Improved LMI Conditions for Local Quadratic Stabilization of Constrained Takagi–Sugeno Fuzzy Systems

Tran Anh-Tu Nguyen, Raymundo Márquez Borbn, Thierry-Marie Guerra, Antoine Dequidt

Improved LMI Conditions for Local Quadratic Stabilization of Constrained Takagi–Sugeno Fuzzy Systems

Article *in* International Journal of Fuzzy Systems · February 2017

DOI: 10.1007/s40815-016-0269-7

CITATIONS
14

READS
404

4 authors:

Anh-Tu Nguyen
Université Polytechnique Hauts-de-France
140 PUBLICATIONS **2,084** CITATIONS
SEE PROFILE

Raymundo Marquez
Instituto Tecnológico de Sonora
32 PUBLICATIONS **328** CITATIONS
SEE PROFILE

Thierry-Marie Guerra
Université Polytechnique Hauts-de-France
349 PUBLICATIONS **9,873** CITATIONS
SEE PROFILE

Antoine Dequidt
Université Polytechnique Hauts-de-France
77 PUBLICATIONS **470** CITATIONS
SEE PROFILE

manuscript No.
(will be inserted by the editor)

Improved LMI Conditions for Local Quadratic Stabilization of Constrained Takagi-Sugeno Fuzzy Systems

Anh-Tu Nguyen · Raymundo Márquez · Thierry-Marie Guerra · Antoine Dequidt

Abstract This paper is devoted to the control synthesis of constrained Takagi-Sugeno (T-S) fuzzy systems. These systems are subject to multiple constraints, namely actuator saturation, system state constraints and also $\mathcal{L}_2-$disturbances. Using descriptor redundancy approach, new design conditions guaranteeing the existence of an input-saturated parallel distributed state feedback compensator are formulated as an LMI optimization problem with extra matrix decision variables. Differently from most of results dealing with the same local control context of constrained T-S systems, a generalized sector condition is used to handle effectively the nonlinearity of the control input. The use of a descriptor redundancy representation in conjunction with a sector condition for actuator saturation allows to reduce significantly the design conservatism while keeping a simple control structure. The effectiveness of the proposed method compared to many recent works is clearly demonstrated by means of numerical examples.

Keywords Takagi-Sugeno fuzzy systems · Local quadratic stabilization · Actuator saturation · State constraints · Descriptor redundancy representation · Linear matrix inequality (LMI)

1 Introduction

Over the past two decades, model-based fuzzy control has been an issue of great interest in control theory and its applications [1–6]. In particular, control technique based on Takagi-Sugeno (T-S) fuzzy models [7] has become one of the most popular and promising approaches for dealing with complex nonlinear systems, see for instance [1, 8–11] and references therein. Indeed, thanks to the polytopic structure of T-S models, T-S fuzzy model-based approaches provide a systematic framework for the studies of stability and stabilization of nonlinear systems [1]. Moreover, the concept of parallel distributed compensation (PDC) offers an effective control design tool. Direct Lyapunov method has been widely applied in the control context of T-S fuzzy systems, leading to the conditions for checking the stability and control design which are usually expressed in terms of linear matrix inequalities (LMIs) [12]. Without a doubt, quadratic approaches based on a common Lyapunov function still remain the most employed in fuzzy control framework and also in practical situations. This comes from the fact that they can provide a stabilizing controller with simple structure in many cases with reasonable computational cost [13]. Sufficient quadratic conditions proving the existence of PDC controllers can be found in [1,14–16]. Recent results dealing with the non-quadratic stabilization can be also mentioned such as [10, 17–21] for continuous-time T-S fuzzy systems and [22, 23] for discrete-time cases.

