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Tran Anh-Tu Nguyen, Michio Sugeno, Victor Campos, Michel Dambrine. LMI-Based Stability Analysis for Piecewise Multi-affine Systems. *IEEE Transactions on Fuzzy Systems*, 2017, 25 (3), pp.707-714. 10.1109/TFUZZ.2016.2566798 . hal-03428735

**HAL Id: hal-03428735**

**<https://uphf.hal.science/hal-03428735v1>**

Submitted on 25 Nov 2023

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Article in IEEE Transactions on Fuzzy Systems · June 2017

DOI: 10.1109/TFUZZ.2016.2566798

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# LMI-based Stability Analysis for Piecewise Multi-Affine Systems

Anh-Tu Nguyen, Michio Sugeno, Victor Campos, Michel Dambrine

**Abstract**—This paper provides a computational method to study the asymptotic stability of piecewise multi-affine systems. Such systems stem from a class of fuzzy systems with singleton consequents and can be used to approximate any smooth nonlinear system with arbitrary accuracy. Based on the choice of piecewise Lyapunov functions, stability conditions are expressed as a feasibility test of a convex optimization with linear matrix inequality constraints. The basic idea behind these conditions is to exploit the parametric expressions of piecewise multi-affine systems by means of Finsler’s lemma. Numerical examples are given to point out the effectiveness of the proposed method.

**Index Terms**—Piecewise multi-affine systems, fuzzy systems, singleton consequents, piecewise Lyapunov functions, stability analysis, linear matrix inequalities (LMIs).

## I. INTRODUCTION

Fuzzy logic is a powerful tool for the control of complex systems [1], [2]. The success of fuzzy control systems comes from their linguistic knowledge representation and their powerful ability to cope with very general nonlinearities [1]. Moreover, experimental results have illustrated their remarkable robustness properties [3]. Since Mamdani’s pioneering works [4], various fuzzy systems have been proposed in the literature which are categorized into three following types according to the consequent parts of IF-THEN rules [1].

- Mamdani fuzzy systems are with fuzzy variables in both premises and consequences so that they can be linguistically understandable [4].
- Mamdani-like fuzzy systems [1] are with singletons in consequent parts. Here, these fuzzy systems are called *piecewise multi-affine* (PMA) systems as explained later.
- For Takagi-Sugeno (T-S) fuzzy systems, the consequences are functions (linear/affine in most cases) of premise variables. T-S fuzzy modeling has been well discussed in [5], [6].

Although Mamdani-type systems are the first fuzzy models used for control purposes and the most meaningful to human beings, they have still received serious criticisms from the control community. This fact is mainly due to the lack of a systematic and rigorous method for studying their stability issues [1]. Indeed, establishing such a method is expected to be very hard because of the novelty of the fuzzy mathematics involved, and the fuzzy language [3]. The situation has been

greatly improved with T-S fuzzy systems whose key feature consists in their polytopic structure with linear systems [6], [7] or affine systems [8], [9] in consequent parts. This feature allows to extend some linear control concepts to deal with nonlinear systems [7]. Indeed, tremendous investigations have been devoted to the studies of T-S systems such that T-S model-based stability analysis and control design have become the most popular research platform in model-based fuzzy control [2]. However, T-S fuzzy systems are not meaningful to human beings and do not allow for an incorporation of the qualitative expertise of human operators into the control design procedure [10]. Notice that until now most of engineering applications are still based on Mamdani fuzzy models. In [1], [11], several outstanding features of PMA systems have been highlighted. First, PMA models can be straightforwardly obtained from mathematical expression of a given nonlinear system. They are also easily obtained from input/output data using the identification method developed in [12]. Second, these systems can be naturally implemented with *look-up tables* which are intensively employed in industry as a powerful tool for dynamic model approximation and controller implementation. Third, PMA systems have universal approximation capability for any smooth nonlinear function. Recently, this outstanding feature has been effectively exploited in [13] (respectively in [14]) to identify unknown nonlinear functions for the design of an adaptive fuzzy backstepping output-feedback tracking controller of nonlinear systems (respectively stochastic nonlinear systems). Finally, both state rate vector and state vector itself of PMA systems can be expressed in a specific form, called *parametric expressions* [1]. This representation of PMA systems allows a simple and efficient mathematical analysis. Up to now, the *simplicity* and the *efficiency* of analysis are less obvious; however this paper aims to demonstrate this fact. These advantages make PMA systems particularly interesting when dealing with nonlinear systems.

Despite these great advantages, only very few theoretical works regarding PMA systems are available in the literature. Using the knowledge on the membership functions (MFs), quadratic approaches for stability analysis of PMA systems were first presented in [1] with only sufficient conditions and then improved in [11] with necessary and sufficient conditions. Sufficient stability conditions regardless of the knowledge on the MFs were suggested in [10] (respectively [15]) with quadratic (respectively piecewise quadratic) Lyapunov functions. Although these works offer some guidelines to study rigorously the stability of PMA systems, it should be stressed that they suffer two serious drawbacks. First, the proposed stability conditions are expressed in terms of bilinear matrix

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inequalities which are hardly tractable with available numerical solvers. Second and more importantly, these conditions are only applicable to second-order systems. We note that state-space representation has been *exclusively* employed to study the stability of PMA systems in all previous works. However, such a representation is hard to be obtained for higher order PMA systems and not convenient for theoretical developments [1]. Therefore, extensions to high dimensional systems of existing approaches are not obvious. These drawbacks make the use of PMA modeling for studying nonlinear systems too restrictive and, without a doubt, have prevented PMA systems to be commonly used in control theory. This paper aims at developing a novel stability framework for PMA systems which can overcome the aforementioned drawbacks. The main contributions can be summarized as follows.

- 1) For the first time in the open literature, the *specific* parametric expressions of PMA systems [1] will be judiciously exploited via Finsler's lemma [16] to derive the stability conditions. We note that differently from the state-space representation, these expressions are *straightforwardly* obtained for high-order systems. Therefore, the proposed results can be applied to PMA systems of any order which is not the case of all previous works [1], [10], [11], [15].
- 2) The proposed method relies on the use of a piecewise Lyapunov function and the  $\mathcal{S}$ -procedure [17] to take fully advantage of the information on piecewise state-space partition associated to PMA systems. This allows to effectively reduce the conservatism of the results.
- 3) The proposed stability conditions are strictly expressed in terms of linear matrix inequalities (LMIs) [17]. Then, stability analysis of PMA systems amounts to solving a set of LMIs which can be easily done with available numerical solvers.

