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# Anti-windup Based Dynamic Output Feedback Controller Design with Performance Consideration for Constrained Takagi-Sugeno Systems

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**Abstract:** This paper proposes an LMI-based method to guarantee the closed-loop regional stability and performance for Takagi-Sugeno systems that are subject to input saturation, state constraints and also amplitude-bounded disturbance. Based on Lyapunov stability tool, the proposed method provides conditions to simultaneously design the dynamic output feedback controller and its anti-windup compensator. By solving a convex optimization problem, these conditions are derived such that a tradeoff between the upper bound on the nominal  $\mathcal{L}_2$  gain for exogenous disturbance and the minimal size of the domain of attraction can be found. This method is simple and systematic, allowing dealing with a very large class of constrained nonlinear systems. The effectiveness of the proposed method is illustrated with numerical examples.

**Keywords:** Nonlinear systems, Takagi-Sugeno fuzzy model, dynamic output feedback controller, anti-windup compensator,  $\mathcal{L}_2$  performance, domain of attraction.

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## 1. Introduction

Over the past two decades, Takagi-Sugeno (T-S) fuzzy models (Takagi & Sugeno, 1985) have been intensively studied in the control community (Tanaka & Wang, 2001). It is motivated by the fact that these models have general approximation capability for complex dynamical system (Cao & Rees, 1997). Moreover, under weak conditions, a nonlinear model can be, globally or more often regionally, rewritten on a fuzzy T-S form. Stability analysis or controller synthesis is then facilitated due to their polytopic structure (Tanaka & Wang, 2001). As a consequence, this approach has become now a very attractive research topic in control theory (Feng, 2006). Stability analysis of a given T-S system is investigated in most cases via the direct Lyapunov method through the use of a quadratic Lyapunov function; the derived stability conditions being expressed as linear matrix inequalities (LMIs) (Boyd et al., 1994) for which efficient solvers are available. For the controller design, the choice of a parallel distributed compensation (PDC) control law is usually done for T-S models (Tanaka & Wang, 2001). An abundant literature is available on this nonlinear state feedback control law; see e.g. (Feng, 2006) for a quick overview.

For technological or economic reasons, the state variables are not all measured in most of real-world applications. In order to deal with this practical problem, output feedback control must be used. In general, observer-based control

scheme is proposed for unconstrained T-S systems (Liu & Zhang, 2003; Guerra et al., 2006). However, the design problem becomes much more complicated when state and/or input constraints have to be explicitly considered. As highlighted in (Ding, 2009), this control issue is not well addressed in the literature.

Due to physical/technical limitations and/or safety constraints, actuator saturation is unavoidable in almost all real applications. This phenomenon can severely degrade the closed-loop system performance and, in some cases, may lead to system instability. Motivated by this practical control aspect, a great deal of effort has been focused on saturated systems (Tarbouriech et al., 2011). In the literature, several methods now exist to handle saturation effects, but the most popular and effective one remains the anti-windup (AW) approach (see for instance (Kothare et al., 1994)). LMI-based synthesis of anti-windup compensators has been proposed recently to synthesize either static (Gomes da Silva & Tarbouriech, 2005; Mulder et al., 2009) or dynamic anti-windup compensators (Cao et al., 2002; Grimm et al., 2003; Hu et al., 2008), an overview of these results can be found in the survey (Tarbouriech & Turner, 2009). Most of these works deal with a two-step method in which the controller and the AW strategy are designed separately. This method often proves to be satisfactory but it has however some drawbacks. First, only sub-optimal solution can be achieved since the controller and its AW compensator are designed separately (Sawada et al., 2009). Second, the effect of the nominal controller on the closed-loop performance under saturation is completely ignored (Mulder et al., 2009). To overcome these drawbacks, an alternative solution called one-step method, which designs simultaneously the controller and the AW compensator, has been proposed. Among few works existing in the literature, we can notably cite (Mulder et al., 2009) in which an LMI solution ensuring global stability and performance is proposed for systems that are stable in open loop; (Sawada et al., 2009) proposes a regional approach based on a change of variable; whereas (Dai et al., 2009) relies on the parameter elimination approach of (Skelton et al., 1998). Both latter approaches are based on some preliminary results proposed in (Hu et al., 2006).

Up to now, a large amount of works on AW-based design are available for linear systems (Tarbouriech et al., 2011), but very few works deal with nonlinear cases, especially when exogenous disturbance signals are actively present (Gomes da Silva Jr. et al., 2013). In the T-S control framework, there are a couple of works devoted to the analysis or control design of saturated systems, for instance (Cao & Lin, 2003; Tseng & Chen, 2006; Du & Zhang, 2009; Bezzaoucha et al., 2013; Ariño et al., 2010). However, very few papers are dedicated to AW synthesis for T-S systems: in (Ting & Chang, 2011), the authors addressed a two-step approach to deal with a continuous time-delay T-S systems; in (Zhang et al., 2009), an interesting one-step approach based on piecewise fuzzy AW dynamic output feedback controller (DOFC) for discrete-time T-S systems has been proposed, note that these results seems to be valid only for systems that are stable in open-loop since no admissible sets of initial conditions are defined; and, at last, (Song et al., 2011) which extends the approach proposed in (Gomes da Silva & Tarbouriech, 2005) to the case of T-S systems. It is noteworthy that an important point is neglected in all these results: besides control input saturation, the T-S model is only valid on a given subset of the state space. This is of course true for any model of real-world systems, but is fundamental in the writing of a T-S model using the nonlinear sector decomposition approach (Tanaka & Wang, 2001). This validity domain can be represented by some constraints on the state variables (see Example 2 for illustration of this fact). It is particularly important to consider explicitly these state constraints in the control design to ensure a good behavior of the closed-loop system in response of disturbances. This fact has been very recently emphasized in (Nguyen, 2013; Nguyen et al., 2014;

Klug et al., 2014). The two first cited references concern stabilization of nonlinear switching systems under saturation. The latter one deals with the design of a dynamic output feedback control for nonlinear systems represented in Takagi-Sugeno form. Apart from the fact that we consider the effect of bounded disturbances on the design, a fundamental difference with the work in (Klug et al., 2014) is that they considered discrete-time systems, whereas this paper concerns continuous-time systems. This allows them to use elegantly non-quadratic Lyapunov functions for the control law design. Multiple or non-quadratic Lyapunov functions are also developed in the continuous-time case allowing to obtain less conservative conditions for stability analysis (Tanaka et al., 2003; Mozelli et al., 2009). Concerning synthesis of control laws, the situation becomes much more complex and remains open due to the presence of the time-derivative of the membership functions in the stability conditions. Results so far developed lead to very complex design conditions or to controllers requiring the inversion in real time of a parameter-dependent matrix (see for instances (Guelton et al., 2009; Guerra et al., 2012; Bouarar et al., 2013)). The goal of this paper is to obtain a controller that may be easily implemented in practice. For this reason, the obtained results rely on the quadratic stabilization approach. However, as will be shown in Section 5 through numerical example, in the context of input-saturated T-S systems, the proposed results may be competitive in some sense with respect to those presented in (Guerra et al., 2012) which actually provides the less conservative results among all non-quadratic approaches found in the open literature.