The present work is motivated by practical control issues. Concretely, for real-time applications, control signals and closed-loop system trajectories are usually limited in given ranges due to physical and/or safety reasons. Notice also that state constraints appear *ubiquitously* when the nonlinear sector-decomposition approach is used to obtain the T-S fuzzy representation of nonlinear systems [15]. Considering *explicitly* these limitations allows preventing destabilizing initial conditions [24]. In particular, this consideration becomes crucial in the case where disturbance signals are involved in the system dynamics. However, state constraints are often neglected in T-S fuzzy model-based control design [15]. Control design of input-saturated systems is another challenging task in control theory and its applica-

A.-T. Nguyen
IRCCyN - UMR CNRS 6597, École des Mines de Nantes, France.
E-mail: nguyen.trananhtu@gmail.com.

R. Márquez
Department of Electrical and Electronics Engineering, Sonora Institute of Technology, Sonora, Mexico.

T.-M. Guerra and A. Dequidt
LAMIH - UMR CNRS 8201, University of Valenciennes, France.

tions. It is worth noting that actuator saturation is *unavoidable* for most of physical systems, especially high performance applications. Neglecting this phenomenon in the control design procedure may lead to the degradation of closed-loop performance, or even unstable behaviors [15, 24, 25]. Over the years, the control synthesis of input-saturated systems has been subject to extensive investigation, see for instance [9, 25–27] and references therein. In T-S fuzzy control framework, notable works can be mentioned as follows. Norm-bounded approach was proposed in [27, 28] to deal with control input constraints. This method may lead to conservative results [25, 29]. Moreover, the resulting *unsaturated* PDC controllers generally offer poor closed-loop performance. Polytopic representation of the saturation nonlinearity [29] has been exploited in [30] for time-delay T-S systems, in [31] for discrete-time T-S systems, and in [15] for a class of switching T-S systems. In [26], the authors proposed a polynomial fuzzy controller for a class of polynomial fuzzy systems subject to both state delay and actuator saturation using polytopic model of the input saturation. In particular, it has been shown in [15] that this polytopic representation approach leads to much less conservative results than those of the input-unsaturated approaches. Very recently, descriptor redundancy approach [32, 33] has been proposed to deal with input-saturated T-S systems in [34]. It is important to stress that the results in [34] can be only applied to a restrictive class of T-S fuzzy systems with all linear subsystems that are open-loop stable. Moreover, this method cannot guarantee the asymptotic stability even exogenous disturbances are not involved in the dynamics of the closed-loop systems. Notice also that state constraints were not considered in most of these above works (except for [15]).

This paper presents a new LMI-based control method for continuous-time T-S fuzzy systems subject to both state and control input constraints and in presence of energy-bounded disturbances. The novelty and contributions of the proposed method can be summarized as follows.

(i) The closed-loop T-S fuzzy systems will be rewritten in an *equivalent* descriptor redundancy form which is compatible with the use of a new parameter-dependent Lyapunov function. This latter allows to avoid the *unknown* time-derivatives of the membership functions involved in the theoretical developments. In particular, a special matrix transformation is judiciously exploited to introduce slack variable matrices for the purpose of conservatism reduction.

(ii) Differently from existing works in the control context of constrained T-S fuzzy systems, an extended parameter-dependent version of the generalized sector condition [9] is applied to deal with the actuator saturation. As a consequence, we can derive conveniently LMI-based design conditions with less design conservatism. The proposed control method is able to deal with *open-loop unstable* T-S fuzzy systems which is not the case of [34]. Therefore, it can be applied to a larger class of nonlinear constrained systems.

(iii) By means of numerical experiments, it is demonstrated that in many cases the new method may lead to much less conservative results than recent works in quadratic [15, 29] and non-quadratic [10, 20, 21] control frameworks dealing with similar *local* control issue. It is also worth noting that for the proposed method, the control structure is very simple and the computational complexity remains reasonable for real-world applications.

The paper is organized as follows. Section 2 defines the considered control problem, and then some useful lemmas for the theoretical developments are also presented. Section 3 discusses on the derivation procedure of the proposed LMI-based design conditions. Numerical examples are given in Section 4 to demonstrate the effectiveness of the new method. Finally, Section 5 provides some concluding remarks.