We wish that these outstanding contributions of the proposed results will contribute to promote the research on PMA systems to the control community.

Notice that general nonlinear systems can be also approximated by piecewise linear (PL) models [18] with any given accuracy. However, there are significant differences between PL and PMA systems. For each piecewise region, PMA models can be *equivalently* expressed as a polytope of affine systems [1] whereas PL model is defined by an affine system. Especially, by construction PMA systems are inherently continuous over piecewise regions and they are structurally simpler because all piecewise regions are partitioned in the same fashion, i.e. hyper-rectangle. Therefore, PMA modeling is more attractive for approximating nonlinear systems since rectangular grids are easier to work with, especially for large dimensional systems. Actually, PL modeling has been mostly used as a powerful tool for a class of hybrid systems [18]. Notice also that piecewise linear differential inclusions have been used to model an important class of affine Takagi-Sugeno fuzzy systems in [8], [9], [18].

The paper is organized as follows. The description of PMA systems and useful preliminaries are presented in Section II. Section III is devoted to the theoretical developments of

the new stability conditions for PMA systems. Numerical examples are given in Section IV to illustrate the effectiveness of the proposed method. Finally, Section V provides some concluding remarks.

**Notation.** For an integer number  $n$ ,  $\Omega_n$  represents the subset  $\{1, 2, \dots, n\}$ . For a vector  $x \in \mathbb{R}^n$  and  $i \in \Omega_n$ ,  $x_{(i)}$  denotes the  $i$ th entry of  $x$ . For a square matrix  $X$ ,  $X^\top$  denotes its transpose, and  $\text{He}(X) = X + X^\top$ . For a symmetric matrix  $X$ ,  $X > 0$  means it is positive definite, and  $X \succeq 0$  denotes the matrix with all nonnegative entries. We denote  $I_n$  as the identity matrix of dimension  $n$  and  $0_{n \times m}$  as the  $n \times m$  zero matrix. Throughout the paper, the explicit time-dependence of the variables is omitted when convenient.

## II. SYSTEM DESCRIPTION AND PRELIMINARIES

In this section, the procedure to derive the parametric expressions of a PMA model from a given dynamical system is reviewed, see [1] for more details. Then, some useful preliminaries for the theoretical developments are also presented.

### A. Description of Piecewise Multi-Affine Systems

Let us consider a system described by the following state-space representation

$$\dot{x} = f(x) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state vector and the vector field  $f(\cdot)$  is sufficiently smooth. As common in practice, the state variables of (1) are assumed to be limited in amplitude for physical and/or safety reasons, i.e.  $x_{i,\min} \leq x_{(i)} \leq x_{i,\max}$ ,  $i \in \Omega_n$ , where  $x_{i,\min}$  and  $x_{i,\max}$  denote respectively the minimal and maximal bounds of the  $i$ th entry of vector  $x$ . As a consequence, the state vector  $x$  belongs to the set  $\mathcal{R} = [x_{1,\min}, x_{1,\max}] \times \dots \times [x_{n,\min}, x_{n,\max}]$ . In order to approximate system (1) by means of PMA modeling, its state space is partitioned as follows

$$x_{j,\min} = \chi_{j,1} < \chi_{j,2} < \dots < \chi_{j,N_j+1} = x_{j,\max} \quad (2)$$

for  $j \in \Omega_n$ . Multi-index notations are used to represent both elementary regions and vertices of each partition. Let  $\mathcal{K}_v = \Omega_{N_1+1} \times \dots \times \Omega_{N_n+1}$  be the set of multi-indexes corresponding to all the vertices induced by the partition (2) and  $\mathcal{K}_r = \Omega_{N_1} \times \dots \times \Omega_{N_n}$  the set of multi-indexes corresponding to the regions. For  $i = (i_1, \dots, i_n) \in \mathcal{K}_r$ , the region  $[\chi_{1,i_1}, \chi_{1,i_1+1}] \times \dots \times [\chi_{n,i_n}, \chi_{n,i_n+1}]$  is denoted by  $\mathcal{R}_i$  and  $\mathcal{K}_i = \{i_1, i_1 + 1\} \times \dots \times \{i_n, i_n + 1\}$  is the set of multi-indexes corresponding to all vertices of  $\mathcal{R}_i$ . For  $k \in \mathcal{K}_i$ ,  $\chi_k$  is the vertex of  $\mathcal{R}_i$  whose  $j$ th component is defined by  $\chi_{k(j)} = \chi_{j,k_j}$  for  $j \in \Omega_n$ . Finally, we denote  $f_k = f(\chi_k)$ . Using these notations, the PMA model derived from system (1) valid on  $\mathcal{R}$  can be expressed as follows

$$\begin{cases} \dot{x} = \sum_{k \in \mathcal{K}_v} \eta_k(x) f_k \\ x = \sum_{k \in \mathcal{K}_v} \eta_k(x) \chi_k \end{cases} \quad (3)$$

where  $\eta_k(x) = \prod_{j=1}^n \eta_{j,k_j}(x_{(j)})$  and the membership functions  $\eta_{j,k_j}(x_{(j)})$ ,  $j \in \Omega_n$ ,  $k_j \in \Omega_{N_j+1}$ , are defined by

$$\eta_{j,k_j}(x_{(j)}) = \begin{cases} \frac{x_{(j)} - \chi_{j,k_j-1}}{\chi_{j,k_j} - \chi_{j,k_j-1}}, & \text{if } x_{(j)} \in [\chi_{j,k_j-1}, \chi_{j,k_j}] \\ & \text{and } j \geq 2 \\ \frac{\chi_{j,k_j+1} - x_{(j)}}{\chi_{j,k_j+1} - \chi_{j,k_j}}, & \text{if } x_{(j)} \in [\chi_{j,k_j}, \chi_{j,k_j+1}] \\ & \text{and } j \leq N_j \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