In this paper, we address a novel one-step method to design simultaneously a DOFC and an AW compensator for a disturbed T-S system subject to control input and system state constraints. The proposed approach has some special features deserving particular attention: quadratic boundedness of the trajectories (see (Ding, 2009) and references therein) is ensured for any admissible initial condition and disturbance signal, as well as a maximal  $\mathcal{L}_2$  gain for the unsaturated system. Note that this  $\mathcal{L}_2$  performance may be ensured regionally for the saturated case as in (Dai et al., 2009), however this decoupling allows reducing the conservatism of the results. Moreover, it will be shown that the control design can be formulated as a multi-objective LMI optimization problem. In such a way, the obtained controller can solve the tradeoff between some predefined closed-loop requirements.

The paper is organized as follows. Section 2 describes the design problem and recalls some preliminaries results. The main result is stated in Section 3. In Section 4, a constructive control design is presented as a multi-objective LMI optimization problem. The results of the paper are effectively illustrated through an example in Section 5. Finally, Section 6 gives some concluding remarks.

The notations and terminology used in this paper are standard. For an integer number  $r$ ,  $\Omega_r$  denotes the set  $\{1, 2, \dots, r\}$ .  $\mathbb{R}^+ = [0, \infty)$  is the set of non-negative real numbers.  $\mathcal{L}_{2e}(\mathbb{R}^n)$  denotes the extended  $\mathcal{L}_2$ -space composed of measurable functions  $f: [t_0, \infty) \mapsto \mathbb{R}^n$  such that  $\int_{t_0}^T \|f(t)\|^2 dt < \infty, \forall T \geq t_0$ ; and  $\|f\|_{2,T} = \left( \int_{t_0}^T \|f(t)\|^2 dt \right)^{1/2}$ .  $x_{(i)}$  is the  $i^{\text{th}}$  element of a vector  $x$ .  $x \succ y$ , with  $x, y \in \mathbb{R}^n$  means that  $x_{(i)} - y_{(i)} > 0$  for all  $i \in \Omega_n$ .  $X_{(i)}$  denotes the  $i^{\text{th}}$  row of a matrix  $X$ , and  $\text{sym}(X) = X + X^T$  (for square matrices).  $X > 0$  means that  $X$  is a symmetric, positive-definite matrix.  $I$  denotes the identity matrix of appropriate dimension, and  $(*)$  stands for matrix blocks that can be deduced by symmetry in a partitioned matrix. For  $P \in \mathbb{R}^{n \times n}$  such that  $P > 0$ ,  $\mathcal{E}(P)$  denotes the ellipsoid  $\{x \in \mathbb{R}^n : x^T P x \leq 1\}$ . For any value of

their arguments, the nonlinear functions  $\eta_1, \dots, \eta_r$  are said to verify the convex sum property if  $\eta_i \geq 0$ ,  $\forall i \in \Omega_r$  and  $\sum_{i=1}^r \eta_i = 1$ . The following notations are occasionally used:

$$Y_\theta = \sum_{i=1}^r \eta_i(\theta) Y_i; \quad Y_\theta^{-1} = \left( \sum_{i=1}^r \eta_i(\theta) Y_i \right)^{-1}; \quad Z_{\theta\theta} = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) Z_{ij} \quad (1)$$

where  $Y_i, Z_{ij}$  are matrices of appropriate dimensions and  $\eta_i(\theta)$ ,  $i \in \Omega_r$  are functions sharing the convex sum property.

## 2. Problem Definition and Preliminaries Results

### 2.1. Control Problem Definition

#### 2.1.1. Closed-loop system description

Consider the following fuzzy T-S model described by (Tanaka & Wang, 2001) valid on a polyhedral domain  $\mathcal{P}_x$ :

$$\begin{cases} \dot{x} = \sum_{i=1}^r \eta_i(\theta) (A_i x + B_i^u u + B_i^w w) \\ z = \sum_{i=1}^r \eta_i(\theta) (C_i^z x + D_i^{zu} u + D_i^{zw} w) \\ y = \sum_{i=1}^r \eta_i(\theta) (C_i^y x + D_i^{yw} w) \end{cases} \quad (2)$$

where  $x \in \mathcal{P}_x \subset \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_u}$ ,  $w \in \mathbb{R}^{n_w}$ ,  $y \in \mathbb{R}^{n_y}$  and  $\theta \in \mathbb{R}^k$  are respectively the state, the control input, the disturbance, the measured output and the scheduling variable vectors of the system. The regulated output vector  $z \in \mathbb{R}^{n_z}$  is used for performance purposes. For  $i \in \Omega_r$ , the real constant matrices of appropriate dimensions  $A_i, B_i^u, B_i^w, C_i^z, D_i^{zu}, D_i^{zw}, C_i^y, D_i^{yw}$  represent the set of  $r$  local linear subsystems and the nonlinear scalar functions  $\eta_i(\theta)$  satisfy the convex sum property. For system (2), we consider the following assumptions:

**Assumption 1.** The scheduling variable vector  $\theta \in \mathbb{R}^k$  is assumed to be a function of all measured signals of interest (states, external disturbances ...) with the exception of the control input value  $u$ .

**Assumption 2.** The input vector  $u$  is subject to symmetric magnitude limitations:

$$-u_{\max(i)} \leq u_{(i)} \leq u_{\max(i)}; \quad u_{\max(i)} > 0; \quad \forall i \in \Omega_{n_u}. \quad (3)$$

**Assumption 3.** The disturbance signal  $w$  is assumed to belong to the set  $W_\delta$  defined by:

$$W_\delta = \left\{ w \in \mathcal{L}_{2e}(\mathbb{R}^{n_w}) : w^T(t) R w(t) \leq \delta, \forall t \geq t_0 \right\}, \quad (4)$$

where the matrix  $R > 0$  and the bound  $\delta > 0$  are known.

**Assumption 4.** The validity domain  $\mathcal{P}_x$  of (2) is a polyhedral set defined by:

$$\mathcal{P}_x = \{x \in \mathbb{R}^{n_x} : x^T h_1^k \leq 1; \quad \forall k \in \Omega_q\}, \quad (5)$$

where the  $q$  vectors  $h_1^k \in \mathbb{R}^{n_x}$  are given.

Consider now the unconstrained dynamic output feedback controller (DOFC) in the form of (Tanaka & Wang, 2001):

$$\begin{cases} \dot{x}_c = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) A_c^{ij} x_c + \sum_{i=1}^r \eta_i(\theta) B_c^i y + v; & x_c(0) = 0 \\ u_c = \sum_{i=1}^r \eta_i(\theta) C_c^i x_c + D_c y \end{cases} \quad (6)$$

where  $x_c \in \mathbb{R}^{n_x}$ ,  $u_c \in \mathbb{R}^{n_u}$  are respectively the state and the output vectors of the controller. The additional term  $v$  is introduced to minimize the undesirable degradation of closed-loop performance caused by input saturation. The controller (6) has to be designed to guarantee the local stability and some performance requirements for the closed-loop system. Because of the input limitation, the actual control signal injected into the system is subject to the saturation effect:

$$u = \text{sat}(u_c), \quad (7)$$

where each component of the saturation function  $\text{sat}(\cdot)$  is given by:

$$\text{sat}(u_c)_{(i)} \triangleq \text{sign}(u_{c(i)}) \min(|u_{c(i)}|, u_{\max(i)}); \quad i \in \Omega_{n_u}. \quad (8)$$

Then, the interactions between the system (2) and the constrained controller are given as:

$$u = \text{sat}(u_c); \quad v = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) E_{ij}^c (\text{sat}(u_c) - u_c). \quad (9)$$