Notation. For an integer number r, Ω_r denotes the set of integer numbers $\{1, 2, \ldots, r\}$. I denotes the identity matrix of appropriate dimensions. For any square matrix X, $\mathrm{He}\,(X) = X + X^\top$. $X > 0$ means that the matrix X is symmetric and positive definite. The ith element of a vector u is denoted $u_{(i)}$ and $X_{(i)}$ denotes the ith row of matrix X. $(*)$ stands for matrix blocks that can be deduced by symmetry. For a matrix $P > 0$ and a positive scalar ρ, we denote $\mathcal{E}(P, \rho) = \{x : x^\top P x \leq \rho\}$ and for brevity $\mathcal{E}(P) \equiv \mathcal{E}(P, 1)$. The scalar functions $\eta_1, \ldots, \eta_r$ are said to verify the convex sum property if $\eta_i(\theta) \geq 0$, $\sum_{i=1}^r \eta_i = 1$, $\forall i \in \Omega_r$. For such functions with any argument θ, and for matrices Y_i and Z_{ij} of appropriate dimensions, the following notation will be used for brevity:

$$\begin{aligned} &Y_\theta = \sum_{i=1}^r \eta_i(\theta) Y_i, \quad Y_\theta^{-1} = \left(\sum_{i=1}^r \eta_i(\theta) Y_i\right)^{-1}, \\ &Z_{\theta\theta} = \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta)\eta_j(\theta) Z_{ij}. \end{aligned} \tag{1}$$

2 Problem Formulation

2.1 System description

Consider the following continuous-time T-S system [1]:

$$\begin{cases} \dot{x} = \sum_{i=1}^r \eta_i(\theta)\left(A_i x + B_i^u v + B_i^w w\right) \\ z = \sum_{i=1}^r \eta_i(\theta)\left(C_i x + D_i^w w\right) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $v \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the system disturbance, and $z \in \mathbb{R}^{n_z}$ is the performance output. All components of the premise variable vector $\theta \in \mathbb{R}^{n_p}$ are supposed to be measured. The real constant matrices A_i, B_i^u, B_i^w, C_i, D_i^w, $i \in \Omega_r$ are of adequate dimensions, and the scalar membership functions (MFs) $\eta_i(\theta)$ satisfy the convex sum property. In this paper, the input vector v is bounded in amplitude:

$$-u_{\max(l)} \leq v_{(l)} \leq u_{\max(l)}, \quad l \in \Omega_{n_u}, \tag{3}$$

where the control bounds $u_{\max(l)} > 0$, $l \in \Omega_{n_u}$ are given. Moreover, we assume that the disturbance signal w in (2) is bounded in energy, i.e.

$$w \in \mathcal{W}_\delta = \left\{ w : \mathbb{R}^+ \to \mathbb{R}^{n_w}, \ \int_0^\infty w(\tau)^\top w(\tau) d\tau \leq \delta \right\},$$

where the bound $\delta > 0$ is given.

In order to respect the control input limitations (3), the following saturated PDC control law is considered:

$$v = \text{sat}(u), \quad u = \sum_{i=1}^{r} \eta_i(\theta) K_i x = K_\theta x, \tag{4}$$

where $\text{sat}(u) \in \mathbb{R}^{n_u}$ and

$$\text{sat}(u)_{(l)} = \text{sign}\left(u_{(l)}\right) \min\left(\left|u_{(l)}\right|, u_{\max(l)}\right),$$

for $l \in \Omega_{n_u}$. Let us denote

$$\tilde{x}^\top = \left[x^\top \ \dot{x}^\top\right], \quad \psi(u) = u - sat(u). \tag{5}$$

