last few decades, a number of methods have been introduced [12, 13, 14, 15, 16, 17] mainly using Lyapunov-based theory. Analysis methods can be categorized into the common, multiple and piecewise Lyapunov function methods. For more details of recent results on stability and stabilization problem for switched dynamical systems, one can see [18, 19, 20, 21, 22, 23, 24].

There are many physical processes such as natural disasters, harvesting and shocks, which are often subject to unexpected abrupt changes in their state [25, 26]. These short-term abrupt changes are treated as having acted instantaneously or in the form of impulses. In recent years, impulsive differential equations have attracted significant attention on both practical and theoretical aspects since they provide an appropriate structure for mathematical modeling of numerous physical phenomena where the systems have some sudden changes. It is found that impulsive differential equations have significant applications in many fields of science and engineering such as mechanical systems, networked control systems with scheduling protocol, sampled-data systems, control systems with communication constraints [27, 28], etc. In the last few years, many authors established different results such as controllability, Lyapunov stability, FTS and stabilization for impulsive differential equations [29, 30, 31, 32, 33]. Further, in many switched differential systems, there exist some impulse effects at the time of switching, henceforth it is very important to study switched systems with impulsive conditions. Recently, switched impulsive systems have received a lot of attention, see for instance [34, 35, 36, 37] and references cited therein.

In general, one studies the discrete and continuous dynamical systems separately and most of the results have to be proved for each case (using discrete analysis or continuous analysis). In many physical models, we often need to consider continuous and discrete evolution at the same time, or evolution on some different time-lines. For instance, to model the growth process of some species such as Pharaoh cicada, Magicicada cassinii and Magicicada septendecim, we need a particular time scale of type $\mathbb{T} = \bigcup_{k=1}^{\infty} [k(a+b), k(a+b)+b]$ with $a, b \in \mathbb{R}^+$, since it depends on both continuous and discrete times. Moreover, in a simple electric circuit with capacitance C , inductance L and resistance R , if we discharge the capacitor periodically at every time unit while assuming that the discharging takes a small $\delta > 0$ time unit, then this process can be modeled by using the following time scale $\mathbb{T} = \bigcup_{k \in \mathbb{N}_0} [k, k+1-\delta]$. Using either a difference equation or a differential equation, we cannot accurately describe the dynamic behaviour of such types of models. Therefore, we need an equation that works simultaneously for continuous and discrete analyses. As a consequence, in 1988, S. Hilger, in his Ph.D. thesis [38], introduced the concept of time scales and dynamical systems on time scales. This theory encapsulates the discrete and continuous analysis into a single analysis and helps to avoid studying the results twice. A time scale is a non-empty closed subset of \mathbb{R} . Therefore, the results obtained on time scales will be true for discrete-time systems (by choosing the time scales to be the set of integers) and continuous-time systems (by considering the time scales as the real numbers), as well as for any non-uniform time domains (combination between continuous intervals and discrete points or a discrete non-uniform domain) which is very useful in the study of various complex dynamical systems. In the last few years, many authors have investigated the study of dynamical systems on time scales and found many applications in various fields of science and engineering such as control theory, population dynamics, thermal physics and so on. For a comprehensive study on time scales and dynamical systems on time scales, one can see [39, 40, 41].

Stability analysis of dynamical systems on arbitrary time scales has been investigated in several works [42, 43, 44, 45, 46, 47]. In [42], the author investigated the exponential stability for linear dynamical systems on

arbitrary time scales with the help of generalized exponential function on time scales. The exponential stability for nonlinear positive dynamic systems on time scales was reported in [43]. In [44], the authors established some sufficient conditions for the uniform exponential stability and h-stability for a class of dynamic systems on time scales. In [45], the authors examined the stability results for a nonlinear dynamic system on time scales by using Lyapunov technique. However, only few works have been reported regarding the stability result of switched systems on time scales [48, 49, 50, 51, 52]. In [48], the authors considered a class of switched linear systems on non-uniform time domains and established the stability results. The stability results by using Lyapunov functions for a class of uncertain switched nonlinear systems have been reported in [49]. In [50], the authors studied the global asymptotic stability results for switched systems on time scales by using Lyapunov technique.

From the author's knowledge there is no manuscript which discussed the FTS and stabilization problem for switched impulsive systems on time scales. Motivated from the above discussion, in this manuscript, we establish the FTS and stabilization results for switched dynamical systems with impulsive conditions on arbitrary time scales. These results are established by using common Lyapunov-quadratic and Lyapunov-like functions.

The primary contribution and advantage of this paper are as follows. The results of this manuscript are devoted to impulsive switched systems evolving on an arbitrary time domain and formulated using time scale theory. It includes continuous-time, discrete-time as well as any combination of these two, henceforth the results of this manuscript are non-trivial extensions of the existing results. There are many applications which have some jumps at some specific time instants. Here, we considered an impulsive switched system and established some FTS results. To establish these results, we constructed a common Lyapunov quadratic function on time scales in which the delta derivative of this function should be negative definite. In addition, we also established FTS results by constructing some Lyapunov like functions on time scales in which we relax the condition of negative definiteness. Some numerical simulations for different time domains are given to verify the proposed theoretical results.

The rest of the paper is organized as follows: In Section 2, we recall some preliminaries, fundamental definitions and important lemmas. The problem of statement is defined in Section 3. In Section 4, we investigate FTS for switched impulsive dynamical systems on time scales. Section 5 is devoted to study the finite time stabilization problem for the considered class of systems. We present some illustrative examples to show the effectiveness of the obtained analytical results.

Notations: Throughout this manuscript, we denote \mathbb{T} for time scales and $I = [0, T]_{\mathbb{T}}$, $T > 0$. For any matrix A , $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ and A^* denotes the minimum, maximum eigenvalue value and matrix transpose of a matrix A respectively. For a matrix $P \in \mathbb{R}^{n \times n}$, $P > (\geq, <, \leq) 0$ means that P is positive (semi-positive, negative, semi-negative) definite matrix. \mathbb{R}^n denotes the n-dimensional Euclidean space. For $x = \text{col}(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the norm of x is $\|x\| = \sup_{t \in I} \|x(t)\|$. $M_{m \times n}(\mathbb{R})$ denotes the set of all real matrices of order $m \times n$. Id denotes the identity matrix of order n .

2 Preliminaries

Here, we recall some basics definitions, important lemmas for time scales theory.

A time scale interval is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. In a similar way, $(a, b)_{\mathbb{T}}$, $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ can be defined. An operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ defined by $\sigma(t) = \inf\{\vartheta \in \mathbb{T} : \vartheta > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ is called the forward jump operator while the positive function $\mu : \mathbb{T} \rightarrow [0, \infty)$ defined by $\mu(t) = \sigma(t) - t, \forall t \in \mathbb{T}$ is called the graininess function. We define a backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup\{\vartheta \in \mathbb{T} : \vartheta < t\}$ with the substitution $\sup \emptyset = \inf \mathbb{T}$. Also, we define the set \mathbb{T}^{κ} as follows:

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}^{\kappa} \setminus (\rho(\sup(\mathbb{T}), \sup(\mathbb{T})) & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.1. [39] A point $t \in \mathbb{T}$ is called

1. right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$;
2. right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$;
3. left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

Remark 2.2. A point t is called a dense point if it is right and left dense at the same time.

In the next definition, we define the delta derivative.

Definition 2.3. [39] Let $f : \mathbb{T} \rightarrow \mathbb{R}^n$ be a function. Then the delta derivative of f at a point t is defined as a number $f^\Delta(t)$ (provided it exists) whenever for each $\epsilon > 0$ there is a neighborhood \mathcal{U} of t such that

$$|[f(\sigma(t)) - f(\vartheta)] - f^\Delta(t)[\sigma(t) - \vartheta]| \leq \epsilon |\sigma(t) - \vartheta|, \quad \forall \vartheta \in \mathcal{U}.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$, which is the usual derivative of $f : \mathbb{R} \rightarrow \mathbb{R}^n$. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t)$, which is the usual forward difference of $f : \mathbb{Z} \rightarrow \mathbb{R}^n$ defined by $\Delta f(t) = f(t+1) - f(t)$.