From (6) and (9), the DOFC combined with the anti-windup strategy can be expressed as:

$$\begin{cases} \dot{x}_c = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) (A_{ij}^c x_c - E_{ij}^c \psi(u_c)) + \sum_{i=1}^r \eta_i(\theta) B_i^c y \\ u_c = \sum_{i=1}^r \eta_i(\theta) C_i^c x_c + D^c y \end{cases} \quad (10)$$

where  $E_{ij}^c$  are the anti-windup gains to be designed and  $\psi(u_c) \triangleq u_c - \text{sat}(u_c)$ . The  $i^{\text{th}}$  component of the decentralized dead-zone nonlinearity  $\psi(u_c)$  is defined as:

$$\psi_{(i)}(u) = \begin{cases} 0 & \text{if } |u_{(i)}| \leq u_{\max(i)} \\ u_{(i)} - \text{sign}(u_{(i)})u_{\max(i)} & \text{if } |u_{(i)}| > u_{\max(i)} \end{cases} \quad (11)$$

Define  $x_{cl}^T \triangleq [x^T \ x_c^T]$ , the constrained closed-loop system is obtained from (2) and (10):

$$\begin{cases} \dot{x}_{cl} = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) (\mathbb{A}_{ij} x_{cl} + \mathbb{B}_{ij}^w w - (\mathbb{B}_i^w + R_\psi E_{ij}^c) \psi(u_c)) \\ z = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) (\mathbb{C}_{ij} x_{cl} + \mathbb{D}_{ij}^w w - \mathbb{D}_i^w \psi(u_c)) \end{cases} \quad (12)$$

where

$$\begin{aligned} \mathbb{A}_{ij} &= \begin{bmatrix} A_i + B_i^u D^c C_j^y & B_i^u C_j^c \\ B_i^c C_j^y & A_j^c \end{bmatrix}; \quad \mathbb{B}_w^{ij} = \begin{bmatrix} B_i^w + B_i^u D^c D_j^{yw} \\ B_i^c D_j^{yw} \end{bmatrix}; \quad \mathbb{B}_\psi^i = \begin{bmatrix} B_i^u \\ 0 \end{bmatrix}; \quad R_\psi = \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \mathbb{C}_{ij} &= \begin{bmatrix} C_i^z + D_i^{zu} D^c C_j^y & D_i^{zu} C_j^c \end{bmatrix}; \quad \mathbb{D}_w^{ij} = \begin{bmatrix} D_i^{zw} + D_i^{zu} D^c D_j^{yw} \end{bmatrix}; \quad \mathbb{D}_i^w = D_i^{zu} \end{aligned} \quad (13)$$

and the controller output is given as:

$$u_c = \sum_{i=1}^r \eta_i(\theta) (\mathbb{K}_i x_{cl} + K_i^w w), \quad (14)$$

where  $\mathbb{K}_i = \begin{bmatrix} D^c C_i^y & C_i^c \end{bmatrix}$  and  $K_i^w = D^c D_i^{yw}$ .

Using notations (1), the closed-loop system (12) is rewritten as:

$$\begin{cases} \dot{x}_{cl} = \mathbb{A}_{\theta\theta} x_{cl} + \mathbb{B}_{\theta\theta}^w w - (\mathbb{B}_\theta^w + R_\psi E_{\theta\theta}^c) \psi(u_c) \\ z = \mathbb{C}_{\theta\theta} x_{cl} + \mathbb{D}_{\theta\theta}^w w - \mathbb{D}_\theta^w \psi(u_c) \end{cases} \quad (15)$$

and the DOFC output (14) is rewritten as:

$$u_c = \mathbb{K}_\theta x_{cl} + K_\theta^w w. \quad (16)$$

### 2.1.2. Control problem definition

Local stability and performance specifications of the closed-loop system (12) will be presented in terms of Lyapunov analysis tools. Our goal is to propose a systematic method to design a DOFC together with its AW compensator of the form (10) such that the closed-loop system satisfies the following properties:

- **Property 1** [Regional Quadratic Stability]. For  $w=0$ , there exist a quadratic function  $V(x_{cl}) = x_{cl}^T P x_{cl}$  with  $P > 0$  and a real number  $\tau_1 > 0$  such that  $\dot{V}(x_{cl}) < -\tau_1 V(x_{cl})$  along the trajectory of the closed-loop system for any initial



augmented state in the ellipsoid  $\mathcal{E}(P)$ . This fact means that, for any  $x_{cl}(t_0) \in \mathcal{E}(P)$ , the corresponding trajectory of the closed-loop system will converge exponentially to the origin with a rate less than  $\tau_1 / 2$ .

• **Property 2 [State Constraints].** For any initial augmented state in  $\mathcal{E}(P)$  and any disturbance signal in  $W_\delta$ , the trajectory of the closed-loop system (12) remains in the polyhedral region described by linear inequalities:

$$\mathcal{P}_{x_{cl}} = \left\{ x_{cl} \in \mathbb{R}^{2n_x} : h_k^T x_{cl} \leq 1; \quad \forall k \in \Omega_q \right\}, \quad (17)$$

where  $h_k = \begin{bmatrix} h_1^{kT} & 0 \end{bmatrix}^T$  (implying that the system state  $x$  remains in the validity domain  $\mathcal{P}_x$  of the model (2)).

• **Property 3 [Finite  $\mathcal{L}_{2e}$ -Gain Performance].** For  $x_{cl}(t_0) = 0$ , and for any  $w \in W_\delta$  such that no saturation occurs, there exists a positive real number  $\gamma$  such that:  $\|z\|_{2,T_f} \leq \gamma \|w\|_{2,T_f}$  for all  $T_f \geq t_0$ .

## 2.2. Preliminaries

In this section, some important preliminaries results needed for design problem in Section 3 will be presented.

### 2.2.1. Generalized sector bound condition for input saturation

Given the matrices  $\mathbb{K}_i \in \mathbb{R}^{n_u \times 2n_x}$  and  $\mathbb{G}_i = \begin{bmatrix} G_1^i & G_2^i \end{bmatrix} \in \mathbb{R}^{n_u \times 2n_x}$ , with  $G_1^i \in \mathbb{R}^{n_u \times n_x}$ ,  $G_2^i \in \mathbb{R}^{n_u \times n_x}$ , for  $i \in \Omega_r$ , we define the polyhedral set  $\mathcal{P}_u$  as follows:

$$\mathcal{P}_u = \bigcap_{i=1}^r \Xi(\mathbb{K}_i - \mathbb{G}_i), \quad (18)$$

where

$$\Xi(\mathbb{K}_i - \mathbb{G}_i) \triangleq \left\{ x_{cl} \in \mathbb{R}^{2n_x} : \left| (\mathbb{K}_{i(l)} - \mathbb{G}_{i(l)}) x_{cl} \right| \leq u_{\max(l)}; \quad \forall l \in \Omega_{n_u} \right\}. \quad (19)$$

The following lemma is an extended version of Lemma 1 of (Gomes da Silva & Tarbouriech, 2005).

**Lemma 1.** Consider the function  $\psi(u_c)$  defined in (11) with  $u_c$  defined in (16). If  $x_{cl} \in \mathcal{P}_u \cap \mathcal{P}_{x_{cl}}$ , then the following condition is verified:

$$\psi^T(u_c) \left( \sum_{i=1}^r \eta_i S_i \right)^{-1} \left[ \psi(u_c) - \left[ \sum_{i=1}^r \eta_i \mathbb{G}_i \quad \sum_{i=1}^r \eta_i K_i^w \right] \begin{bmatrix} x_{cl} \\ w \end{bmatrix} \right] \leq 0, \quad (20)$$

where  $S_i \in \mathbb{R}^{n_u \times n_u}$  are any positive diagonal matrix and  $\eta_1, \dots, \eta_r$  are scalar functions satisfying the convex sum propriety.