On a region  $\mathcal{R}_i$ , the expressions in (3) can be simplified to

$$\begin{cases} \dot{x} = \sum_{k \in \mathcal{K}_i} \eta_k(x) f_k \\ x = \sum_{k \in \mathcal{K}_i} \eta_k(x) \chi_k \end{cases} \quad (5)$$

and the membership functions (4) satisfy the following properties [1]:

$$\begin{cases} \eta_{j,k_j}(x_{(j)}) \geq 0, & \sum_{k_j=i_j}^{i_j+1} \eta_{j,k_j}(x_{(j)}) = 1, \quad j \in \Omega_n \\ \sum_{k \in \mathcal{K}_i} \eta_k(x) = 1 \end{cases} \quad (6)$$

Note that system (1) can be approximated by PMA model (3) with arbitrary accuracy on  $\mathcal{R}$  by increasing the number of partition points in (2). In this paper, we exclusively focus on the stability analysis of PMA system (3) for a *given* state-space partition. For engineering applications, such a partition is often carried out with experiments, for example in automotive and aerospace industries [19].

**Remark 1.** When  $n = 2$ , PMA model (3) with membership functions (4) corresponds to the parametric expressions presented in [1]. This model is associated to a class of fuzzy systems whose rules are defined by singleton consequents. Here, the term *piecewise multi-affine* system is originated from the fact that both  $x$  and  $\dot{x}$  in each region  $\mathcal{R}_i$  can be expressed on the multi-affine form (5).

**Remark 2.** The choice of triangular membership functions (4) for PMA systems seems restrictive since several other types of MFs are available in the literature. However, this type of MFs allows conveniently deriving canonical forms, namely parametric expression and also state-space representation, of PMA systems [1]. In addition, this is the most employed in fuzzy control systems and also in practice. The advantages of triangular MFs are discussed in detail in [20].

The following notations are introduced for convenience:

$$\begin{aligned} \bar{x} &= \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \quad \bar{f}_k = \begin{bmatrix} f_k \\ 0 \end{bmatrix}, \quad \bar{\chi}_k = \begin{bmatrix} \chi_k \\ 1 \end{bmatrix} \\ \mathbb{X}_i &= \sum_{k \in \mathcal{K}_i} \eta_k(x) \chi_k, \quad \bar{\mathbb{X}}_i = \sum_{k \in \mathcal{K}_i} \eta_k(x) \bar{\chi}_k \\ \mathbb{F}_i &= \sum_{k \in \mathcal{K}_i} \eta_k(x) f_k, \quad \bar{\mathbb{F}}_i = \sum_{k \in \mathcal{K}_i} \eta_k(x) \bar{f}_k \end{aligned} \quad (7)$$

With the notations defined in (7), system (3) on  $\mathcal{R}_i$  can be rewritten on the following compact form:

$$\begin{cases} \dot{\bar{x}} = \sum_{k \in \mathcal{K}_i} \eta_k(x) \bar{f}_k = \bar{\mathbb{F}}_i \\ \bar{x} = \sum_{k \in \mathcal{K}_i} \eta_k(x) \bar{\chi}_k = \bar{\mathbb{X}}_i \end{cases} \quad (8)$$

For stability analysis, without loss of generality we assume that the origin  $x = 0$  is the equilibrium of system (3) which corresponds to the vertex  $\chi_{k_0}$  of the state-space partition, for a given  $k_0 \in \mathcal{K}_v$ . Let  $\mathcal{K}_Z$  be the set of multi-indexes for regions containing the origin which will be called *zero-regions*, and  $\mathcal{K}_{NZ} = \mathcal{K}_r \setminus \mathcal{K}_Z$  (called *non-zero regions*). In each zero-region, every point can be computed by interpolating its  $2^n - 1$  vertices (except for the origin). Denoting  $\mathcal{K}_i^* = \mathcal{K}_i \setminus \{k_0\}$ , the interpolation procedure for zero-regions can be expressed by

$$\begin{cases} \dot{x} = \sum_{k \in \mathcal{K}_i^*} \eta_k(x) f_k = \mathbb{F}_i^0 \\ x = \sum_{k \in \mathcal{K}_i^*} \eta_k(x) \chi_k = \mathbb{X}_i^0 \end{cases} \quad (9)$$

## B. Preliminaries

Since the state space of PMA systems is partitioned into hyper-rectangle regions, it is natural to consider piecewise quadratic Lyapunov functions [8] and the  $\mathcal{S}$ -procedure [17] for their stability analysis. In this way, the information on the region structure of PMA systems can be *fully* exploited to reduce the conservatism of the results. To this end, let us construct the following matrices:

$$\bar{L}_i = [L_i \quad l_i], \quad \bar{M}_i = [M_i \quad m_i] \quad (10)$$

such that

$$\bar{L}_i \bar{x} \geq 0, \quad x \in \mathcal{R}_i, \quad i \in \mathcal{K}_r \quad (11)$$

$$\bar{M}_i \bar{x} = \bar{M}_j \bar{x}, \quad x \in \mathcal{R}_i \cap \mathcal{R}_j, \quad i, j \in \mathcal{K}_r \quad (12)$$

where  $l_i = 0$ , and  $m_i = 0$ , for  $i \in \mathcal{K}_Z$ .

**Remark 3.** Systematic methods to compute  $\bar{L}_i$  and  $\bar{M}_i$  are available in [8], [18]. It is worth remarking that the state-space partition (2) of PMA systems is structurally simple with  $n$ -rectangular regions, therefore in contrast to [18], the dimensions of all constraint matrices in (11)-(12) are exactly known, i.e.  $L_i \in \mathbb{R}^{2n \times n}$ ,  $\bar{L}_i \in \mathbb{R}^{2n \times (n+1)}$ ,  $M_i \in \mathbb{R}^{N \times n}$  and  $\bar{M}_i \in \mathbb{R}^{N \times (n+1)}$  with  $N = \sum_{i=1}^n N_i$ .