Definition 2.4. [39] Let $f : \mathbb{T} \rightarrow \mathbb{R}^n$ be a function. Then a function F is called antiderivative of f if $F^\Delta(t) = f(t), \forall t \in \mathbb{T}^k$. Also, the delta integral is given by

$$\int_{t_0}^t f(\xi) \Delta \xi = F(t) - F(t_0).$$

The next properties of functions defined on time scale are often used.

Definition 2.5. [39] A function q from \mathbb{T} to \mathbb{R} is called regressive if $1 + \mu(t)q(t) \neq 0$ for all $t \in \mathbb{T}$ and \mathcal{R} denotes the collection of all regressive functions.

Definition 2.6. [39] For $p, q \in \mathcal{R}$, the following holds

$$(i) \quad p \oplus q = p + q + \mu(t)pq. \quad (ii) \quad \ominus p = \frac{-p}{1 + \mu(t)p}. \quad (iii) \quad p \ominus q = p \oplus (\ominus q).$$

The generalized exponential function of scalar function $q \in \mathcal{R}$ is defined hereafter.

Definition 2.7. [39] If $q \in \mathcal{R}$, then we generalize the classical exponential function on arbitrary time scales as

$$e_q(t, \vartheta) = \exp \left(\int_{\vartheta}^t \zeta_{\mu(\xi)}(q(\xi)) \Delta \xi \right), \quad \text{for } t, \vartheta \in \mathbb{T},$$

with

$$\zeta_{\mu(\vartheta)}(q(\vartheta)) = \begin{cases} \frac{1}{\mu(\vartheta)} \text{Log}(1 + q(\vartheta)\mu(\vartheta)), & \text{if } \mu(\vartheta) \neq 0 \\ q(\vartheta), & \text{if } \mu(\vartheta) = 0. \end{cases}$$

A function $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is said to be regulated if the right-hand limit exists (finite) at all right-dense points of \mathbb{T} and the left-hand limit exists (finite) at all left-dense points of \mathbb{T} . Also, f is called rd-continuous, if f is regulated and continuous at all right-dense points of \mathbb{T} . The collection of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

A function $A : \mathbb{T} \rightarrow M_{m \times n}(\mathbb{R})$ is said rd-continuous, if each of its scalar entry functions are rd-continuous. Also, if $m = n$, then A is said to be regressive if $\det(Id + \mu(t)A(t)) \neq 0, \forall t \in \mathbb{T}$, where Id is the $n \times n$ identity matrix. The collection of all regressive functions $A : \mathbb{T} \rightarrow M_n(\mathbb{R})$ is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

Lemma 2.8. [39] Let $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, then the unique solution of the following system

$$x^\Delta(t) = A(t)x(t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad t_0 \in \mathbb{T}, \quad (2.1)$$

is given by $x(t) = \Phi_A(t, t_0)x_0$, where $\Phi_A(\cdot, t)$ is called the fundamental matrix of system (2.1). If $A(t) = A$ then $\Phi_A(\cdot, t) = e_A(\cdot, t)$.

Lemma 2.9. [39] If $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$. Then for any $t, \vartheta, r \in \mathbb{T}$, we have

- (i) $e_A(t, t) = Id$ and $e_0(t, \vartheta) = Id$; (ii) $e_A(t, \vartheta) = e_A^{-1}(\vartheta, t)$.
- (iii) $e_A(t, \vartheta)e_A(\vartheta, r) = e_A(t, r)$; (iv) $e_A(\sigma(t), \vartheta) = (Id + \mu(t)A(t))e_A(t, \vartheta)$.

3 Problem of statement

It is well known that impulsive differential equations arise in many physical phenomena of science and engineering. For instance, consider the following population growth model with impulses [33]

$$\begin{aligned} N^\Delta(t) &= r_k N(t) + c_k U(t), \quad t \neq t_k \\ N(t_k^+) &= (r_{k+1} - r_k)N(t_k), \quad k = 1, 2, \dots, \\ N(t_0) &= N_0, \end{aligned}$$

where $N(t)$ is the number of population at the time t , N_0 denotes the initial condition at time $t = 0$, r_k is the rate of population growth between two consecutive impulsive points, t_k is the point of impulses and $U(t)$ is the control input. Such model can describe the evolution of cicada magicicada septendecim. In this case, it is needed to consider the time scale $\mathbb{T} = \mathbb{P}_{1,1}$ (see [33], Example 5.1). Another application is an impulsive model in nonelectronic as follows

$$\begin{aligned} \theta^\Delta(t) &= -\frac{\gamma}{\pi}\theta(t) + \gamma(a - b \cos t), \quad t \neq \tau_k, \\ \theta(\tau_k^+) &= -3\pi, \quad t = \tau_k, \\ \theta(0) &= \theta_0, \\ |\theta(0)| &< \pi, \end{aligned}$$

where $\theta = \frac{2\pi C v_c}{e}$, $\gamma = \frac{\pi}{RC\omega_p}$, $a = \frac{V_b}{V_T}$, $b = \frac{V_p}{V_T}$, $\tau = \omega_p t$, $V_T = \frac{e}{2C}$ is the quantum-mechanical tunneling voltage, $v_p = V_p \cos \omega_p$ is the sinusoidal voltage source, V_p is the voltage of the source, ω_p is the angular velocity of sinusoidal voltage source v_p , V_b is the DC voltage source, v_c is the junction voltage, R denotes the resistance, C denotes the capacitor, τ is the normalized time, e is the electron charge, t_k , $k = 1, 2, \dots$ are time instants when $\theta(t) = \pi$. For more details, please see [33], example 5.2. Therefore, in this manuscript, for the FTS analysis, we consider the following switched impulsive dynamical system on time scales

$$\begin{aligned} x^\Delta(t) &= A_k x(t), \quad t \in (t_{k-1}, t_k]_{\mathbb{T}}, k = 1, 2, \dots, \\ x(t_k^+) &= J_k(t_k, x(t_k)), \quad k = 1, 2, \dots, \\ x(0) &= x_0 \end{aligned} \quad (3.2)$$

and for the stabilization analysis, we consider the following impulsive system

$$\begin{aligned} x^\Delta(t) &= A_k x(t) + B_k u(t), \quad t \in (t_{k-1}, t_k]_{\mathbb{T}}, k = 1, 2, \dots, \\ x(t_k^+) &= J_k(t_k, x(t_k)), \quad k = 1, 2, \dots, \\ x(0) &= x_0, \end{aligned} \quad (3.3)$$

where \mathbb{T} is a time scale, $x(t) \in \mathbb{R}^n$ is the state variable. There exist a natural number $m > 1$, a sequence of times t_k and $T > 0$ such that $0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m = T < t_{m+1} < \dots$, $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t_k - h)$, denotes the right and left limit of $x(t)$ at $t = t_k$. $A_k \in C_{rd}\mathcal{R}(I, \mathbb{R}^n)$, $B_k \in C_{rd}\mathcal{R}(I, \mathbb{R}^{n \times m})$, $u(t) \in \mathbb{R}^m$ is the control function. J_k are continuous functions which satisfy some conditions to be specified later.

The definition of FTS in [32] is extended here to the impulsive switched systems on time scales as follows.

Definition 3.1. For three positive scalars c_1, c_2, T such that $c_1 < c_2$, a positive definite matrix R and a positive definite matrix valued function $\Lambda(\cdot)$ defined over $[0, T]_{\mathbb{T}}$, with $\Lambda(0) < R$, system (3.2) is called FTS w.r.t. $(c_1, c_2, T, R, \Lambda(\cdot))$ if

$$x_0^* R x_0 \leq c_1 \implies x^*(t) \Lambda(t) x(t) < c_2, \quad \forall t \in (0, T]_{\mathbb{T}}. \quad (3.4)$$

Remark 3.2. In the above Definition 3.1, if we take $\mathbb{T} = \mathbb{R}$, $c_1 = 1, c_2 = 1$, then we can obtain Definition 1 of [32].

Remark 3.3. The FTS dealt with in this paper should not be confused with the FTS concept adopted in some other papers [53, 54, 55]. In these works, the authors focus on the Lyapunov stability analysis of nonlinear systems whose trajectories converge to an equilibrium point in finite time and on the characterization of the associated settling-time. Thus, it should be remarked that the definition of FTS considered in this paper is unrelated to the one given in [53, 54, 55]. For more details please see [5].