**Proof.** The proof of this result can be done in the same fashion as the one given in (Nguyen, 2013).

### 2.2.2. Other preliminary results

Some other results that will be useful in the proof of our main result are recalled below.

**Lemma 2.** (Tuan et al., 2001) Let  $\Upsilon_{ij}$ ,  $(i, j) \in \Omega_r \times \Omega_r$  be symmetric matrices of appropriate dimensions and  $\eta_1, \dots, \eta_r$  be a family of numbers satisfying the convex sum property. Condition  $\sum_{i=1}^r \sum_{j=1}^r \eta_i \eta_j \Upsilon_{ij} < 0$  is satisfied if:

$$\begin{cases} \Upsilon_{ii} < 0, \quad \forall i \in \Omega_r \\ \frac{2}{r-1} \Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i, j) \in \Omega_r \times \Omega_r \text{ and } i \neq j \end{cases} \quad (21)$$

**Remark 1.** Note that more performing relaxation result exists in (Liu & Zhang, 2003; Ariño & Sala, 2007). However, Lemma 2 constitutes a good tradeoff between complexity and conservatism since it does not require the introduction of auxiliary variables.

**Lemma 3.** (Boyd et al., 1994) The ellipsoid  $\mathcal{E}(P)$  is included in the polyhedral set  $\mathcal{P}_u$  defined in (18) if and only if:

$$\left( \mathbb{K}_{i(l)} - \mathbb{G}_{i(l)} \right) P^{-1} \left( \mathbb{K}_{i(l)} - \mathbb{G}_{i(l)} \right)^T \leq u_{\max(l)}^2; \quad \forall (i, l) \in \Omega_r \times \Omega_{n_u}, \quad (22)$$

where  $\mathbb{K}_{i(l)}$ ,  $\mathbb{G}_{i(l)}$  are respectively the  $l^{th}$  rows of the matrices  $\mathbb{K}_i$  and  $\mathbb{G}_i$ .

**Lemma 4.** (Boyd et al., 1994) The ellipsoid  $\mathcal{E}(P)$  is included in the polyhedral set  $\mathcal{P}_{x_{cl}}$  defined in (17) if and only if:

$$\begin{bmatrix} P & h_k \\ h_k^T & 1 \end{bmatrix} \geq 0; \quad \forall k \in \Omega_p. \quad (23)$$

## 3. Main Results

The theorem below provides LMI conditions to design the DOFC together with its anti-windup strategy of the form (10) which solves the control problem defined in Subsection 2.1.2.

**Theorem 1.** Given positive real numbers  $\delta$  and  $\tau_1$ , assume there exists positive definite matrices  $P_{11} \in \mathbb{R}^{n_x \times n_x}$ ,  $X_{11} \in \mathbb{R}^{n_x \times n_x}$ , positive diagonal matrices  $S_i \in \mathbb{R}^{n_u \times n_u}$ , matrices  $W_{ij} \in \mathbb{R}^{n_x \times n_u}$ ,  $U_i \in \mathbb{R}^{n_u \times n_x}$ ,  $V_i \in \mathbb{R}^{n_u \times n_x}$ ,  $\hat{\mathcal{A}}_{ij} \in \mathbb{R}^{n_x \times n_x}$ ,  $\hat{\mathcal{B}}_i \in \mathbb{R}^{n_x \times n_y}$ ,  $\hat{\mathcal{C}}_i \in \mathbb{R}^{n_u \times n_x}$  and  $\hat{\mathcal{D}} \in \mathbb{R}^{n_u \times n_y}$  for  $(i, j) \in \Omega_r \times \Omega_r$ , and positive real numbers  $\tau_2$ ,  $\gamma$  such that:

$$\begin{bmatrix} X_{11} & I \\ I & P_{11} \end{bmatrix} > 0; \quad (24)$$

$$\begin{bmatrix} X_{11} & I & X_{11}h_1^k \\ * & P_{11} & h_1^k \\ * & * & 1 \end{bmatrix} \geq 0; \quad \forall k \in \Omega_q \quad (25)$$

$$\begin{bmatrix} X_{11} & * & * \\ I & P_{11} & * \\ \mathcal{C}_{i(l)} - U_{i(l)} & \left( \mathcal{DC}_y^i \right)_{(l)} - V_{i(l)} & u_{\max(l)}^2 \end{bmatrix} \geq 0; \quad \forall (i, l) \in \Omega_r \times \Omega_{n_u} \quad (26)$$

$$\tau_1 - \tau_2 \delta > 0 \quad (27)$$

$$\begin{cases} \Psi_{ii} < 0, \quad \forall i \in \Omega_r \\ \frac{2}{r-1} \Psi_{ii} + \Psi_{ij} + \Psi_{ij} < 0, \quad \forall (i, j) \in \Omega_r \times \Omega_r \text{ and } i \neq j \end{cases} \quad (28)$$

$$\begin{cases} \Lambda_{ii} < 0, \quad \forall i \in \Omega_r \\ \frac{2}{r-1} \Lambda_{ii} + \Lambda_{ij} + \Lambda_{ij} < 0, \quad \forall (i, j) \in \Omega_r \times \Omega_r \text{ and } i \neq j \end{cases} \quad (29)$$

where  $\Psi_{ij}$  in (28) and  $\Lambda_{ij}$  in (29) are given as follows:

$$\Psi_{ij} \triangleq \begin{bmatrix} \mathbb{H}_{ij} & \mathbb{I}_{ij} & U_i^T - B_i^u S_j & B_i^u \hat{\mathcal{D}} D_j^{yw} + B_i^w \\ * & \mathbb{J}_{ij} & V_i^T - W_{ij} & P_{11} B_i^w + \hat{\mathcal{B}}_i D_j^{yw} \\ * & * & -2S_j & \hat{\mathcal{D}} D_i^{yw} \\ * & * & * & -\tau_2 R \end{bmatrix} \quad (30)$$

$$\Lambda_{ij} \triangleq \begin{bmatrix} \mathbb{L}_{ij} & \mathbb{M}_{ij} & B_i^u \hat{\mathcal{D}} D_j^{yw} + B_i^w & C_i^z X_{11} + D_i^{zu} \hat{\mathcal{C}}_j \\ * & \mathbb{N}_{ij} & P_{11} B_i^w + \hat{\mathcal{B}}_i D_j^{yw} & C_i^z + D_i^{zu} \hat{\mathcal{D}} C_j^y \\ * & * & -\gamma I & D_i^{zw} + D_i^{zu} \hat{\mathcal{D}} D_j^{yw} \\ * & * & * & -\gamma I \end{bmatrix} \quad (31)$$

with

$$\begin{aligned} \mathbb{H}_{ij} &\triangleq \text{sym}(A_i X_{11} + B_i^u \hat{\mathcal{C}}_j) + \tau_1 X_{11}; & \mathbb{I}_{ij} &\triangleq \hat{\mathcal{A}}_{ij}^T + A_i + B_i^u \hat{\mathcal{D}} C_j^y + \tau_1 I; & \mathbb{J}_{ij} &\triangleq \text{sym}(P_{11} A_i + \hat{\mathcal{B}}_i C_j^j) + \tau_1 P_{11}; \\ \mathbb{L}_{ij} &\triangleq \text{sym}(A_i X_{11} + B_i^u \hat{\mathcal{C}}_j); & \mathbb{M}_{ij} &\triangleq \hat{\mathcal{A}}_{ij}^T + A_i + B_i^u \hat{\mathcal{D}} C_j^y; & \mathbb{N}_{ij} &\triangleq \text{sym}(P_{11} A_i + \hat{\mathcal{B}}_i C_j^j). \end{aligned} \quad (32)$$