We will consider a piecewise quadratic Lyapunov function parameterized as follows

$$\mathbb{V}(x) = \begin{cases} x^\top P_i x, & \text{for } x \in \mathcal{R}_i, i \in \mathcal{K}_Z \\ \bar{x}^\top \bar{P}_i \bar{x}, & \text{for } x \in \mathcal{R}_i, i \in \mathcal{K}_{NZ} \end{cases} \quad (13)$$

where

$$\begin{cases} P_i = M_i^\top T M_i, & \text{for } i \in \mathcal{K}_Z \\ \bar{P}_i = \bar{M}_i^\top T \bar{M}_i, & \text{for } i \in \mathcal{K}_{NZ} \end{cases} \quad (14)$$

**Remark 4.** All free parameters of the Lyapunov function  $\mathbb{V}(x)$  are collected in the symmetric matrix  $T$  and the expression of  $P_i$  is linear in  $T$ . This allows an LMI formulation

for the stability analysis of PMA systems. Moreover, since the piecewise quadratic Lyapunov function (13) combines the power of quadratic functions near an equilibrium point with the flexibility of piecewise functions in the large, it can lead to less conservative results compared to those based on a common quadratic Lyapunov function. Indeed, the latter can be regarded as a special case of piecewise quadratic Lyapunov function (13) by taking  $M_i = I$  and  $L_i = 0$ .

The following lemma will be useful for the theoretical developments in next section.

**Lemma 1** (Finsler's Lemma [16]). For a vector  $\xi \in \mathbb{R}^n$  and two matrices  $Q = Q^\top \in \mathbb{R}^{n \times n}$ ,  $\mathcal{R} \in \mathbb{R}^{m \times n}$  such that  $\text{rank}(\mathcal{R}) < n$ , the following statements are equivalent

- i)  $\xi^\top Q \xi < 0$ ,  $\forall \xi \in \{\xi \in \mathbb{R}^n : \xi \neq 0, \mathcal{R}\xi = 0_{m \times 1}\}$
- ii)  $\exists \mathcal{F} \in \mathbb{R}^{n \times m}$  such that  $Q + \mathcal{F}\mathcal{R} + \mathcal{R}^\top \mathcal{F}^\top < 0$

Lemma 1 can be used to convert checking the sign of a quadratic form over a subspace into solving an LMI problem. By this lemma, the parametric expressions in (5) will be effectively exploited for the stability study.

### III. STABILITY ANALYSIS OF PIECEWISE MULTI-AFFINE SYSTEMS

Hereafter, a novel LMI-based method for studying the stability of PMA systems is presented. This method relies on the choice of the piecewise quadratic Lyapunov function (13). Moreover, the  $\mathcal{S}$ -procedure [17] is also introduced to take advantage of the piecewise-region feature of PMA systems via constraint matrices (11). These facts contribute to reduce the conservatism inherent to this type of Lyapunov-based method.

**Theorem 1** (Piecewise Quadratic Stability for PMA Systems). Given PMA system (3), assume there exist a symmetric matrix  $T \in \mathbb{R}^{N \times N}$ , symmetric matrices with nonnegative entries  $U_i \in \mathbb{R}^{2n \times 2n}$  and  $W_i \in \mathbb{R}^{2n \times 2n}$  (for  $i \in \mathcal{K}_r$ ), matrices  $Y_{1i} \in \mathbb{R}^{n \times n}$ ,  $Y_{2i} \in \mathbb{R}^{1 \times n}$  (for  $i \in \mathcal{K}_Z$ ),  $\bar{Y}_{1i} \in \mathbb{R}^{(n+1) \times (n+1)}$  and  $\bar{Y}_{2i} \in \mathbb{R}^{1 \times (n+1)}$  (for  $i \in \mathcal{K}_{NZ}$ ) satisfying the following linear matrix inequalities:

$$P_i - L_i^\top U_i L_i > 0, \quad i \in \mathcal{K}_Z \quad (15)$$

$$\text{He} \left( \begin{bmatrix} Y_{1i} + L_i^\top W_i L_i / 2 & -Y_{1i} \chi_k + P_i f_k \\ Y_{2i} & -Y_{2i} \chi_k \end{bmatrix} \right) < 0, \quad (16)$$

$i \in \mathcal{K}_Z, k \in \mathcal{K}_i^*$

$$\bar{P}_i - \bar{L}_i^\top U_i \bar{L}_i > 0, \quad i \in \mathcal{K}_{NZ} \quad (17)$$

$$\text{He} \left( \begin{bmatrix} \bar{Y}_{1i} + \bar{L}_i^\top W_i \bar{L}_i / 2 & -\bar{Y}_{1i} \bar{\chi}_k + \bar{P}_i \bar{f}_k \\ \bar{Y}_{2i} & -\bar{Y}_{2i} \bar{\chi}_k \end{bmatrix} \right) < 0, \quad (18)$$

$i \in \mathcal{K}_{NZ}, k \in \mathcal{K}_i$

where  $P_i$  and  $\bar{P}_i$  are defined in (14). Then, the equilibrium  $x = 0$  of PMA system (3) is asymptotically stable. In addition, the piecewise-quadratic function (13) is a Lyapunov function of this system.

*Proof.* Consider the piecewise function (13) which can be rewritten in a more compact form

$$\mathbb{V}(x) = \bar{x}^\top \bar{P}_i \bar{x}, \quad x \in \mathcal{R}_i, i \in \mathcal{K}_r \quad (19)$$

The construction of the constraint matrices  $\bar{L}_i$  in (11) and the satisfaction of conditions (15) and (17) imply that there exist positive real numbers  $\alpha$  and  $\beta$  such that [8]

$$\alpha|x|^2 \leq \mathbb{V}(x) \leq \beta|x|^2, \quad x \in \mathcal{R} \quad (20)$$

Now, we will first consider the *non-zero regions*, i.e.  $x \in \mathcal{R}_i$ , for  $i \in \mathcal{K}_{NZ}$ . To derive the stability conditions, the time-derivative of  $\mathbb{V}(x)$  along the solution of system (8) is required to be negative definite, i.e.

$$\dot{\mathbb{V}}(x) = \dot{\bar{x}}^\top \bar{P}_i \bar{x} + \bar{x}^\top \bar{P}_i \dot{\bar{x}} < 0.$$