Assumption 1: There exist positive constants $\beta_k, k = 1, 2, \dots, m - 1$, such that

$$\|J_k(t_k, x(t_k))\|^2 \leq \beta_k \|x(t_k)\|^2.$$

For a finite time interval I , we set $\mu_{\max} = \max_{t \in I} \mu(t)$.

Lemma 3.4. Let Assumption 1 hold, then system (3.2) has a unique solution of the form

$$x(t) = \begin{cases} e_{A_1}(t, t_0)x_0, & t \in (0, t_1]_{\mathbb{T}} \\ e_{A_2}(t, t_1)J_1(t_1, x(t_1)), & t \in (t_1, t_2]_{\mathbb{T}} \\ e_{A_3}(t, t_2)J_2(t_2, x(t_2)), & t \in (t_2, t_3]_{\mathbb{T}} \\ \vdots & \vdots \\ e_{A_{m-1}}(t, t_{m-2})J_{m-2}(t_{m-2}, x(t_{m-2})), & t \in (t_{m-2}, t_{m-1}]_{\mathbb{T}} \\ e_{A_m}(t, t_{m-1})J_{m-1}(t_{m-1}, x(t_{m-1})), & t \in (t_{m-1}, t_m]_{\mathbb{T}} \end{cases} \quad (3.5)$$

Proof. From Lemma 2.8, if $t \in (0, t_1]_{\mathbb{T}}$, then the unique solution of system (3.2) is given by

$$x(t) = e_{A_1}(t, t_0)x_0, \quad t \in (0, t_1]_{\mathbb{T}}.$$

For $t \in (t_1, t_2]_{\mathbb{T}}$, the initial value problem

$$\begin{aligned} x^\Delta(t) &= A_2 x(t), \quad t \in (t_1, t_2]_{\mathbb{T}}, \\ x(t_1^+) &= J_1(t_1, x(t_1)), \end{aligned}$$

has the unique solution

$$x(t) = e_{A_2}(t, t_1)x(t_1^+) = e_{A_2}(t, t_1)J_1(t_1, x(t_1)), \quad t \in (t_1, t_2]_{\mathbb{T}}.$$

Similarly, repeating this process, we get the desired solution (3.5). \square

4 Finite time stability analysis of impulsive switched systems

This section is devoted to the study of FTS results for the considered impulsive system (3.3).

Let $P_k(t) > 0$, $k = 1, 2, \dots, m$, be continuously differentiable matrices. We set

$$\rho_k = \frac{\lambda_{\max}(P_{k+1}(t_k))}{\lambda_{\min}(P_k(t_k))}, \quad k = 1, 2, \dots, m-1. \quad (4.6)$$

Theorem 4.1. *Suppose that Assumption 1 holds and the following inequalities*

$$A_k^* P_k(t) + P_k(t) A_k + \mu_{\max} A_k^* P_k(t) A_k + (I + \mu_{\max} A_k^*) P_k^\Delta(t) (I + \mu_{\max} A_k) < 0, \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad k = 1, 2, \dots, m, \quad (4.7)$$

$$\rho_k \beta_k < 1, \quad k = 1, 2, \dots, m-1, \quad (4.8)$$

$$\Lambda(t) < P_k(t), \quad \forall t \in [0, T]_{\mathbb{T}}, \quad (4.9)$$

$$P_k(0) < R, \quad (4.10)$$

are satisfied. Then, system (3.2) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda(\cdot))$.

Proof. Let us consider the candidate switched Lyapunov functions $V_k(t, x) = x^* P_k(t) x$, $k = 1, 2, \dots, m$. Then, for any $t \in (t_{k-1}, t_k]_{\mathbb{T}}$, we have

$$\begin{aligned} V_k^\Delta(t) &= (x^*)^\Delta P_k(t) x + (x^*)^\sigma P_k^\Delta(t) x + (x^*)^\sigma P_k^\sigma(t) x^\Delta \\ &= x^* A_k^* P_k(t) x + (x^* + \mu(t) x^* A_k^*) P_k^\Delta(t) x + (x^* + \mu(t) x^* A_k^*) (P_k(t) + \mu(t) P_k^\Delta(t)) A_k x \\ &\leq x^* (A_k^* P_k(t) + P_k^\Delta(t) + \mu_{\max} A_k^* P_k^\Delta(t) + P_k(t) A_k + \mu_{\max} (P_k^\Delta(t) + A_k^* P_k(t) A_k + \mu_{\max} A_k^* P_k^\Delta(t) A_k)) x \\ &= x^* (A_k^* P_k(t) + P_k(t) A_k + \mu_{\max} A_k^* P_k(t) A_k + (I + \mu_{\max} A_k^*) P_k^\Delta(t) (I + \mu_{\max} A_k)) x, \end{aligned}$$

which is negative definite due to (4.7). Now, when $t = t_k$, $k = 1, 2, \dots, m-1$, we have

$$\begin{aligned} V_{k+1}(t_k^+) - V_k(t_k) &= x(t_k^+)^* P_{k+1}(t_k) x(t_k^+) - x(t_k)^* P_k(t_k) x(t_k) \\ &= J_k(t_k, x(t_k))^* P_{k+1}(t_k) J_k(t_k, x(t_k)) - x(t_k)^* P_k(t_k) x(t_k) \\ &\leq \lambda_{\max}(P_{k+1}(t_k)) J_k(t_k, x(t_k))^* J_k(t_k, x(t_k)) - x(t_k)^* P_k(t_k) x(t_k) \\ &\leq \beta_k \lambda_{\max}(P_{k+1}(t_k)) x(t_k)^* x(t_k) - x(t_k)^* P_k(t_k) x(t_k) \\ &\leq (\beta_k \rho_k - 1) V_k(t_k). \end{aligned}$$

which is non-positive due to (4.8). Hence, $V_k(t, x)$ is decreasing along system (3.2). Furthermore, thanks to inequalities (4.9) and (4.10), for given x_0 such that $x_0^* R x_0 \leq c_1$, we have, for all $t \in I$,

$$\begin{aligned} x^*(t) \Lambda(t) x(t) &\leq x^*(t) P_k(t) x(t) \\ &\leq x^*(0) P_k(0) x(0) \\ &< x_0^* R x_0 \\ &\leq c_1 < c_2. \end{aligned}$$

Hence, the result follows. \square

Remark 4.2. *If we set $\mathbb{T} = \mathbb{R}$, $A_k = A(t)$ and $J_k(t_k, x(t_k)) = A_d(t_k) x(t_k)$, then, we can obtain Theorem 5 of [32]. Hence, the outcomes of this manuscript are non-trivial extensions of the existing results in the literature.*

Next, we provide an example to illustrate Theorem 4.1.

Example 4.3. Let us consider system (3.2) with the following matrices

$$A_i = \begin{bmatrix} -0.3 & 0 \\ 1 & -1.1 \end{bmatrix}, A_j = \begin{bmatrix} -1.2 & 1 \\ 0 & -0.4 \end{bmatrix}, J_k(t_k, x(t_k)) = \begin{bmatrix} (0.3 - 0.03k) \sin(x(t_k)) \\ 0.25/k \cos(x(t_k)) \end{bmatrix},$$

$i = 1, 2, 3, j = 1, 2, k = 1, 2, 3, 4, x_0 = [1 \ 1]^*, t_0 = 0, t_1 = 1.5, t_2 = 2, t_3 = 3, t_4 = 4.5, t_5 = T = 7$. We want to study the FTS with $c_1 = 8, c_2 = 10$ and

$$\Lambda = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

For this, we choose

$$P_i = \begin{bmatrix} 0.3682 & -0.00944 \\ -0.00944 & 0.101 \end{bmatrix}, P_j = \begin{bmatrix} 0.1042 & -0.0186 \\ -0.0186 & 0.2841 \end{bmatrix}, i = 1, 2, 3, j = 1, 2,$$

then we can find $\rho_1 = \rho_2 = 2.841$ and $\rho_3 = \rho_4 = 3.60$. Also, we can see that $\beta_1 = 0.27, \beta_2 = 0.24, \beta_3 = 0.21, \beta_4 = 0.18$. It is clear that conditions (4.8), (4.9) and (4.10) are satisfied. Now, we need to check condition (4.7). For this, we consider the following cases