Let  $P_{12}$  and  $X_{12}$  be two nonsingular matrices satisfying the condition:

$$P_{11} X_{11} + P_{12} X_{12}^T = I. \quad (33)$$

Then, the DOFC together with its anti-windup compensation (10) given by,  $\forall (i, j) \in \Omega_r \times \Omega_r$ :

$$\begin{aligned}
E_{ij}^c &= P_{12}^{-1} (W_{ij} S_j^{-1} - P_{11} B_i^u) \\
D^c &= \hat{D} \\
C_i^c &= (\hat{C}_i - D^c C_i^y X_{11}) X_{12}^{-T} \\
B_i^c &= P_{12}^{-1} (\hat{B}_i - P_{11} B_i^u D^c) \\
A_{ij}^c &= P_{12}^{-1} (\hat{A}_{ij} - P_{11} (A_i + B_i^u D^c C_j^y) X_{11} - P_{12} B_i^c C_j^y X_{11} - P_{11} B_i^u C_j^c X_{12}^T) X_{12}^{-T}
\end{aligned} \tag{34}$$

solves the control problem stated in Subsection 2.1.2.

**Proof sketch.** The conditions (24) and (33) imply the existence of two matrices  $P_{22}$  and  $X_{22}$  such that the block matrices  $P$  and  $X$  given by (Scherer et al., 1997):

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}; \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \tag{35}$$

are definite positive and  $X = P^{-1}$ . Note also that the same properties imply that the matrices  $P_{12}$  and  $X_{12}$  are regular. Let us introduce the matrices:

$$\Pi_1 \triangleq \begin{bmatrix} X_{11} & I \\ X_{12}^T & 0 \end{bmatrix}; \quad \Pi_2 \triangleq P \Pi_1 = \begin{bmatrix} I & P_{11} \\ 0 & P_{12}^T \end{bmatrix}. \tag{36}$$

By congruence transformation with  $\text{diag}(\Pi_1^T, I)$ , the inequality (23) is shown to be equivalent to (25). This implies that the ellipsoid  $\mathcal{E}(P)$  is included in the polyhedral set  $\mathcal{P}_{x_{cl}}$  defined in (17). Similarly, by Schur complement lemma and congruence transformation with  $\text{diag}(\Pi_1^T, I)$ , the inequality (22) is shown to be equivalent to (26), so that the ellipsoid  $\mathcal{E}(P)$  is included in the polyhedral set  $\mathcal{P}_u$  defined in (18) with  $G_1^i \triangleq V_i$  and  $G_2^i \triangleq U_i X_{12}^{-T} - G_1^i X_{11} X_{12}^{-T}$ .

By Lemma 2, the inequalities (28) imply that  $\Psi_{\theta\theta} \triangleq \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta) \eta_j(\theta) \Psi_{ij} < 0$ . Using a congruence transformation with

$\text{diag}(\Pi_1^T, S_\theta^{-1}, I)$ , this inequality is proved to be equivalent to:

$$\begin{bmatrix} \mathbb{A}_{\theta\theta}^T P + P \mathbb{A}_{\theta\theta} + \tau_1 P & \mathbb{G}_\theta^T S_\theta^{-1} - P(\mathbb{B}_\theta^w + R_\psi E_{\theta\theta}^c) & P \mathbb{B}_{\theta\theta}^w \\ * & -2S_\theta^{-1} & S_\theta^{-1} K_\theta^w \\ * & * & -\tau_2 R \end{bmatrix} < 0. \tag{37}$$

Pre- and post-multiplying (37) by  $\begin{bmatrix} x_{cl}^T & \psi^T(u_c) & w^T \end{bmatrix}$  and its transpose, the following condition can be obtained after some algebraic manipulations:

$$\dot{V}(x_{cl}) + \tau_1 x_{cl}^T P x_{cl} - \tau_2 w^T R w - 2\psi^T(u_c) S_\theta^{-1} \left[ \psi(u_c) - \begin{bmatrix} \mathbb{G}_\theta & K_\theta^w \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix} \right] < 0. \tag{38}$$

Since  $\mathcal{E}(P) \subset \mathcal{P}_u \cap \mathcal{P}_{x_{cl}}$ , by Lemma 1, the condition (38) implies that the following condition is verified along the trajectories of the closed-loop system (12):

$$\dot{V}(x_{cl}) + \tau_1 x_{cl}^T P x_{cl} - \tau_2 w^T R w < 0. \quad (39)$$

From (27) and (39), it follows that:

$$\dot{V}(x_{cl}) + \tau_1 (x_{cl}^T P x_{cl} - 1) + \tau_2 (\delta - w^T R w) < 0. \quad (40)$$

Satisfaction of (40) ensures the ellipsoid  $\mathcal{E}(P)$  is a robustly positively invariant set (Blanchini, 1999) with respect to the closed-loop system (15). In additional, since  $\mathcal{E}(P) \subset \mathcal{P}_{x_{cl}}$ , these facts prove **Property 2**.

Moreover, when  $w = 0$  (or  $w$  vanishes), it follows from (39) that  $\forall x_{cl} \in \mathcal{E}(P)$ :

$$\dot{V}(x_{cl}) < -\tau_1 x_{cl}^T P x_{cl}, \quad (41)$$

which proves **Property 1**.

Furthermore, in the absence of input saturation i.e.  $\psi(u_c) = 0$ , by the well-known Bounded Real Lemma (Scherer et al., 1997), the  $\mathcal{L}_{2e}$ -gain from the admissible disturbance  $w$  to the regulated output  $z$  will be less than  $\gamma$  if:

$$\begin{bmatrix} \mathbb{A}_{\theta\theta}^T P + P \mathbb{A}_{\theta\theta} & P \mathbb{B}_{\theta\theta}^w & \mathbb{C}_{\theta\theta}^T \\ * & -\gamma I & (\mathbb{D}_{\theta\theta}^w)^T \\ * & * & -\gamma I \end{bmatrix} < 0. \quad (42)$$

Applying again Lemma 2 and then using a congruence transformation with  $\text{diag}(\Pi_1^T, I, I)$ , it can be found that (29) implies inequality (42). This fact shows that **Property 3** holds which completes the proof.

#### 4. Anti-Windup Based Dynamic Output Feedback Controller Synthesis

The feasibility problem stated in Theorem 1 provides the conditions to design the controller satisfying some predefined requirements in Subsection 2.1.2. In this section, the design method will be formulated as a multi-objective optimization problem. Two objectives are considered: maximization of the disturbance rejection and of the estimated domain of attraction.