From (2) and (4), the closed-loop system can be *equivalently* represented in the following descriptor form:

$$\mathbb{E}\dot{\tilde{x}} = \mathbb{A}_{\theta\theta}\tilde{x} - \mathbb{B}_\theta^\psi \psi(u) + \mathbb{B}_\theta^w w, \tag{6}$$

where

$$\mathbb{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbb{A}_{\theta\theta} = \mathbb{A}_\theta + \mathbb{B}_\theta^u \mathbb{K}_\theta, \quad \mathbb{K}_\theta = \begin{bmatrix} K_\theta & 0 \end{bmatrix},$$
$$\mathbb{A}_\theta = \begin{bmatrix} 0 & I \\ A_\theta & -I \end{bmatrix}, \quad \mathbb{B}_\theta^u = \mathbb{B}_\theta^\psi = \begin{bmatrix} 0 \\ B_\theta^u \end{bmatrix}, \quad \mathbb{B}_\theta^w = \begin{bmatrix} 0 \\ B_\theta^w \end{bmatrix}. \tag{7}$$

2.2 Problem definition

In this paper, we search for a PDC control law (4) for the constrained T-S fuzzy system (2) such that the following closed-loop properties are satisfied:

(i) *State constraints:* Given vectors $h_k \in \mathbb{R}^{n_x}$, $k \in \Omega_q$, there exist a matrix $P > 0$ and a positive scalar ρ such that, for any $x(0) \in \mathcal{E}(P, \rho)$ and for any $w \in \mathcal{W}_\delta$, the closed-loop trajectory of (2) remains inside the following polyhedral set which defines the state constraints of the T-S fuzzy system (2):

$$\mathcal{P}_x = \left\{ x \in \mathbb{R}^{n_x} : \quad h_k^\top x \leq 1, \quad k \in \Omega_q \right\}.$$

(ii) *Regional quadratic stability and $\mathcal{L}_2-$gain performance:* In the absence of disturbances, i.e. when $w = 0$, any closed-loop trajectory initialized inside the set $\mathcal{E}(P)$ converges exponentially to the origin. When $w \neq 0$ and for $\forall w \in \mathcal{W}_\delta$, if $x(0) \in \mathcal{E}(P, \rho)$, then there exists a positive scalar γ such that the performance output z of the closed-loop system (2) is in $\mathcal{L}_2$ and its norm satisfies the following condition:

$$\int_0^\infty z(\tau)^\top z(\tau) d\tau \leq \gamma \int_0^\infty w(\tau)^\top w(\tau) d\tau + \rho.$$

2.3 Useful lemmas

In what follows, some useful preliminaries needed for the control design in Section 3 are presented.

Lemma 1 *Given matrices $K_i \in \mathbb{R}^{n_u \times n_x}$ and $G_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \Omega_r$. Let us define the following set:*

$$\mathcal{P}_u = \left\{ x \in \mathbb{R}^{n_x} : \ \left| \sum_{i=1}^{r} \eta_i \left(K_{i(l)} - G_{i(l)}\right) x \right| \leq u_{\max(l)} \right\}$$

for $l \in \Omega_{n_u}$. Consider the PDC control law u of the form (4). If $x \in \mathcal{P}_u$, then the following inequality on the dead-zone nonlinearity $\psi(u)$:

$$\psi(u)^\top \left(\sum_{i=1}^{r} \eta_i S_i \right)^{-1} \left[\psi(u) - \sum_{i=1}^{r} \eta_i G_i x \right] \leq 0,$$

holds for any positive diagonal matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, and for any scalar functions η_i, $i \in \Omega_r$, satisfying the convex sum property.

Lemma 1 presents an extension of the generalized sector condition for control input nonlinearity proposed in [9]. Notice that for this extended version, all involved matrices are parameter-dependent which helps to reduce the conservatism of the results.