Case 1: When $\mathbb{T} = \mathbb{R}$, then $\mu = 0$, we have

$$A_i^* P_i + P_i A_i = \begin{bmatrix} -0.2398 & 0.1142 \\ 0.1142 & -0.2222 \end{bmatrix}, A_j^* P_j + P_j A_j = \begin{bmatrix} -0.2500 & 0.1339 \\ 0.1339 & -0.26450 \end{bmatrix}, i = 1, 2, 3, j = 1, 2.$$

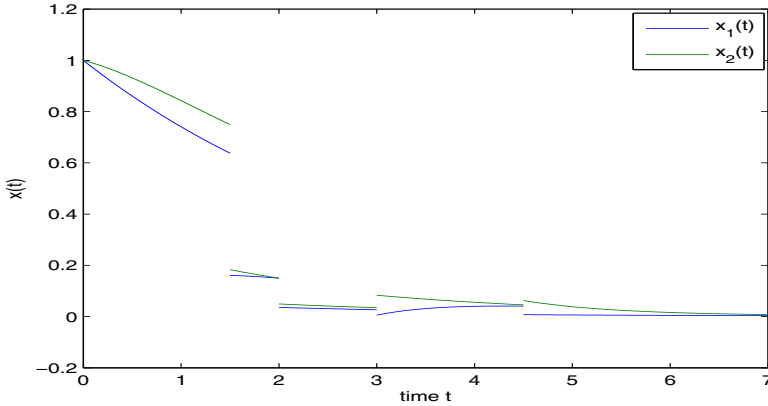


Figure 1: State trajectory of the system when $\mathbb{T} = \mathbb{R}$.

Case 2: When $\mathbb{T} = \frac{1}{2}\mathbb{Z}$, then $\mu_{\max} = \frac{1}{2}$, we have

$$A_i^* P_i + P_i A_i + \frac{1}{2} A_i^* P_i A_i = \begin{bmatrix} -0.1699 & 0.0571 \\ 0.0571 & -0.1611 \end{bmatrix}, A_j^* P_j + P_j A_j + \frac{1}{2} A_j^* P_j A_j = \begin{bmatrix} -0.17500 & 0.0669 \\ 0.0669 & -0.1822 \end{bmatrix},$$

$i = 1, 2, 3, j = 1, 2$.

Case 3: When $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, then $\mu_{\max} = \frac{1}{4}$, we have

$$A_i^* P_i + P_i A_i + \frac{1}{4} A_i^* P_i A_i = \begin{bmatrix} -0.2048 & 0.0857 \\ 0.0857 & -0.1917 \end{bmatrix}, A_j^* P_j + P_j A_j + \frac{1}{4} A_j^* P_j A_j = \begin{bmatrix} -0.2125 & 0.10044 \\ 0.10044 & -0.2233 \end{bmatrix},$$

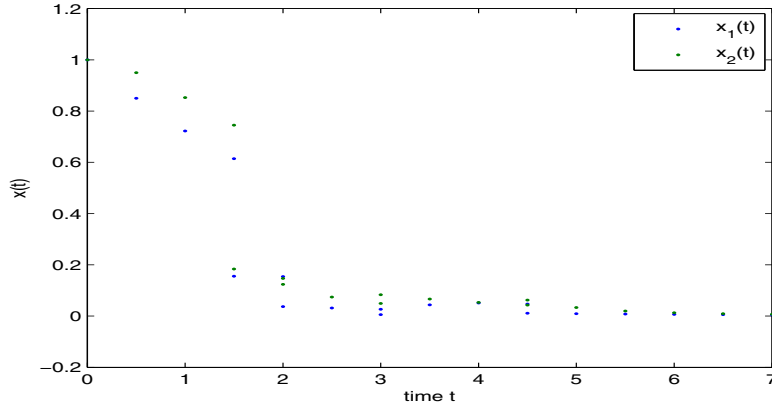


Figure 2: State trajectory of the system when $\mathbb{T} = \frac{1}{2}\mathbb{Z}$.

$i = 1, 2, 3, j = 1, 2$.

Clearly, for the above three cases, condition (4.7) of Theorem 4.1 holds. Hence, system (3.2) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda)$. Also, for all the three cases, the state trajectory is shown in Figure 1, Figure 2 and Figure 3, respectively. We can see that the state trajectory is finite-time stable. Moreover, for all the three cases the switching signal is shown in Figure 4.

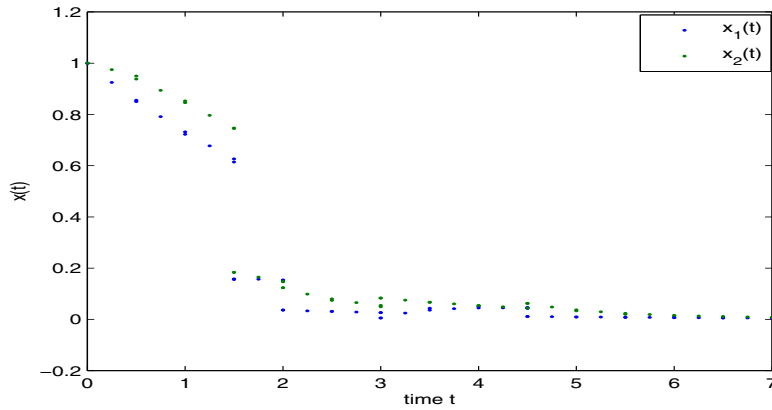


Figure 3: State trajectory of the system when $\mathbb{T} = \frac{1}{4}\mathbb{Z}$.

Remark 4.4. In Theorem 4.1, we constructed the switched Lyapunov functions such that their Δ -derivative should be negative definite along system (3.2). This may be a restrictive condition. Therefore, in the next theorem, we built up a new set of less restrictive conditions for FTS.

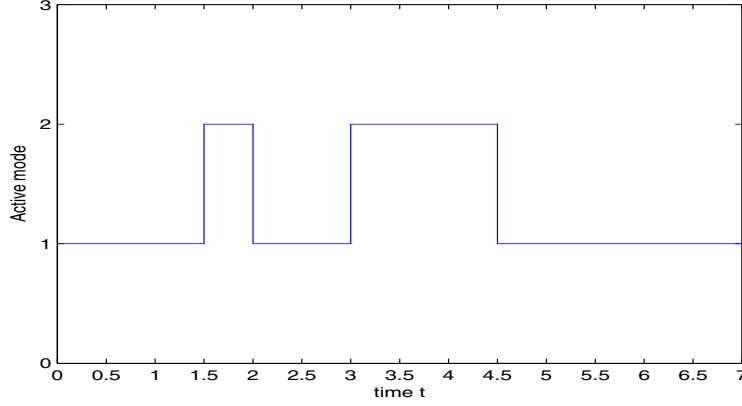


Figure 4: Switching signal.

Theorem 4.5. *Suppose that Assumption 1 holds and the following conditions*

$$A_k^* P_k(t) + P_k(t) A_k + \mu_{\max} A_k^* P_k(t) A_k + (I + \mu_{\max} A_k^*) P_k^\Delta(t) (I + \mu_{\max} A_k) - \alpha P_k(t) < 0, \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad (4.11)$$

$$\prod_{j=1}^{k-1} \rho_j \beta_j e_\alpha(T, 0) < \frac{c_2}{c_1}, \quad k = 1, 2, \dots, m, \quad (4.12)$$

$$\Lambda(t) < P_k(t), \quad \forall t \in [0, T]_{\mathbb{T}}, \quad (4.13)$$

$$P_k(0) < R. \quad (4.14)$$

are satisfied, where $\alpha > 0$ is a constant. Then, the impulsive switched system (3.2) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda(\cdot))$.