In order to obtain a sufficiently large estimated domain of attraction, we can consider, for example, the reference ellipsoid set  $X_R \subset \mathbb{R}^{2n_x}$  containing the origin:

$$X_R = \{x_{cl} \in \mathbb{R}^{2n_x} : x_{cl}^T P_0 x_{cl} \leq 1\}, \quad (43)$$

where  $P_0 > 0$ . Then, finding the largest ellipsoid  $\mathcal{E}(P)$  can be formulated as the following optimization problem:

$$\max_{\tau, P > 0} \tau \text{ subject to: } \tau X_R \subseteq \mathcal{E}(P). \quad (44)$$

It is worth noting that  $X_R$  defines the directions in which we want to maximize  $\mathcal{E}(P)$ . Condition  $\tau X_R \subseteq \mathcal{E}(P)$  is equivalent to  $P_0 \geq \tau^2 P$ . By Schur complement lemma, this condition is equivalent to, with  $X = P^{-1}$  and  $\mu \triangleq 1/\tau^2$ :

$$\begin{bmatrix} \mu P_0 & I \\ I & X \end{bmatrix} \geq 0. \quad (45)$$

Note that maximization  $\tau$  amounts to minimize  $\mu$ . Pre- and post-multiplying condition (45) by  $\text{diag}(I, \Pi_2^T)$  and its transpose, the condition (45) is equivalent to:

$$\begin{bmatrix} \mu P_0 & I & P_{11} \\ & 0 & P_{12}^T \\ * & X_{11} & I \\ & I & P_{11} \end{bmatrix} \geq 0. \quad (46)$$

Now, based on the result of Theorem 1, the following LMI optimization problem solves the control design problem stated in Subsection 2.1.2 while maximizing the disturbance rejection level and the estimated domain of attraction of the system (12):

$$\min_{P_{11}, X_{11}, S, U_i, V_i, W_i, \hat{A}_{ij}, \hat{B}_i, \hat{C}_i, \hat{D}, \tau_2, \gamma, \mu} \lambda \gamma + (1 - \lambda) \mu, \quad (47)$$

such that the following LMI conditions hold:  $\gamma > 0$ ,  $\mu > 0$ ; LMIs (24)-(29); LMI (46).

The positive weighting factor  $0 \leq \lambda \leq 1$  is chosen according to desired trade-off between the closed-loop performance (characterized by  $\gamma$ ) and the size of the domain of attraction  $\mathcal{E}(P)$  of the closed-loop system (characterized by  $\mu$ ).

**Remark 2.** It is important to stress that equation (33) has an infinite number of solutions and the choice of matrices  $P_{12}$ ,  $X_{12}$  is totally irrelevant for closed-loop performance (El Ghaoui & Scorletti, 1996). As remarked in (Dai et al., 2009), up to now this degree of freedom is not yet exploited for the selection of the most desirable controller. However, we have clearly pointed out in (46) that the matrix  $P_{12}$  can be considered as a slack variable contributing to enlarge the domain of attraction. For matrix inversion property, the condition  $P_{12} + P_{12}^T > 0$  should be added to guarantee that  $P_{12}$  is nonsingular. Then  $X_{12}$  is deduced from condition (33). Now, the DOFC with its AW strategy (10) can be computed from (34).

**Remark 3.** Note that the condition  $X > 0$  is equivalent to  $P_{11} > X_{11}^{-1}$  and if  $x(0) \in \mathcal{E}(P_{11})$  and  $x_c(0) = 0$ , then we have  $x_{cl}(0) \in \mathcal{E}(P)$ . Since the set  $\mathcal{E}(P)$  is robustly positively invariant with respect to the closed-loop system, it follows that

$x_{cl}(t) \in \mathcal{E}(P)$  for all  $w \in W_\delta$  and  $t > 0$ . Therefore, the trajectories  $x$  will never leave out the projection of  $\mathcal{E}(P)$  onto the plane defined by  $x_c = 0$ . This projection is nothing but the ellipsoid  $\mathcal{E}(X_{11}^{-1})$ .

## 5. Illustrative Examples

In this section, numerical examples are given for illustrating the effectiveness of the proposed method. The implementations are done using the SeDuMi solver and the YALMIP toolbox (Lofberg, 2004) with option `sdpsettings('solver','sedumi','shift',1e-4)` for the resolution of all LMI problems.

**Example 1.** Consider the T-S system (2) adapted from Example 1 in (Guerra et al., 2012) with,  $\forall i \in \Omega_2$  :

$$\begin{aligned} A_1 &= \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}; B_1^u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; A_2 = \begin{bmatrix} a & -5 \\ 1 & 2 \end{bmatrix}; B_2^u = \begin{bmatrix} b \\ 2 \end{bmatrix}; B_1^w = B_2^w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}; \\ C_i^y &= C_i^z = \begin{bmatrix} 1 & 0 \end{bmatrix}; D_i^{yw} = D_i^{zw} = 0; D_i^{zu} = 0; \eta_1(x_1) = (1 - \sin(x_1))/2; \eta_2(x_1) = 1 - \eta_1(x_1). \end{aligned} \quad (48)$$

The feasibility of (48) is checked using the proposed quadratic anti-windup based approach (QAWA) and the non-quadratic approach (NQA) (Theorem 3 in (Guerra et al., 2012)) for all the 806 points of a 31 by 26 rectangle grid on the parameter space  $(a, b)$ ,  $a \in [-20, 25]$ ,  $b \in [0, 25]$ . It is noteworthy that in (Guerra et al., 2012) static state-feedback controller is considered for undisturbed T-S systems and closed-loop performance was not taken into account for the control design. Thus, it is not objective to compare these two approaches since their control contexts are different. However, checking the feasibility of their conditions provides us an idea about the conservatism of each approach. For this study, the following data are considered:  $\delta = 0.01$ ,  $\tau_1 = 0.5$ ,  $u_{\max} = 10$  and  $\mathcal{P}_x = \{x \in \mathbb{R}^2 : |x_1| \leq \pi/2; |x_2| \leq \pi/2\}$ . Note also that all considered T-S systems are open-loop unstable. As can be observed in Figure 1, the feasibility space obtained with QAWA is larger than that of NQA for the considered grid. This interesting fact can be explained by the over-conservativeness of the unsaturated control approach in (Guerra et al., 2012) which is not the case when applying the parameter-dependent sector condition proposed in Lemma 1. Moreover, the degree of relaxation of the NQA depends strongly on the control bound of the system. These serious drawbacks of the NQA are inherently due to the handling of the time-derivative of membership functions involved in the derivation of the design conditions as stated in Section 1. Note that if closed-loop performance is considered for NQA, the obtained results are expected to be much more conservative (Guerra et al., 2012).

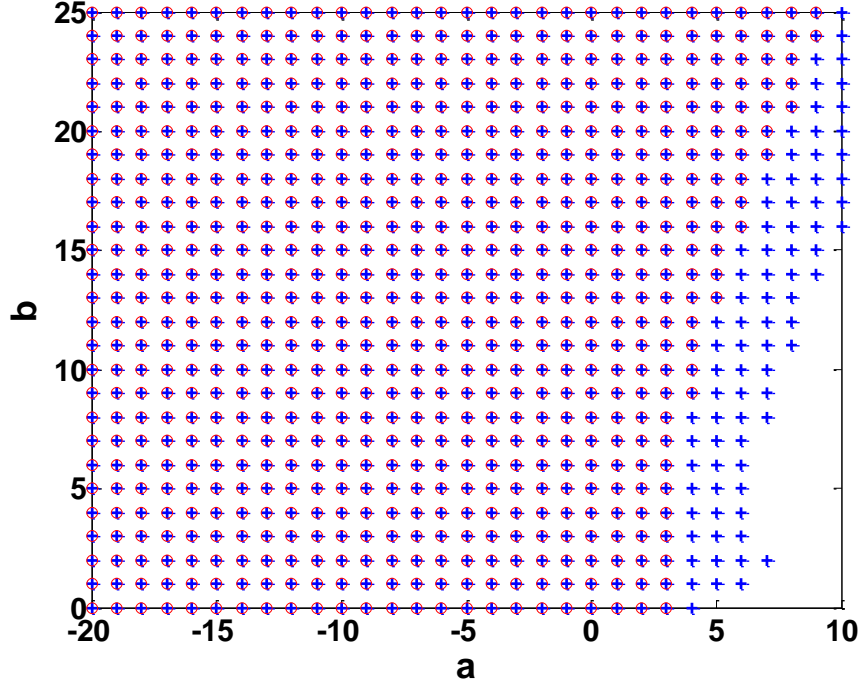


Figure1. Feasibility spaces obtained with QAWA (+) and NQA (o)