From the expression of  $\dot{\bar{x}}$  in (8), this inequality can be rewritten as follows

$$\dot{\mathbb{V}}(x) = \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}^\top \begin{bmatrix} 0 & \bar{P}_i \bar{\mathbb{F}}_i \\ * & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} < 0 \quad (21)$$

Let  $\bar{Q}_i = \begin{bmatrix} \bar{L}_i^\top W_i \bar{L}_i & \bar{P}_i \bar{\mathbb{F}}_i \\ * & 0 \end{bmatrix}$  and  $\bar{\mathcal{F}}_i = [I_{n+1} \quad -\bar{\mathbb{X}}_i]$ . Since  $W_i \succeq 0$  and by the construction of  $\bar{L}_i$  in (11), the condition

$$\begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}^\top \bar{Q}_i \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} < 0, \quad x \in \mathcal{R}_i, i \in \mathcal{K}_{NZ} \quad (22)$$

implies obviously (21). Moreover, from (8) it follows that

$$\bar{\mathcal{F}}_i \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} = 0, \quad x \in \mathcal{R}_i, i \in \mathcal{K}_{NZ} \quad (23)$$

By Finsler's lemma, inequality (22) holds under condition (23) if there exist matrices  $\bar{\mathcal{M}}_i$  such that

$$\bar{Q}_i + \bar{\mathcal{M}}_i \bar{\mathcal{F}}_i + \bar{\mathcal{F}}_i^\top \bar{\mathcal{M}}_i^\top < 0, \quad i \in \mathcal{K}_{NZ} \quad (24)$$

Let us partition  $\bar{\mathcal{M}}_i = \begin{bmatrix} \bar{Y}_{1i} \\ \bar{Y}_{2i} \end{bmatrix}$ , then condition (24) can be rewritten in the form

$$\text{He} \left( \begin{bmatrix} \bar{Y}_{1i} + \bar{L}_i^\top W_i \bar{L}_i / 2 & -\bar{Y}_{1i} \bar{\mathbb{X}}_i + \bar{P}_i \bar{\mathbb{F}}_i \\ \bar{Y}_{2i} & -\bar{Y}_{2i} \bar{\mathbb{X}}_i \end{bmatrix} \right) < 0, \quad i \in \mathcal{K}_{NZ} \quad (25)$$

By the convexity property (6), if condition (18) is verified, then (25) holds as well. In other words, condition (18) implies that  $\dot{\mathbb{V}}(x) < 0$  for all  $x$  in non-zero regions.

The proof for *zero-regions* will slightly differ to that of non-zero regions since the parametric expressions given in (9) do not correspond to convex combinations. This issue can be addressed via some special changes of variables as follows. Let  $x \in \mathcal{R}_i \setminus \{0\}$  for  $i \in \mathcal{K}_Z$ , and  $\alpha(x) = \sum_{k \in \mathcal{K}_i^*} \eta_k(x)$ . Since  $x \neq 0$  and  $\alpha(x) = 1 - \eta_{k_0}(x)$ , it follows that  $\alpha(x) > 0$ . Denoting

$$x^* = x/\alpha(x), \quad \bar{x}^* = [x^{*\top} \quad 1]^\top, \quad \eta_k^*(x) = \eta_k(x)/\alpha(x),$$

$$\mathbb{F}_i^* = \sum_{k \in \mathcal{K}_i^*} \eta_k^*(x) f_k, \quad \mathbb{X}_i^* = \sum_{k \in \mathcal{K}_i^*} \eta_k^*(x) \chi_k.$$

By the convexity property (6), inequality (16) implies clearly that

$$\text{He} \left( \begin{bmatrix} Y_{1i} + L_i^\top W_i L_i / 2 & -Y_{1i} \mathbb{X}_i^* + P_i \mathbb{F}_i^* \\ Y_{2i} & -Y_{2i} \mathbb{X}_i^* \end{bmatrix} \right) < 0, \quad i \in \mathcal{K}_Z \quad (26)$$

which can be also represented in the following form:

$$\mathcal{Q}_i + \mathcal{M}_i \mathcal{F}_i + \mathcal{F}_i^\top \mathcal{M}_i^\top < 0, \quad i \in \mathcal{K}_Z \quad (27)$$

with

$$\mathcal{Q}_i = \begin{bmatrix} L_i^\top W_i L_i & P_i \mathbb{F}_i^* \\ \mathbb{F}_i^{*\top} P_i & 0 \end{bmatrix}, \quad \mathcal{F}_i = [I_n \quad -\mathbb{X}_i^*], \quad \mathcal{M}_i = \begin{bmatrix} Y_{1i} \\ Y_{2i} \end{bmatrix}.$$

We note that  $\mathcal{F}_i \bar{x}^* = 0$ . From this latter and inequality (27), applying again Finsler's lemma it follows that

$$\bar{x}^{*\top} \mathcal{Q}_i \bar{x}^* < 0, \quad x \in \mathcal{R}_i \setminus \{0\}, \quad i \in \mathcal{K}_Z \quad (28)$$

Remarking that

$$\alpha^2(x) \bar{x}^{*\top} \mathcal{Q}_i \bar{x}^* = \dot{V}(x) + x^\top L_i^\top W_i L_i x \quad (29)$$

for  $x \in \mathcal{R}_i \setminus \{0\}$ ,  $i \in \mathcal{K}_Z$ . Since  $W_i \geq 0$ , it is easily deduced from (28)-(29) that  $\dot{V}(x) < 0$  for all  $x$  in zero-regions,  $x \neq 0$ . This completes the proof of Theorem 1.  $\square$

The stability conditions in Theorem 1 are expressed in strict LMIs. The terms involving the matrices  $U_i$ ,  $W_i$ ,  $i \in \mathcal{K}_r$ , are related to the  $\mathcal{S}$ -procedure to reduce the conservatism of those inequalities.

**Remark 5.** It has been proved in [1] that the parametric expression  $x = \sum_{k \in \mathcal{K}_i} \eta_k(x) \chi_k$  in (3) is strictly *equivalent* to the membership functions (4). For the results of Theorem 1, we fully exploit this expression of  $x$  to derive the stability conditions. This fact means that the knowledge on the MFs of PMA systems is taken into account in the stability analysis to reduce the conservatism of the proposed method.