Proof. Let us consider the candidate switched Lyapunov functions $V_k(t, x) = x^* P_k(t) x$, $k = 1, 2, \dots, m$. Then, for any $t \in (t_{k-1}, t_k]_{\mathbb{T}}$, we have

$$\begin{aligned} V_k^\Delta(t) &= (x^*)^\Delta P_k(t) x + (x^*)^\sigma P_k^\Delta(t) x + (x^*)^\sigma P_k^\sigma(t) x^\Delta \\ &= x^* A_k^* P_k(t) x + (x^* + \mu(t) x^* A_k^*) P_k^\Delta(t) x + (x^* + \mu(t) x^* A_k^*) (P_k(t) + \mu(t) P_k^\Delta(t)) A_k x \\ &\leq x^* (A_k^* P_k(t) + P_k(t) A_k + \mu_{\max} A_k^* P_k(t) A_k + (I + \mu_{\max} A_k^*) P_k^\Delta(t) (I + \mu_{\max} A_k)) x \\ &< \alpha x^* P_k(t) x, \end{aligned}$$

which implies that

$$V_k(t) < V_k(t_{k-1}^+) e_\alpha(t, t_{k-1}), \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad (4.15)$$

which immediately gives

$$V_k(t_k) < V_k(t_{k-1}^+) e_\alpha(t_k, t_{k-1}).$$

On the other side, we have

$$\begin{aligned}
V_{k+1}(t_k^+) &= x(t_k^+)^* P_{k+1}(t_k) x(t_k^+) \\
&= J_k(t_k, x(t_k))^* P_{k+1}(t_k) J_k(t_k, x(t_k)) \\
&\leq \lambda_{\max}(P_{k+1}(t_k)) J_k(t_k, x(t_k))^* J_k(t_k, x(t_k)) \\
&\leq \beta_k \lambda_{\max}(P_{k+1}(t_k)) x(t_k)^* x(t_k) \\
&\leq \beta_k \rho_k V_k(t_k).
\end{aligned} \tag{4.16}$$

Now, for $t \in (0, t_1]_{\mathbb{T}}$, we have

$$V_1(t) < V_1(0) e_\alpha(t, 0),$$

which gives

$$V_1(t_1) < V_1(0) e_\alpha(t_1, 0).$$

Also, from (4.16), we get

$$V_2(t_1^+) \leq \beta_1 \rho_1 V_1(t_1).$$

Similarly, for any $t \in (t_1, t_2]_{\mathbb{T}}$

$$\begin{aligned}
V_2(t) &< V_2(t_1^+) e_\alpha(t, t_1) \\
&< \beta_1 \rho_1 V_1(t_1) e_\alpha(t, t_1) \\
&< \beta_1 \rho_1 V_1(0) e_\alpha(t_2, 0).
\end{aligned}$$

In general, for any $t \in (t_{k-1}, t_k]_{\mathbb{T}}$, we have

$$\begin{aligned}
V_k(t) &< V_k(t_{k-1}^+) e_\alpha(t, t_{k-1}) \\
&< \beta_{k-1} \rho_{k-1} V_{k-1}(t_{k-1}) e_\alpha(t, t_{k-1}) \\
&< \beta_{k-2} \rho_{k-2} V_{k-2}(t_{k-2}^+) e_\alpha(t_{k-1}, t_{k-2}) e_\alpha(t, t_{k-1}) \\
&< \prod_{j=1}^{k-1} \beta_j \rho_j V_1(0) e_\alpha(t, 0).
\end{aligned}$$

Next, we have

$$V_1(0) = x(0)^* P_1(0) x(0) < x(0)^* R x(0) < c_1.$$

Given x_0 such that $x_0^* R x_0 \leq c_1$, we have, for all $t \in [t_0, T]_{\mathbb{T}}$,

$$\begin{aligned}
x^*(t) \Lambda(t) x(t) &\leq x^*(t) P_k(t) x(t) \\
&< \prod_{j=1}^{k-1} \beta_j \rho_j V_1(0) e_\alpha(t, 0) \\
&< c_1 \prod_{j=1}^{k-1} \beta_j \rho_j e_\alpha(T, 0) \\
&< c_2.
\end{aligned}$$

Hence, the result follows. □

Remark 4.6. One can see that in Theorem 4.5, the designed switched Lyapunov functions decrease along system (3.2) (this adds some conservatism). Such property for the switched Lyapunov functions is no longer required in Theorem 4.5. Furthermore, we have introduced a new variable α in Theorem 4.5. Thanks to this variable, it is easy to check inequalities (4.11) and (4.12) of Theorem 4.5 as compared to inequalities (4.7) and (4.8) of Theorem 4.1. At last, it should be mention that, Theorem 4.5 cannot be degenerated to Theorem 4.1.

Now, we present an example to illustrate Theorem 4.5.

Example 4.7. Consider system (3.2) with the following matrices

$$A_{2k-1} = \begin{bmatrix} -1.1 & 1 \\ 0 & -0.5 \end{bmatrix}, A_{2k} = \begin{bmatrix} -0.7 & 0 \\ 1 & -1 \end{bmatrix}, J_k(t_k, x(t_k)) = \begin{bmatrix} (0.37) \sin(x(t_k)) \\ 0.32 - 0.08k \cos(x(t_k)) \end{bmatrix}, k = 1, 2,$$

$x_0 = [1 \ 0]^*$, time sequences $t_0 = 0, t_1 = 0.8, t_2 = 2.5, t_3 = 4.3, t_4 = T = 5$. Then, we want to analyse the FTS for $c_1 = 5, c_2 = 11$ and

$$\Lambda = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this we choose

$$P_{2k-1} = \begin{bmatrix} 0.2294 & -0.01994 \\ -0.01994 & 0.5405 \end{bmatrix}, P_{2k} = \begin{bmatrix} 0.4725 & 0 \\ 0 & 0.2273 \end{bmatrix}, k = 1, 2,$$

then we can find that $\rho_1 = 2.0718, \rho_2 = 2.3837, \rho_3$. We can see that $\beta_i = 0.37, i = 1, 2, 3$. Clearly, we can see that conditions (4.12), (4.13) and (4.14) are fulfilled. To apply Theorem 4.5, we need to check condition (4.11) only. Now, for $\alpha = 0.1$, we consider the following cases of time scales

Case 1: When $\mathbb{T} = \mathbb{R}$, then $\mu = 0$, we have

$$A_{2k-1}^* P_{2k-1} + P_{2k-1} A_{2k-1} - \alpha P_{2k-1} = \begin{bmatrix} -0.5275 & 0.2632 \\ 0.2632 & -0.6344 \end{bmatrix}, A_{2k}^* P_{2k} + P_{2k} A_{2k} - \alpha P_{2k} = \begin{bmatrix} -0.7088 & 0.2272 \\ 0.2272 & -0.4772 \end{bmatrix},$$

$k = 1, 2$.

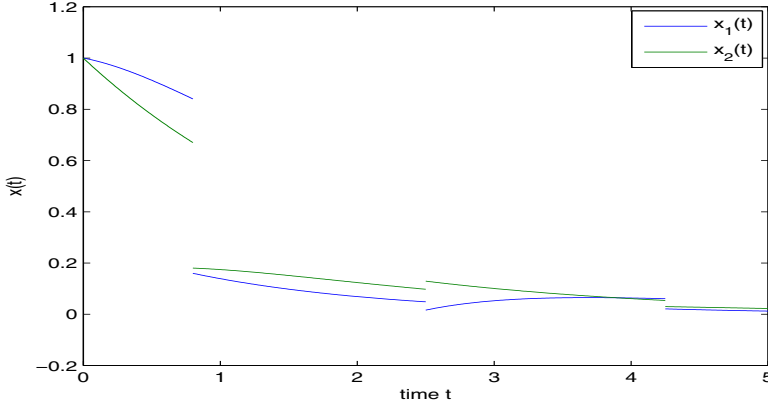


Figure 5: State trajectory of the system when $\mathbb{T} = \mathbb{R}$.

Case 2: When $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, then $\mu_{\max} = \frac{1}{4}$, we have

$$A_{2k-1}^* P_{2k-1} + P_{2k-1} A_{2k-1} - \alpha P_{2k-1} = \begin{bmatrix} -0.4581 & 0.1974 \\ 0.1974 & -0.5383 \end{bmatrix}, A_{2k}^* P_{2k} + P_{2k} A_{2k} - \alpha P_{2k} = \begin{bmatrix} -0.5941 & 0.1704 \\ 0.1704 & -0.4204 \end{bmatrix},$$

$k = 1, 2$.