Lemma 2 *[35] Let $\mathcal{A}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{Q}$ be matrices of appropriate dimensions. The following statements are equivalent:*

a) $\mathcal{A}^\top \mathcal{P} + \mathcal{P}^\top \mathcal{A} + \mathcal{Q} < 0$,

b) $\exists \mathcal{R}, \mathcal{L} : \begin{bmatrix} \mathcal{A}^\top \mathcal{L} + \mathcal{L}^\top \mathcal{A} + \mathcal{Q} & \mathcal{P}^\top - \mathcal{L}^\top + \mathcal{A}^\top \mathcal{R} \\ \mathcal{P} - \mathcal{L} + \mathcal{R}^\top \mathcal{A} & -\mathcal{R} - \mathcal{R}^\top \end{bmatrix} < 0.$

Lemma 3 *[14] Let Υ_{ij}, $i, j \in \Omega_r$ be symmetric matrices of appropriate dimensions and $\eta_1, \ldots, \eta_r$ be a family of scalar functions satisfying the convex sum property. The following condition:*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \eta_i \eta_j \Upsilon_{ij} < 0 \tag{8}$$

is verified if

$$\begin{cases} \Upsilon_{ii} < 0, \quad i \in \Omega_r \\ \dfrac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i,j \in \Omega_r, \ \ i \neq j \end{cases} \tag{9}$$

Notice that more efficient relaxation results than (9) can be found in [36] at the expense of high computational cost. At last, the results presented in this paper are *independent* to the relaxation methods in the sense that different relaxations can be straightforwardly applied to convert the parameter-dependent condition (8) to a finite set of LMIs.

3 Main Results

The following theorem provides LMI conditions that allow the design of a PDC controller (4) solving the control problem defined in Section 2.

Theorem 1 *Given a T-S fuzzy system (2) and a scalar $\delta > 0$ where $w \in \mathcal{W}_\delta$. Assume there exist positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, positive diagonal matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, regular matrices $X_{22}^{ij} \in \mathbb{R}^{n_x \times n_x}$, matrices $X_{21}^{ij} \in \mathbb{R}^{n_x \times n_x}$, $H_{11}^{j} \in \mathbb{R}^{n_x \times n_x}$, $H_{12}^{j} \in \mathbb{R}^{n_x \times n_x}$, $H_{21}^{ij} \in \mathbb{R}^{n_x \times n_x}$, $H_{22}^{ij} \in \mathbb{R}^{n_x \times n_x}$, $R_{11}^{j} \in \mathbb{R}^{n_x \times n_x}$, $R_{12}^{j} \in \mathbb{R}^{n_x \times n_x}$, $R_{21}^{ij} \in \mathbb{R}^{n_x \times n_x}$, $R_{22}^{ij} \in \mathbb{R}^{n_x \times n_x}$, $V_i \in \mathbb{R}^{n_u \times n_x}$, $W_i \in \mathbb{R}^{n_u \times n_x}$ for $i,j \in \Omega_r$, and positive scalars ρ, γ such that*

$$\rho + \gamma\delta \leq 1, \tag{10}$$

$$\begin{bmatrix} X & * \\ V_{i(l)} - W_{i(l)} & u_{\max(l)}^2 \end{bmatrix} \geq 0, \quad i \in \Omega_r, \quad l \in \Omega_{n_u}, \tag{11}$$

$$\begin{bmatrix} X & Xh_k \\ * & 1 \end{bmatrix} \geq 0, \quad k \in \Omega_q, \tag{12}$$

$$\Psi_{ii} < 0, \quad i \in \Omega_r, \tag{13}$$

$$\frac{2}{r-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} < 0, \quad i,j \in \Omega_r \text{ and } i \neq j, \tag{14}$$

where the quantities Ψ_{ij} are defined in (15). Then, the PDC controller (4) with the feedback gains given by

$$K_i = V_i X^{-1}, \quad i \in \Omega_r,$$

solves the control design problem stated in Section 2.