**Remark 6.** A natural question arises is whether conditions of Theorem 1 imply the stability of the original system (1). In fact, if the approximation errors between the system (1) and PMA model (3) can be characterized, then it could be possible to take these errors into account in the analysis so that the successful analysis of PMA model (3) guarantees the stability of system (1). This error characterization is left for future study. By now, it is worth noting that a fine repartition of a PMA model decreases the approximation error and increases the flexibility of the piecewise Lyapunov function (13), but leads to additional LMI conditions to be solved. Thus, there is a trade-off between computational effort and precision in the stability analysis problem.

**Remark 7.** It is noteworthy that the computational cost of an LMI optimization problem can be estimated as being proportional to  $\mathcal{N}_{\text{var}}^3 \mathcal{N}_{\text{row}}$ , where  $\mathcal{N}_{\text{var}}$  is the total number of scalar decision variables and  $\mathcal{N}_{\text{row}}$  the total row size of the LMIs [17]. In the case of Theorem 1, one has

$$\begin{aligned} \mathcal{N}_{\text{row}} &= 2^n [(2n+1)(2^n-1) + (2n+3)(N_r-2^n)] \\ \mathcal{N}_{\text{var}} &= N(N+1)/2 + (5n^2+5n+2)N_r - (2n+2)2^n \end{aligned}$$

$$\text{where } N = \sum_{i=1}^n N_i \text{ and } N_r = \prod_{i=1}^n N_i.$$

## IV. NUMERICAL EXAMPLES

We will demonstrate the effectiveness of the proposed method through three examples. All computations are done in Matlab R2013b with YALMIP toolbox and MOSEK solver.

**Example 1.** Consider the following nonlinear pendulum system with friction:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l} \sin x_1 - b x_2 \end{cases} \quad (30)$$

where  $x_1$  is the pendulum angle,  $x_2$  is its angular velocity,  $g = 9.87$  is the gravity acceleration,  $l = 4$  is the length of the pendulum and  $b = 0.5$  is the friction coefficient. Let us construct the PMA model (3) of system (30) from 121 points of a 11 by 11 rectangular grid on the state-space  $(x_1, x_2)$ :  $\mathcal{R} = [-3.1, 3.1] \times [-3.1, 3.1]$ . The stability conditions in Theorem 1 are feasible in this case and the corresponding Lyapunov function is depicted in Figure 1 (a). Thus, the local asymptotic stability of the considered PMA system can be proved. Some simulated trajectories of system (30) and the estimated domain of attraction of the equilibrium  $x = 0$  of its PMA model are plotted in Figure 1 (b). All decision matrices obtained from Theorem 1 for this example are not shown here for brevity.

**Example 2.** Consider the following family of nonlinear systems taken from [21]:

$$\dot{x} = \sum_{i=1}^2 \sum_{j=1}^2 \eta_{1i}(x) \eta_{2j}(x) A_{ij} x \quad (31)$$

where  $i = 1, 2$  and

$$\begin{aligned} \eta_{1i}(x_i) &= \begin{cases} (1 - \sin(x_i))/2, & |x_i| \leq \pi/2 \\ 0, & x_i > \pi/2 \\ 1, & x_i < -\pi/2 \end{cases} \\ \eta_{2i}(x_i) &= 1 - \eta_{1i}(x_i). \end{aligned}$$

The system matrices in (31) are given by

$$\begin{aligned} A_{11} &= \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -4 & -4 \\ (3b-2)/5 & (3a-4)/5 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} -3 & -4 \\ (2b-3)/5 & (2a-6)/5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix} \end{aligned}$$

for given constant parameters  $a$  and  $b$ . This example aims at studying the conservatism of the proposed method. To this end, the stability of system (31) for different parameter pairs  $(a, b)$  will be studied via three approaches:

- 1) PMA approximation approach with the conditions in Theorem 1,
- 2) Approximate T-S representation approach with the conditions in [8],
- 3) Exact T-S representation approach with recent contributions in [21], [22].

Notice that at the present stage of progress, the feasibility of the conditions in Theorem 1 does not directly imply the stability of (31), see Remark 6. However, the comparison carried out within this example provides an idea on the conservatism of the proposed method. Moreover, the PMA

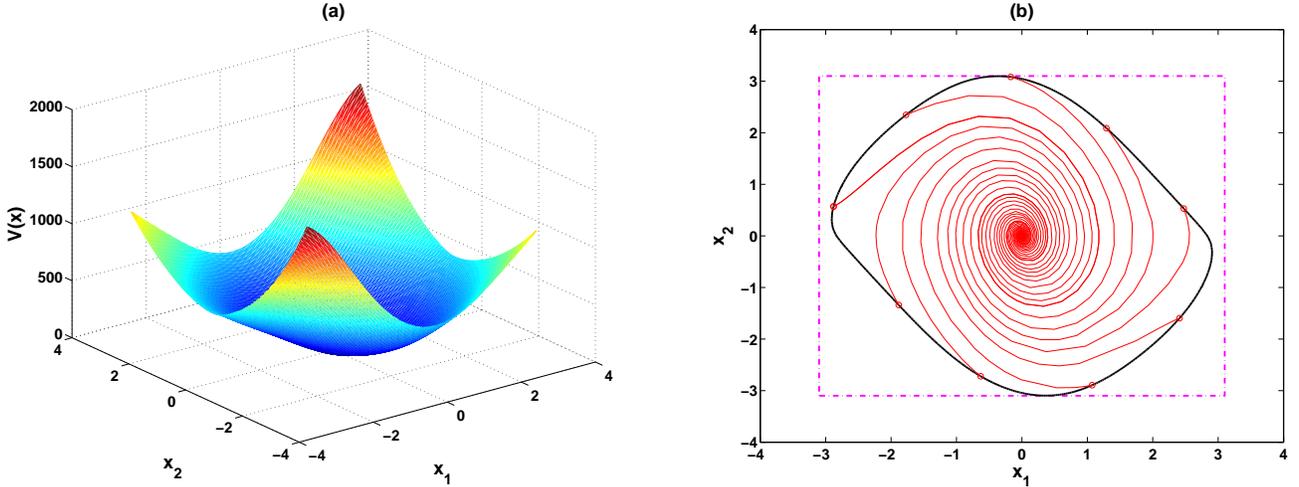


Fig. 1. (a) Lyapunov function obtained with the PMA model of system (30); (b) Estimate of the domain of attraction for equilibrium  $x = 0$  of PMA model and simulated trajectories of nonlinear system (30).

models of (31) will be constructed with very high number of piecewise regions such that they are tightly closed.