Case 3: When $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, then $I = [0, 1] \cup [2, 3] \cup [4, 5]$ and $\mu_{\max} = 1$. Now, we can calculate

$$A_{2k-1}^* P_{2k-1} + P_{2k-1} A_{2k-1} - \alpha P_{2k-1} = \begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix}, \quad A_{2k}^* P_{2k} + P_{2k} A_{2k} - \alpha P_{2k} = \begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix},$$

$k = 1, 2$.

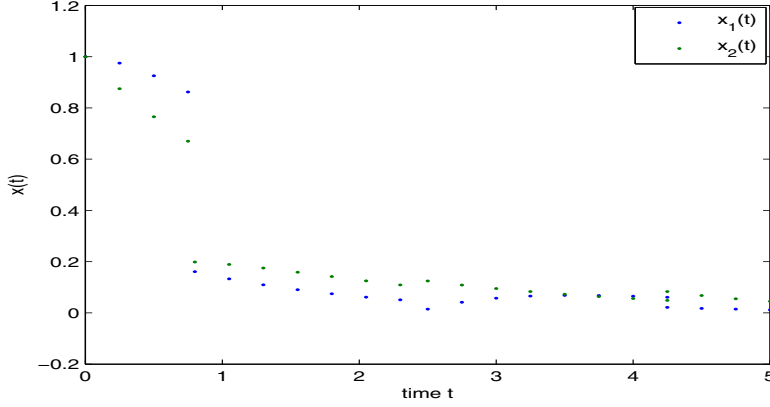


Figure 6: State trajectory of the system when $\mathbb{T} = \frac{1}{4}\mathbb{Z}$.

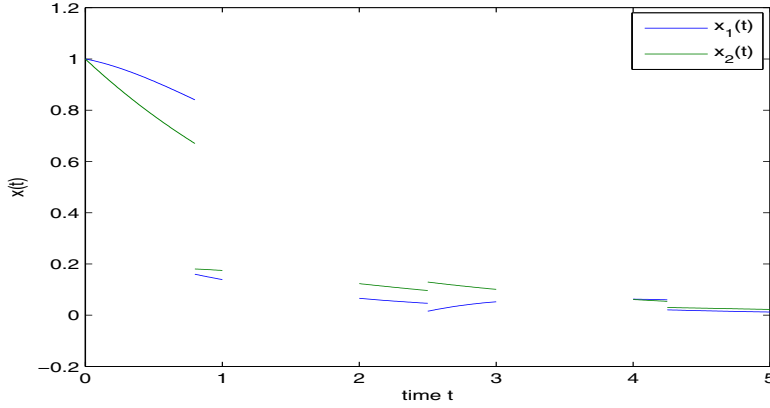


Figure 7: State trajectory of the system when $\mathbb{T} = \mathbb{P}_{1,1}$.

Clearly, for the above three cases, condition (4.11) of Theorem 4.5 is satisfied. Hence, system (3.2) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda)$. Also, when the time domain is smooth, i.e., $\mathbb{T} = \mathbb{R}$, the state trajectory of the considered system is shown in Figure 5. The state trajectory of the considered system is shown in Figure 6 when the time domain is discrete, i.e., $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, while the state trajectory for the case 3 is shown in Figure 7. In all these figures, we can see that the state trajectory is finite-time stable. Moreover, for all the three cases the switching signal is shown in Figure 8.

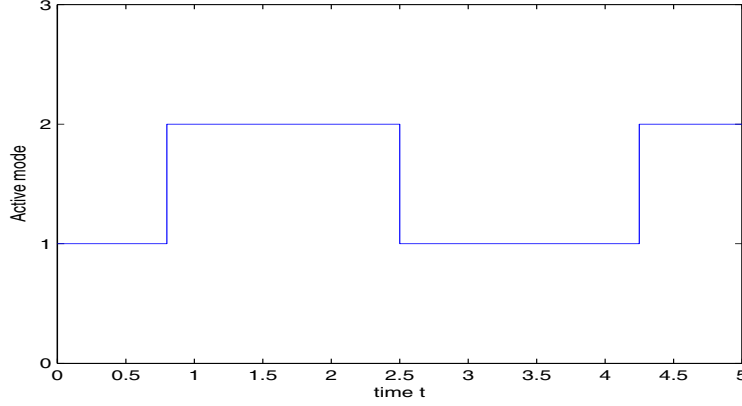


Figure 8: Switching signal.

Remark 4.8. In the literature, some authors established the FTS results with $\Lambda(t) = R$. In this case, equation (3.4) becomes

$$x_0^* R x_0 \leq c_1 \implies x^*(t) R x(t) < c_2, \quad \forall t \in I. \quad (4.17)$$

In the next theorem, we establish the FTS results using Remark 4.8. In this theorem, we will use multiple Lyapunov-like functions for which their Δ -derivative does not need to be negative definite.

For $Q_k > 0, k = 1, 2, \dots, m$, we set

$$\bar{Q}_k = R^{-\frac{1}{2}} Q_k R^{-\frac{1}{2}}, \quad k = 1, 2, \dots, m,$$

$$\lambda_1 = \min_{k=1,2,\dots,m} \{\lambda_{\min}(Q_k)\}, \quad \lambda_2 = \max_{k=1,2,\dots,m} \{\lambda_{\max}(Q_k)\} \text{ and } \delta_k = \frac{\lambda_{\max}(\bar{Q}_{k+1}^{-1})}{\lambda_{\min}(\bar{Q}_k^{-1})}, \quad k = 1, 2, \dots, m.$$

Theorem 4.9. Let Assumption 1 hold and the following inequalities are satisfied

$$\begin{aligned} \bar{Q}_k A_k^* + A_k \bar{Q}_k + \mu_{\max} \bar{Q}_k A_k^* (\bar{Q}_k)^{-1} A_k \bar{Q}_k - \alpha \bar{Q}_k < 0, \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}}, \\ \frac{\lambda_2}{\lambda_1} \prod_{j=1}^{k-1} \beta_j \delta_j e_{\alpha}(T, 0) < \frac{c_2}{c_1}, \quad k = 1, 2, \dots, m, \end{aligned}$$

where $\alpha \geq 0$. Then, the impulsive system (3.2) is FTS w.r.t. (c_1, c_2, T, R) .

Proof. Let us choose Lyapunov-like functions $V_k(t) = x^*(\bar{Q}_k)^{-1}x$, $k = 1, 2, \dots, m$. The rest of the proof can be done in a similar way to Theorem 4.5, hence, we omit the proof. \square

Remark 4.10. It is clear that, the conditions obtained from Theorem 4.9 are less in number as compared to the conditions of Theorem 4.1 and 4.5.

5 Finite time stabilization results

In this section, we study the finite-time stabilization problem for the impulsive system (3.3) with the state feedback controllers

$$u(t) = K_k x(t), \quad t \in (t_{k-1}, t_k]_{\mathbb{T}},$$

where $K_k \in C_{rd}\mathcal{R}(I, \mathbb{R}^{m \times n})$. Now, substituting these controllers into system (3.2), we get the following closed-loop system

$$\begin{aligned} x^\Delta(t) &= \tilde{A}_k x(t), \quad t \in (t_{k-1}, t_k]_{\mathbb{T}}, k = 1, 2, \dots, \\ x(t_k^+) &= J_k(t_k, x(t_k)), \quad k = 1, 2, \dots, \\ x(0) &= x_0, \end{aligned} \tag{5.18}$$

where $\tilde{A}_k = (A_k + B_k K_k)$. Hereafter, some sufficient conditions for finite-time stabilization of the impulsive closed-loop system (5.18) are introduced.

Theorem 5.1. *Suppose that Assumption 1 holds and the following inequalities*

$$\begin{aligned} \tilde{A}_k^* P_k(t) + P_k(t) \tilde{A}_k + \mu_{\max} \tilde{A}_k^* P_k(t) \tilde{A}_k + (I + \mu_{\max} \tilde{A}_k^*) P_k^\Delta(t) (I + \mu_{\max} \tilde{A}_k) < 0, \\ \forall t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad k = 1, 2, \dots, m, \end{aligned} \tag{5.19}$$

$$\rho_k \beta_k < 1, \quad k = 1, 2, \dots, m - 1, \tag{5.20}$$

$$\Lambda(t) < P_k(t), \quad \forall t \in [0, T]_{\mathbb{T}}, \tag{5.21}$$

$$P_k(0) < R, \tag{5.22}$$

are satisfied. Then, the system (5.18) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda(\cdot))$.