Proof Let us introduce the following parameter-dependent matrix:

$$\mathbb{X}_{\theta\theta} = \sum_{i=1}^{r}\sum_{i=1}^{r} \eta_i(\theta)\,\eta_j(\theta) \begin{bmatrix} X & 0 \\ X_{21}^{ij} & X_{22}^{ij} \end{bmatrix}.$$

Since $X = X^\top > 0$, it follows that

$$\mathbb{X}_{\theta\theta}^\top \mathbb{E}^\top = \mathbb{E}\mathbb{X}_{\theta\theta} \geq 0. \tag{16}$$

Since $X > 0$ and the matrices X_{22}^{ij} are regular, for $i,j \in \Omega_r$, then the parameter-dependent matrix $\mathbb{X}_{\theta\theta}$ is also regular. Denoting $\mathbb{P}_{(\theta)} = \mathbb{X}_{\theta\theta}^{-1}$, it follows from (16) that

$$\mathbb{E}^\top \mathbb{P}_{(\theta)} = \mathbb{P}_{(\theta)}^\top \mathbb{E} \geq 0, \tag{17}$$

where the matrix $\mathbb{P}_{(\theta)}$ is of the following form:

$$\mathbb{P}_{(\theta)} = \left(\sum_{i=1}^{r}\sum_{i=1}^{r} \eta_i(\theta)\,\eta_j(\theta) \begin{bmatrix} X & 0 \\ X_{21}^{ij} & X_{22}^{ij} \end{bmatrix} \right)^{-1}. \tag{18}$$

Considering the Lyapunov function

$$\mathbb{V}(\tilde{x}) = \tilde{x}^\top \mathbb{E}^\top \mathbb{P}_{(\theta)} \tilde{x}.$$

From the definitions of $\tilde{x}$ in (5), $\mathbb{E}$ in (7), and $\mathbb{P}_{(\theta)}$ in (18), it can be easily observed that

$$\mathbb{V}(\tilde{x}) = x^\top P x,$$

where $P = X^{-1}$. It follows from Lemma 1 that the following inequality:

$$\begin{aligned} \Xi =& \dot{\mathbb{V}}(\tilde{x}) + z^\top z - \gamma w^\top w - 2\psi(u)^\top S_\theta^{-1}\left[\psi(u) - G_\theta x\right] \\ =& \operatorname{He}\left(\dot{\tilde{x}}^\top \mathbb{E}^\top \mathbb{P}_{(\theta)} \tilde{x} - \psi(u)^\top S_\theta^{-1}\left[\psi(u) - G_\theta x\right] \right) \\ & + z^\top z - \gamma w^\top w < 0, \end{aligned} \tag{19}$$

implies clearly that

$$\dot{\mathbb{V}}(\tilde{x}) + z^\top z - \gamma w^\top w < 0, \quad \forall x \in \mathcal{P}_u. \tag{20}$$

Moreover, with the expression of $\mathbb{E}\dot{\tilde{x}}$ defined in (6), using the property (17) and the notations in (1), the quantity Ξ in (19) can be rewritten in the following form:

$$\Xi = \begin{bmatrix} \tilde{x} \\ \psi(u) \\ w \end{bmatrix}^\top \Phi \begin{bmatrix} \tilde{x} \\ \psi(u) \\ w \end{bmatrix} < 0, \tag{21}$$

$$\Psi_{ij} = \operatorname{He}\left(\begin{bmatrix} H_{21}^{ij} & H_{22}^{ij} & 0 & 0 & 0 & X + R_{21}^{ij} & R_{22}^{ij} \\ A_i H_{11}^{j} + B_i^u V_j - H_{21}^{ij} & A_i H_{12}^{j} - H_{22}^{ij} & -B_i^u S_j & B_i^w & 0 & A_i R_{11}^{j} - R_{21}^{ij} & A_i R_{12}^{j} - R_{22}^{ij} \\ W_j & 0 & -S_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma I/2 & 0 & 0 & 0 \\ C_i X & 0 & 0 & D_i^w & -I/2 & 0 & 0 \\ -H_{11}^{j} & -H_{12}^{j} & 0 & 0 & 0 & -R_{11}^{j} & -R_{12}^{j} \\ X_{21}^{ij} - H_{21}^{ij} & X_{22}^{ij} - H_{22}^{ij} & 0 & 0 & 0 & -R_{21}^{ij} & -R_{22}^{ij} \end{bmatrix} \right) \tag{15}$$