**Example 2.** Consider the following nonlinear open-loop unstable system:

$$\begin{cases} \dot{x}_1 = -x_1 + (0.1 + 0.12x_2^2)x_2 + (1.48 + 0.16x_2^3)u + 0.1w \\ \dot{x}_2 = x_1 + 0.1w \\ z = x_2 + u + 0.1w \\ y = x_2 + 0.2w \end{cases} \quad (49)$$

Since the two nonlinearities  $f(x_2) = 0.1 + 0.12x_2^2$  and  $g(x_2) = 1.48 + 0.16x_2^3$  are unbounded functions of  $x_2$ , it is impossible to obtain a T-S model that is globally equivalent to (49). One solution consists in restricting the range of evolution of variable  $x_2$ , for instance  $|x_2| \leq 1.5$ . We also consider an additional constraint on the first state variable, leading to the validity domain  $\mathcal{P}_x = \{x \in \mathbb{R}^2 : |x_1| \leq 1.5; |x_2| \leq 1.5\}$ . The control saturation limit will be taken as:  $u_{\max} = 1$ .

On  $\mathcal{P}_x$ , functions  $f$  and  $g$  are such that:

$$\begin{cases} f_{\min} \leq f(\cdot) \leq f_{\max} \\ g_{\min} \leq g(\cdot) \leq g_{\max} \end{cases}, \quad \begin{cases} f(\cdot) = \omega_1 f_{\min} + \omega_2 f_{\max} \\ g(\cdot) = \omega_3 g_{\min} + \omega_4 g_{\max} \end{cases} \quad (50)$$

with  $f_{\min} = 0.1$ ;  $f_{\max} = 0.37$ ;  $g_{\min} = 0.94$ ;  $g_{\max} = 2.02$ . The normalized nonlinear functions of T-S model are given by:

$$\eta_1 = \omega_1 \omega_3; \quad \eta_2 = \omega_1 \omega_4; \quad \eta_3 = \omega_2 \omega_3; \quad \eta_4 = \omega_2 \omega_4; \quad (51)$$

where



$$\omega_1 = \frac{f_{\max} - f(\cdot)}{f_{\max} - f_{\min}}; \quad \omega_2 = 1 - \omega_1; \quad \omega_3 = \frac{g_{\max} - g(\cdot)}{g_{\max} - g_{\min}}; \quad \omega_4 = 1 - \omega_3. \quad (52)$$

Then, with the expressions (51)-(52), the nonlinear system (49) can be exactly represented on the polyhedral set  $\mathcal{P}_x$  by the T-S model (2) with subsystem matrices given by,  $\forall i \in \Omega_4$  :

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & -f_{\min} \\ 1 & 0 \end{bmatrix}; \quad B_1^u = \begin{bmatrix} g_{\min} \\ 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -1 & f_{\min} \\ 1 & 0 \end{bmatrix}; \quad B_2^u = \begin{bmatrix} g_{\max} \\ 0 \end{bmatrix}; \quad B_i^w = [0.1 \quad 0.1]^T; \quad C_i^y = C_i^z = [0 \quad 1]; \\ A_3 &= \begin{bmatrix} -1 & -f_{\max} \\ 1 & 0 \end{bmatrix}; \quad B_3^u = \begin{bmatrix} g_{\min} \\ 0 \end{bmatrix}; \quad A_4 = \begin{bmatrix} -1 & f_{\max} \\ 1 & 0 \end{bmatrix}; \quad B_4^u = \begin{bmatrix} g_{\max} \\ 0 \end{bmatrix}; \quad D_i^{yw} = 0.2; \quad D_i^{zw} = 0.1; \quad D_i^{zu} = 1. \end{aligned} \quad (53)$$

In what follows, it will be shown that the proposed DOFC satisfies all predefined properties stated in Subsection 2.1.2. To that end, let us take  $\tau_1 = 0.1$ , the reference shape  $X_R = \mathcal{E}(P_0)$  with  $P_0 = 100I$  and the amplitude-bounded disturbance is defined as  $w(t) = 0.5 \sin(3\pi t + \pi/4)$ , with  $\delta = 0.25$ . First, solving the convex optimization problem (47) with the weighting factor  $\lambda = 0$ , which aims at maximizing the invariant set  $\mathcal{E}(P)$ , yields:

$$P = \begin{bmatrix} 0.6459 & 0.0155 & 0.0406 & -0.0401 \\ 0.0155 & 0.6459 & -0.0401 & 0.0406 \\ 0.0406 & -0.0401 & 0.0111 & -0.0110 \\ -0.0401 & 0.0406 & -0.0110 & 0.0111 \end{bmatrix}; \quad \mu = 0.0066; \quad \gamma = 4.7607. \quad (54)$$

The estimated domain of attraction of the closed-loop system (in the absence of disturbance)  $\mathcal{S}_1 \triangleq \mathcal{E}(P_1)$  is depicted in Figure 2. It is noteworthy that this set is not invariant. However, since the ellipsoid  $\mathcal{E}(P)$  is robustly invariant for all  $w \in W_\delta$ , if  $x(0) \in \mathcal{S}_1$  and  $x_c(0) = 0$  then the corresponding closed-loop trajectory will stay in the set  $\mathcal{S}_2 \triangleq \mathcal{E}(X_{11}^{-1})$ , see Remark 3. It can be also observed in Figure 2 that the ellipsoid  $\mathcal{S}_2$  is maximized along the direction of the polyhedral set  $\mathcal{S}_3 \triangleq \mathcal{P}_u \cap \mathcal{P}_{x_c}$  which is, in turn, included in the  $\mathcal{S}_4 \triangleq \Xi \left( \sum_{i=1}^4 \eta_i (\mathbb{K}_i - \mathbb{G}_i) \right)$ .