Proof. This result can be easily obtained by applying Theorem 4.1 to system (5.18). Hence, we omit the proof. \square

Theorem 5.2. *Suppose that Assumption 1 hold and the following conditions*

$$\tilde{A}_k^* P_k(t) + P_k(t) \tilde{A}_k + \mu_{\max} \tilde{A}_k^* P_k(t) \tilde{A}_k + (I + \mu_{\max} \tilde{A}_k^*) P_k^\Delta(t) (I + \mu_{\max} \tilde{A}_k) - \alpha P_k(t) < 0, \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}},$$

$$\prod_{j=1}^{k-1} \rho_j \beta_j e_\alpha(T, 0) < \frac{c_2}{c_1}, \quad k = 1, 2, \dots, m,$$

$$\Lambda(t) < P_k(t), \quad \forall t \in [0, T]_{\mathbb{T}},$$

$$P_k(0) < R.$$

are satisfied, where $\alpha > 0$ is a constant. Then, the impulsive switched system (5.18) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda(\cdot))$.

Proof. This result can be easily obtained by applying Theorem 4.5 to system (5.18). \square

Next theorem is due to Remark 4.8.

Theorem 5.3. *Let Assumption 1 hold and the following inequalities are satisfied*

$$\bar{Q}_k A_k^* + A_k \bar{Q}_k + M_k^* B_k^* + B_k M_k + \mu_{\max} (\bar{Q}_k A_k^* + M_k^* B_k^*) (\bar{Q}_k)^{-1} (A_k \bar{Q}_k + B_k M_k) - \alpha \bar{Q}_k < 0, \quad \forall t \in (t_{k-1}, t_k]_{\mathbb{T}},$$

$$\frac{\lambda_2}{\lambda_1} \prod_{j=1}^{k-1} \beta_j \delta_j e_\alpha(T, 0) < \frac{c_2}{c_1}, \quad k = 1, 2, \dots, m,$$

where $\alpha \geq 0$. Then, under the feedback controllers $u(t) = M_k (\bar{Q}_k)^{-1} x(t)$, the switched impulsive system (3.2) is FTS w.r.t. (c_1, c_2, T, R) .

Proof. This result can be easily obtained by applying Theorem 4.9 to system (5.18) while changing variables as $M_k = K_k \bar{Q}_k$. \square

Next, we present an example to illustrate the obtained analytical results.

Example 5.4. We consider the following matrices

$$A_1 = A_3 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.4 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 0.3 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B_1 = B_3 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$J_k(t_k, x(t_k)) = \begin{bmatrix} 0.3 \sin(x(t_k)) \\ \frac{0.25}{k} \cos(x(t_k)) \end{bmatrix}, \quad k = 1, 2, 3, 4,$$

$x_0 = [1 \ 2]^*$, time sequence $t_1 = 0.5, t_2 = 2, t_3 = 3.5, t_4 = T = 5$. We want to study the FTS results with $I = [0, 5]_{\mathbb{T}}$, $c_1 = 10, c_2 = 13$ and

$$\Lambda = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

For this we set

$$P_1 = P_3 = \begin{bmatrix} 1.082 & 0.3127 \\ 0.3127 & 0.4995 \end{bmatrix}, \quad P_2 = P_4 = \begin{bmatrix} 0.433 & -0.1576 \\ 0.1576 & 0.7979 \end{bmatrix},$$

$$K_1 = K_3 = [-0.4722 \ -1.3139], \quad K_2 = K_4 = [-1.0500 \ -1.0500],$$

then we can find that $\rho_1 = \rho_3 = 2.3572$ and $\rho_2 = 3.2535$. We can see that $\beta_k = 0.30, k = 1, 2, 3$. Clearly, conditions (5.20), (5.21) and (5.22) are satisfied. Now, it remains to check condition (5.19). For this, we choose the different cases of time scales

Case 1: When $\mathbb{T} = \mathbb{R}$, then $\mu = 0$, we have

$$\tilde{A}_1^* P_1 + P_1 \tilde{A}_1 = \tilde{A}_3^* P_3 + P_3 \tilde{A}_3 = \begin{bmatrix} -0.3734 & -0.1618 \\ -0.1618 & -0.6984 \end{bmatrix},$$

$$\tilde{A}_2^* P_2 + P_2 \tilde{A}_2 = \tilde{A}_4^* P_4 + P_4 \tilde{A}_4 = \begin{bmatrix} -0.6337 & -0.2896 \\ -0.2896 & -0.5466 \end{bmatrix}.$$

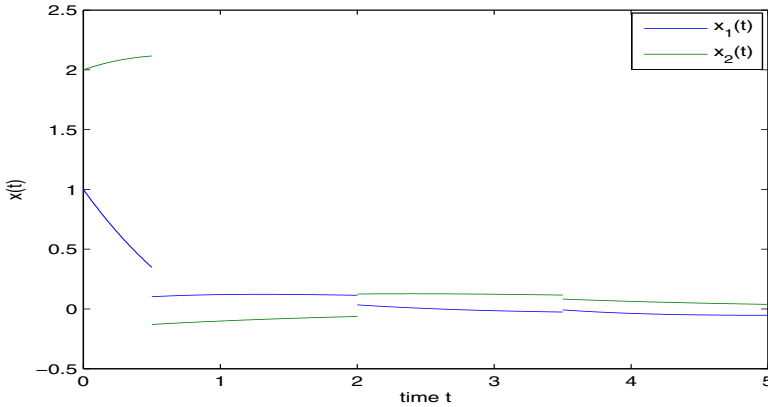


Figure 9: State trajectory of the system when $\mathbb{T} = \mathbb{R}$.

Case 2: When $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, then $\mu_{\max} = \frac{1}{4}$, we have

$$\begin{aligned}\tilde{A}_1^* P_1 + P_1 \tilde{A}_1 + \tilde{A}_1^* P_1 \tilde{A}_1 &= \tilde{A}_3^* P_3 + P_3 \tilde{A}_3 + \tilde{A}_3^* P_3 \tilde{A}_3 = \begin{bmatrix} -0.3425 & -0.1213 \\ -0.1213 & -0.6113 \end{bmatrix}, \\ \tilde{A}_2^* P_2 + P_2 \tilde{A}_2 + \tilde{A}_2^* P_2 \tilde{A}_2 &= \tilde{A}_4^* P_4 + P_4 \tilde{A}_4 + \tilde{A}_4^* P_4 \tilde{A}_4 = \begin{bmatrix} -0.5753 & -0.2172 \\ -0.2172 & -0.4125 \end{bmatrix}.\end{aligned}$$

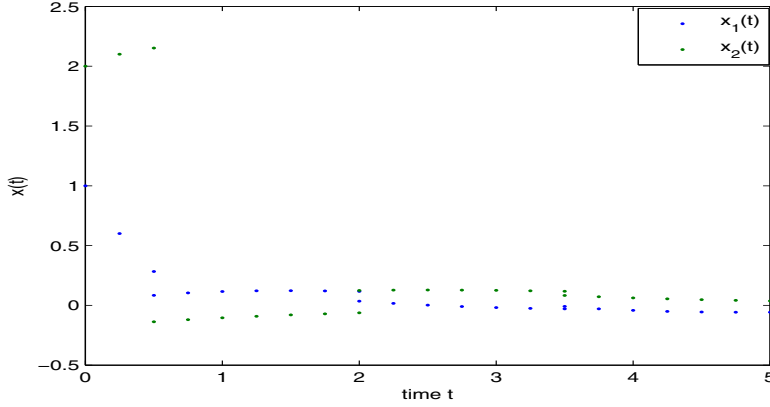


Figure 10: State trajectory of the system when $\mathbb{T} = \frac{1}{4}\mathbb{Z}$.