Now, the PMA approximation of (31) is constructed from 961 points of a 31 by 31 rectangular grid on the state-space  $(x_1, x_2)$ :  $\mathcal{R} = [-1.5, 1.5] \times [-1.5, 1.5]$  resulting in 900 piecewise regions. Therefore, for comparison purposes the approximate T-S fuzzy model used with the stability conditions in [8] has been constructed with 900 membership functions, corresponding to the number of regions of the PMA approximation.

Using Theorem 1, the asymptotic stability of PMA model is checked for different parameter pairs  $(a, b)$ . For illustration, in Figure 2 circles correspond to pairs  $(a, b)$  for which stability conditions of Theorem 1 are satisfied whereas the grey area corresponds to feasibility space of parameters  $(a, b)$  obtained through Theorem 1 in [22] assuming that  $|\dot{h}_{ij}(x)| \leq 0.85$ , with  $h_{ij}(x) = \eta_{1i}(x)\eta_{2j}(x)$ ,  $\forall (i, j) \in \Omega_2$ . We note that the method in [22] is based on multiple Lyapunov functions and less conservative than those of Theorem 3 in [21] (which is based on line-integral Lyapunov function). It is observed from Figure 2 that the proposed method allows proving (approximately) the stability of system (31) for a larger parameter domain. Notice that the stability conditions given in [8] provide the same  $(a, b)$  parameter stability domain as those of Theorem 1. Therefore, in order to compare the conservativeness of these results, we have searched for the largest values of  $b$ , denoted by  $b^*$ , for which the stability conditions are still feasible for two cases  $a = 0$  and  $a = -2$ . The obtained results are presented in Table I.

**Remark 8.** Notice that one of the drawbacks of Takagi-Sugeno fuzzy modeling is the lack of a systematic procedure for selecting, among the *infinite* possibilities, a single parametrization for  $f(x)$  in (1) (in the form of  $\dot{x} = A(\theta(t))x$  where  $\theta(t)$  is premise vector) which achieves stability. The use of PMA approximation approach allows overcoming this major drawback.

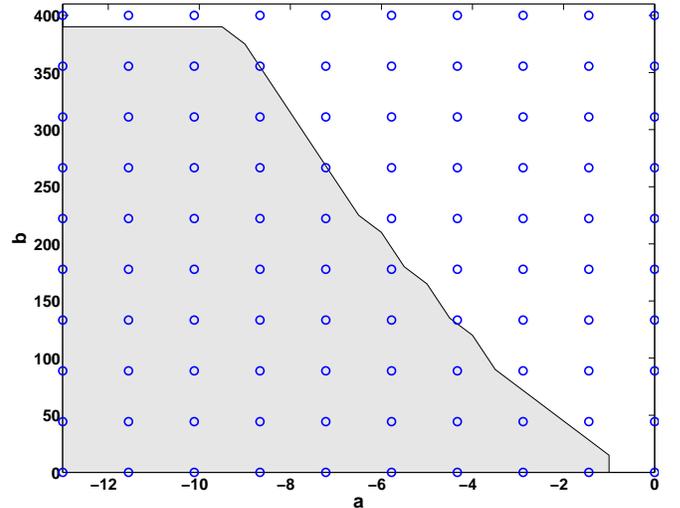


Fig. 2. Parameter stability domain obtained with Theorem 1 (circles) and Theorem 1 in [22] (grey area).

TABLE I  
VALUES OF  $b^*$  OBTAINED IN EXAMPLE 2 FOR DIFFERENT APPROACHES

Conditions	$a = 0$	$a = -2$
Theorem 1 in [8]	$b^* = 2681$	$b^* = 4534$
Theorem 1	$b^* > 20000$	$b^* > 20000$

Taking  $a = 0$ ,  $b = 0$  as a particular case, neither [21] nor [22] are capable of proving the stability of system (31). Figure 3 (a) depicts the phase plane of system (31) and the level curves of the Lyapunov function obtained with Theorem 1. The corresponding estimate of the domain of attraction and some simulated trajectories of are depicted in Figure 3 (b). It is important to note that all simulations in Figure 3 are done with the original system (31).