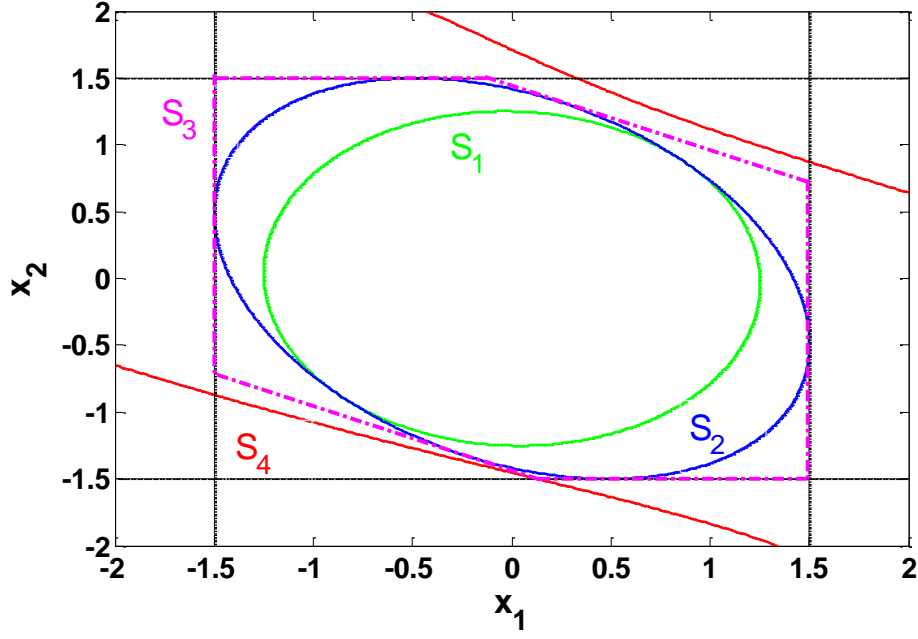


Figure 2. Projection of  $\mathcal{E}(P)$  onto the plane defined by  $x_c = 0$  and system trajectories

Figure 3 shows that the closed-loop system is stabilized and the disturbance is well attenuated. By means of simulation it can be also checked that the obtained controller provides stable behaviors for all closed-loop trajectories initialized in  $\mathcal{P}_x$  and these trajectories remain inside this validity domain.

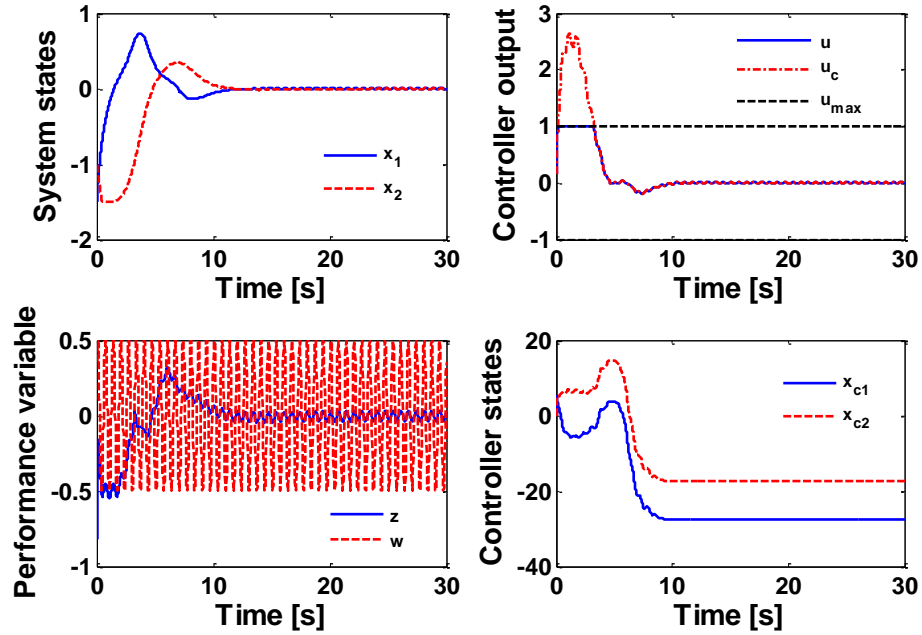


Figure 3. Closed-loop system trajectories and disturbance attenuation (left); corresponding control signal response and evolution of the controller state of the DOFC (right)

Now, solving the convex optimization problem (47) with the aim of minimizing the disturbance effects ( $\lambda = 1$ ) yields:

$$P = \begin{bmatrix} 35.0764 & 23.4791 & 12.2088 & 0.2352 \\ 23.4791 & 61.2175 & 0.3027 & 8.3083 \\ 12.2088 & 0.3027 & 6.1661 & -3.0119 \\ 0.2352 & 8.3083 & -3.0119 & 14.1328 \end{bmatrix}; \quad \mu = 0.8938; \quad \gamma = 0.2432. \quad (55)$$

Figure 4 shows the evolution of the performance variable for two designs of the controller. The first design (Case 1) corresponds to the minimization of  $\mathcal{L}_{2e}$ -gain performance whose solution is given in (55), while in Case 2 this performance is not taken into account in the control design: controller parameters are obtained solving the feasibility LMIs of Theorem 1 without condition (29).

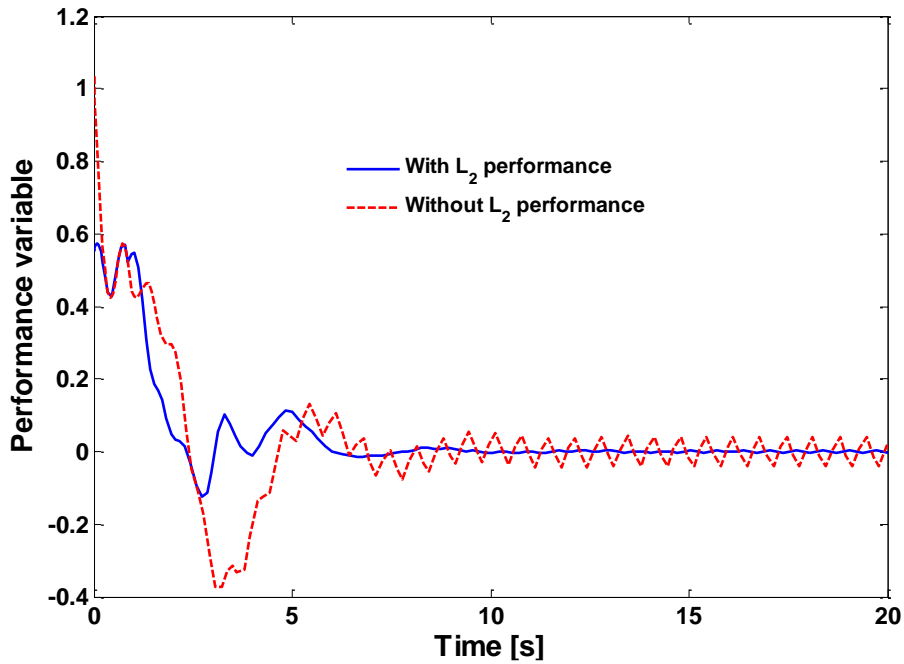


Figure 4. Performance variable of Case 1 (solid line) and Case 2 (dashed line)

From this figure, the interest of the consideration of  $\mathcal{L}_{2e}$ -gain performance in the control design is clearly confirmed.

## 6. Concluding Remarks

In this paper, a novel LMI-based approach has been proposed to design simultaneously a dynamic output feedback controller together with an anti-windup compensator for nonlinear systems under input and state constraints and subject to bounded persistent disturbances. Based on Lyapunov stability theory, conditions for the stability in closed-loop are given, which allows formulating the design of the control as a multi-objective convex optimization problem. In this way, the controller and anti-windup compensator gains can be efficiently computed with semidefinite programming solvers. By means of examples, the paper has pointed out a very interesting fact, namely, the combination of a generalized sector condition with quadratic Lyapunov approach for input-saturated T-S systems leads to simple and efficient conditions which may be competitive with respect to recent non-quadratic approaches. This is particularly true when the control

bounds are not excessively large. It has been also shown that the resulting controller satisfies several predefined closed-loop properties. The proposed approach can be applied to a large class of constrained nonlinear systems and the obtained controllers are without real difficulties in terms of implementation for real-time applications.

### Acknowledgements

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