where

$$\Phi = \begin{bmatrix} \mathbb{A}_{\theta\theta}^{\top}\mathbb{P}_{(\theta)} + \mathbb{P}_{(\theta)}^{\top}\mathbb{A}_{\theta\theta} & -\mathbb{P}_{(\theta)}^{\top}\mathbb{B}_{\theta}^{\psi} + \mathbb{G}_{\theta}^{\top}S_{\theta}^{-1} & \mathbb{P}_{(\theta)}^{\top}\mathbb{B}_{\theta}^{w} \\ * & -2S_{\theta}^{-1} & 0 \\ * & * & -\gamma I \end{bmatrix} + \begin{bmatrix} \mathbb{C}_{\theta}^{\top} \\ 0 \\ D_{\theta}^{w\top} \end{bmatrix} \begin{bmatrix} \mathbb{C}_{\theta} & 0 & D_{\theta}^{w} \end{bmatrix},$$

and

$$\mathbb{C}_{\theta} = \begin{bmatrix} C_{\theta} & 0 \end{bmatrix}, \quad \mathbb{G}_{\theta} = \begin{bmatrix} G_{\theta} & 0 \end{bmatrix}, \quad \mathbb{A}_{\theta\theta} = \mathbb{A}_{\theta} + \mathbb{B}_{\theta}^{u}\mathbb{K}_{\theta}.$$

It can be observed that the condition (21) holds if $\Phi < 0$. By the well-known Schur complement lemma [12], this latter can be proved to be equivalent to the following inequality:

$$\begin{bmatrix} \mathrm{He}\left(\mathbb{A}_{\theta\theta}^{\top}\mathbb{P}_{(\theta)}\right) & -\mathbb{P}_{(\theta)}^{\top}\mathbb{B}_{\theta}^{\psi} + \mathbb{G}_{\theta}^{\top}S_{\theta}^{-1} & \mathbb{P}_{(\theta)}^{\top}\mathbb{B}_{\theta}^{w} & \mathbb{C}_{\theta}^{\top} \\ * & -2S_{\theta}^{-1} & 0 & 0 \\ * & * & -\gamma I & D_{\theta}^{w\top} \\ * & * & * & -I \end{bmatrix} < 0 \quad (22)$$

Pre- and post-multiplying (22) with $\mathrm{diag}\left(\mathbb{X}_{\theta\theta}^{\top}, S_{\theta}, I, I\right)$ and its transpose, this inequality is found to be equivalent to

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & -\mathbb{B}_{\theta}^{\psi}S_{\theta} + \mathbb{X}_{\theta\theta}^{\top}\mathbb{G}_{\theta}^{\top}S_{\theta} & \mathbb{B}_{\theta}^{w} & \mathbb{X}_{\theta\theta}^{\top}\mathbb{C}_{\theta}^{\top} \\ * & -2S_{\theta} & 0 & 0 \\ * & * & -\gamma I & D_{\theta}^{w\top} \\ * & * & * & -I \end{bmatrix} < 0, \quad (23)$$

where $\Upsilon_{11} = \mathrm{He}\left(\mathbb{A}_{\theta}\mathbb{X}_{\theta\theta} + \mathbb{B}_{\theta}^{u}\mathbb{K}_{\theta}\mathbb{X}_{\theta\theta}\right)$. Applying Lemma 2 with $\mathcal{P} = \mathbb{X}_{\theta\theta}$, $\mathcal{A} = \mathbb{A}_{\theta}^{\top}$, $\mathcal{L} = \mathbb{H}_{\theta\theta}$, $\mathcal{R} = \mathbb{R}_{\theta\theta}$ and $\mathcal{Q} = \mathrm{He}\left(\mathbb{B}_{\theta}^{u}\mathbb{K}_{\theta}\mathbb{X}_{\theta\theta}\right)$, to the block-matrix entry Υ_{11} of Υ defined in (23), we can show that this inequality is equivalent to the condition (24). Now, let us denote