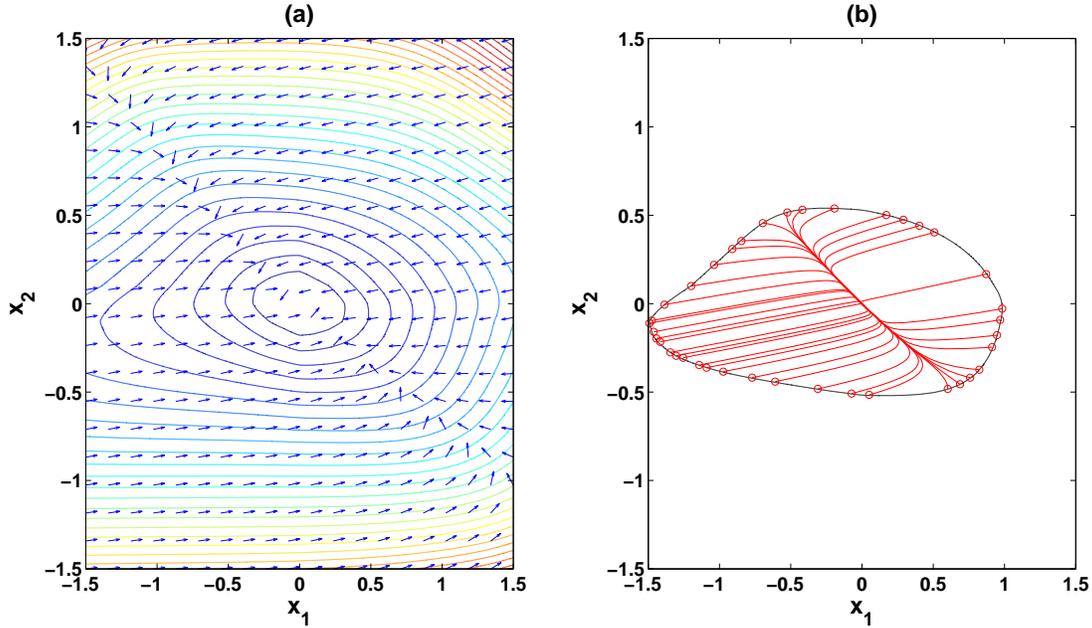


Fig. 3. (a) Phase plane of system (31) for  $a = 0$ ,  $b = 0$  and level curves associated to the obtained Lyapunov function; (b) Estimate of the domain of attraction of PMA model and some simulated trajectories of system (31).

**Example 3.** Consider an electric circuit represented by the following Hopfield artificial neural network model [23]:

$$\dot{x}_i = \frac{\lambda}{C_i} \frac{1}{1 + \tan^2\left(\frac{\pi x_i}{2V_m}\right)} \left( \sum_{j=1}^3 T_{ij} x_j - \frac{2V_m}{\lambda \pi R_i} \tan\left(\frac{\pi x_i}{2V_m}\right) \right) \quad (32)$$

where the state variables  $x_i$ ,  $i \in \Omega_3$ , are the voltages at the amplifier outputs. These voltages can only take values in the set  $\mathbf{S} = \{x \in \mathbb{R}^3 : -V_m < x_i < V_m\}$ . The system data are given as follows  $C_1 = 3.3\mu\text{F}$ ,  $C_2 = 1.5\mu\text{F}$ ,  $C_3 = 5.6\mu\text{F}$ ,  $R_1 = 1.0\text{k}\Omega$ ,  $R_2 = 3.3\text{k}\Omega$ ,  $R_3 = 1.2\text{k}\Omega$ ,  $\lambda = 7.0 \times 10^{-2}$ ,  $V_m = 5\text{V}$  and

$$T_{ij} = 10^{-3} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

The physically motivated system (32) is used to illustrate the outstanding feature of the proposed method in terms of dealing with complex nonlinearities. We note also that existing results in [1], [10], [11], [15] can not be applied to this three-dimensional system.

Let us construct the PMA model (3) of system (32) from 1331 points of a  $11 \times 11 \times 11$  hyper-rectangular grid on the state-space  $(x_1, x_2, x_3)$ :  $\mathcal{R} = [-4.99, 4.99] \times [-4.99, 4.99] \times [-4.99, 4.99]$ . The stability conditions in Theorem 1 are feasible in this case and we are able to find a piecewise Lyapunov function proving the local asymptotic stability of the considered PMA model. Some simulated trajectories of system (32) are depicted in Figure 4 (a) and the time evolution of the Lyapunov function for these same trajectories are given in Figure 4 (b). All decision matrices obtained from Theorem 1 for this example are not shown here for brevity.

## V. CONCLUDING REMARKS

Based on the use of piecewise quadratic Lyapunov functions, a constructive method for stability analysis of PMA systems has been proposed. By exploiting the parametric expressions of PMA systems, stability conditions are derived and expressed in terms of LMIs efficiently solvable with numerical solvers. Moreover, the proposed method is able to deal with PMA systems of any order. The effectiveness of the method is clearly demonstrated via numerical examples.

Despite many outstanding contributions of the proposed method, there still remain some challenging issues which will be the focus of future works. The first topic consists in considering, in the stability analysis, the modeling error between the original system and its PMA model. To this end, PMA model-based methods can be directly employed to study general nonlinear systems. Second, Lyapunov-based control design for PMA systems is another important research topic for future investigation.

## ACKNOWLEDGMENT

We would like to thank anonymous reviewers for their helpful comments and for bringing to our attention important references.

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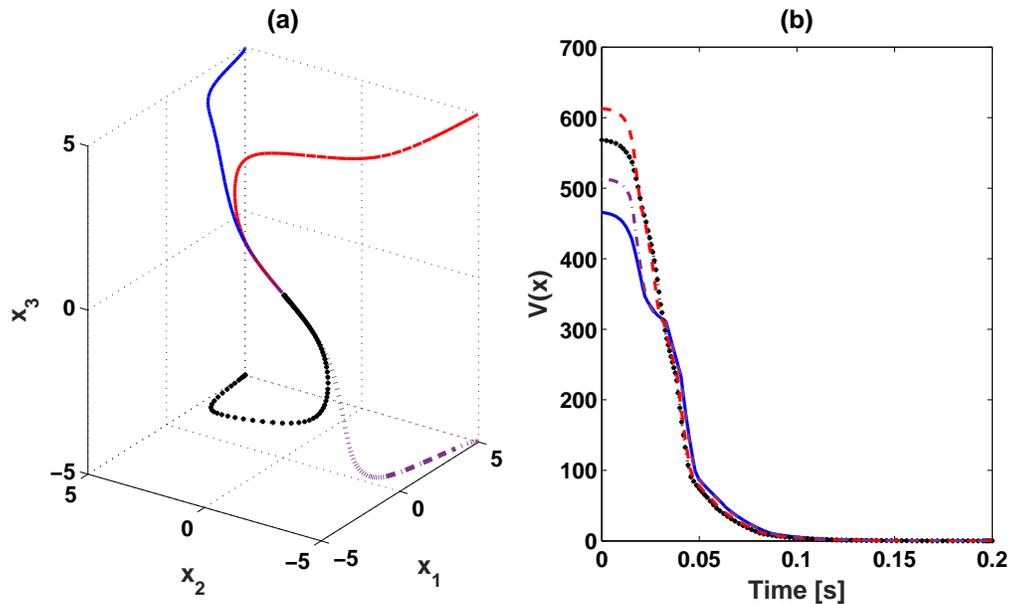


Fig. 4. (a) Simulated trajectories of system (32) with different initial conditions; (b) Time evolution of the Lyapunov function found in Example 3 for the corresponding trajectories in (a).

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