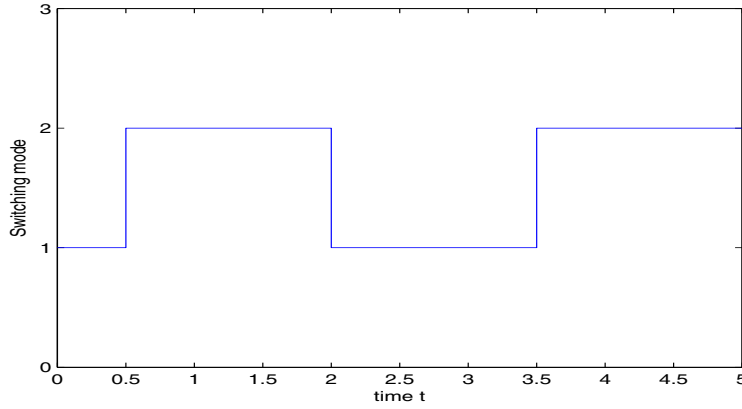


Figure 11: Switching signal.

Clearly, for the above two cases, condition (5.19) of Theorem 5.1 holds. Hence, system (5.18) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda)$. Also, for the above two cases, the state trajectory is shown in Figure 9 and Figure 10, respectively. In these figures, we can see that the state trajectory is finite-time stable. Moreover, for the above two cases the switching signal is shown in Figure 11.

Remark 5.5. *In the existing literature on FTS, many authors established the FTS results by using different techniques for the continuous and discrete-time domain, but they are studied separately. Particularly, in [5],*

authors examined the FTS analysis for continuous-time, time-varying linear systems, while the authors in [9], investigated the problem of FTS analysis of linear discrete-time systems with time-varying delay. In [10], the author studied the FTS results for discrete-time systems with time-varying delay and the effect of nonlinear perturbations. Moreover, in [32], the authors considered a continuous-time linear time-varying system with finite jumps and established some necessary and sufficient conditions for FTS. Furthermore, in the above mentioned works, authors used a common Lyapunov quadratic function which should be a negative function, whereas we relax this condition by introducing Theorem 4.5 and Theorem 4.9. Moreover, this is the first attempt to deal with FTS on an arbitrary time scale (other methods can only treat real line or discrete-time set) and these results can be applied to different time domains.

6 Application to population growth model

In this section, we consider a population growth model with impulses of two species x and y on time scale $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$ given as follows

$$\begin{aligned} N^\Delta(t) &= r_k N(t) + b_k U(t), \quad t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad k = 1, 2, 3, \\ N(t_k^+) &= (r_{k+1} - r_k) N(t_k), \quad k = 1, 2, \\ N(0) &= N_0, \end{aligned} \tag{6.23}$$

where $N(t) = [x(t) \ y(t)]^*$ is the number of population at the time t , N_0 denotes the initial condition at time $t = 0$, r_k is the rate of population growth between two consecutive impulsive points, t_k is the point of impulses and $U(t)$ is the control input, time sequences $t_0 = 0, t_1 = 0.5, t_2 = 2.5, t_3 = T = 5$,

$$\begin{aligned} N_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r_1 = \begin{bmatrix} 0.2 & 0 \\ 1 & 0.1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0.4 & 0.1 \\ 1.1 & 0.3 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0.5 & 0.25 \\ 1.25 & 0.7 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \\ J_1(t_1, N(t_1)) &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} x(t_1) \\ y(t_1) \end{bmatrix}, \quad J_2(t_2, N(t_2)) = \begin{bmatrix} 0.1 & 0.15 \\ 0.15 & 0.4 \end{bmatrix} \begin{bmatrix} x(t_2) \\ y(t_2) \end{bmatrix}. \end{aligned}$$

Then, we want to analyse the FTS for $c_1 = 3, c_2 = 5$ and

$$\Lambda = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this, we choose

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.1961 & -0.07182 \\ -0.07182 & 1.617 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.8087 & 0.03043 \\ 0.03043 & 0.573 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.463 & -0.2409 \\ -0.2409 & 0.1762 \end{bmatrix}, \\ K_1 &= [-1.0000 \ -0.9000], \quad K_2 = [-0.8931 \ -0.3369], \quad K_3 = [-0.6492 \ -1.7554], \end{aligned}$$

then we can find that $\rho_1 = 4.2220$ and $\rho_2 = 1.0542$. We can see that $\beta_1 = 0.0900$ and $\beta_2 = 0.2136$. Clearly, conditions (5.20), (5.21) and (5.22) are satisfied. Now, its remains to check condition (5.19). Since $\mu_{\max} = 1$, then we have

$$\begin{aligned} \tilde{A}_1^* P_1 + P_1 \tilde{A}_1 + \tilde{A}_1^* P_1 \tilde{A}_1 &= \begin{bmatrix} -0.1 & 0 \\ 0 & -1.5 \end{bmatrix}, \\ \tilde{A}_2^* P_2 + P_2 \tilde{A}_2 + \tilde{A}_2^* P_2 \tilde{A}_2 &= \begin{bmatrix} -0.57 & 0 \\ 0 & -0.01 \end{bmatrix}, \\ \tilde{A}_3^* P_3 + P_3 \tilde{A}_3 + \tilde{A}_3^* P_3 \tilde{A}_3 &= \begin{bmatrix} -0.1 & 0 \\ 0 & -0.01 \end{bmatrix}. \end{aligned}$$

Clearly, condition (5.19) of Theorem 5.1 holds. Hence, system (6.23) is FTS w.r.t. $(c_1, c_2, T, R, \Lambda)$. Also, the state trajectory is shown in Figure 12. In the figure, we can see that the state trajectory is finite-time stable. Moreover, the switching signal is shown in Figure 13.

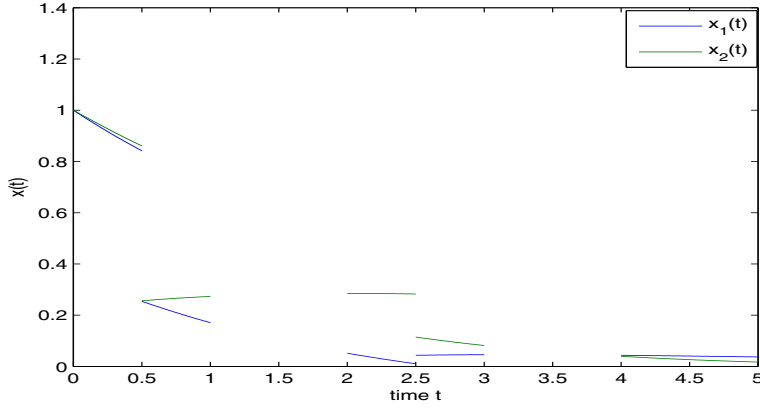


Figure 12: State trajectory of the model (6.23) on $\mathbb{T} = \mathbb{P}_{1,1}$.

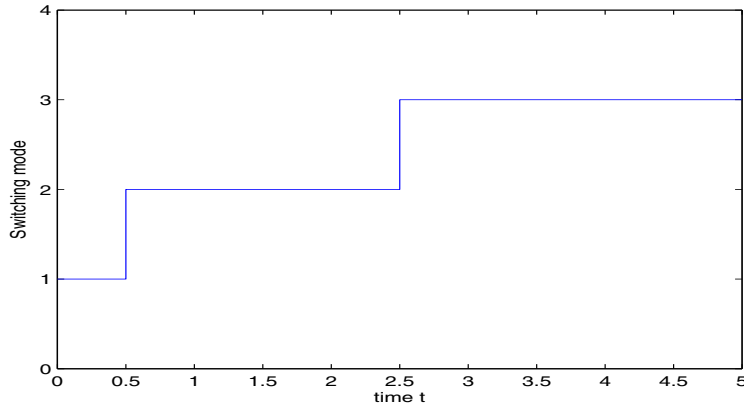


Figure 13: Switching signal.

Conclusion

In this manuscript, we have investigated the FTS and finite time stabilization problem for a class of switched impulsive systems on time scales. We proved these results using common switched Lyapunov functions in which it is required that the delta derivative of this function should be negative definite. Further, we used the Lyapunov-like functions in which there is no requirement of negative definiteness on the delta derivative of the Lyapunov functions. Also, we have presented some examples to illustrate the applications of these obtained results. In future, one can extend the results of this manuscript to nonlinear switched impulsive system on time scales by introducing the dwell time approach.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.