$$\mathbb{H}_{\theta\theta} = \begin{bmatrix} H_{11}^{\theta} & H_{12}^{\theta} \\ H_{21}^{\theta\theta} & H_{22}^{\theta\theta} \end{bmatrix}, \quad \mathbb{R}_{\theta\theta} = \begin{bmatrix} R_{11}^{\theta} & R_{12}^{\theta} \\ R_{21}^{\theta\theta} & R_{22}^{\theta\theta} \end{bmatrix}, \quad W_i = G_i X. \quad (25)$$

Then, the inequality (24) becomes (26) after some algebraic manipulations. Notice that the use of full parameter dependent matrix $\mathbb{H}_{\theta\theta}$ defined in (25) with extra degrees of freedom instead of $\begin{bmatrix} X & 0 \\ X_{21}^{\theta\theta} & X_{22}^{\theta\theta} \end{bmatrix}$ explains the relaxation idea under this new control scheme.

By the relaxation result of Lemma 3, we can prove that if $x \in \mathcal{P}_u$, then the satisfaction of the conditions (13)-(14) with Ψ_{ij}, $i, j \in \Omega_r$, defined in (15) implies clearly the condition (20). Moreover, by Schur complement lemma, (11) is found to be equivalent to

$$X - \frac{\left(V_{i(l)} - W_{i(l)}\right)^{\top}\left(V_{i(l)} - W_{i(l)}\right)}{u_{\max(l)}^{2}} \geq 0, \quad (27)$$

for $i \in \Omega_r$, $l \in \Omega_{n_u}$. Pre- and post- multiplying (27) with the matrix P yields

$$P - \frac{\left(K_{i(l)} - G_{i(l)}\right)^{\top}\left(K_{i(l)} - G_{i(l)}\right)}{u_{\max(l)}^{2}} \geq 0, \quad (28)$$

for $i \in \Omega_r$ and $l \in \Omega_{n_u}$. Since the positive scalar functions η_i, $i \in \Omega_r$, satisfy the convex sum property, it is then deduced that the satisfaction of (28) implies the inclusion $\mathcal{E}(P) \subseteq \mathcal{P}_u$. Similarly, the inequality (12) implies the inclusion $\mathcal{E}(P) \subseteq \mathcal{P}_x$.

Now, we distinguish two following cases:

(i) If $w = 0$, it follows from (20) that

$$\dot{\mathbb{V}}(\tilde{x}) < 0, \quad x \in \mathcal{E}(P),$$

which guarantees the regional quadratic internal stability of the closed-loop system (2).

(ii) If $w \neq 0$ and $w \in \mathcal{W}_{\delta}$, integrating both sides of (20) from 0 to any time $T > 0$ yields

$$\mathbb{V}(x(T)) < \mathbb{V}(x(0)) + \gamma \int_0^T w(\tau)^{\top}w(\tau)d\tau - \int_0^T z(\tau)^{\top}z(\tau)d\tau. \quad (29)$$

Then, it follows from (10) and (29) that

$$\mathbb{V}(x(T)) \leq \rho + \gamma\delta \leq 1, \quad \forall x(0) \in \mathcal{E}(P, \rho),$$

which means that the corresponding closed-loop trajectory remains inside the set $\mathcal{E}(P) \subseteq \mathcal{P}_x$. This fact proves that state constraints are satisfied. Furthermore, from (29) and considering the limit case $T \to \infty$, we obtain

$$\int_0^{\infty} z(\tau)^{\top}z(\tau)d\tau \leq \gamma \int_0^{\infty} w(\tau)^{\top}w(\tau)d\tau + \mathbb{V}(x(0)),$$

which implies the $\mathcal{L}_2-$gain performance of the closed-loop system. Thus, the proof can be now concluded.

$$\begin{bmatrix} \mathrm{He}\left(\mathbb{A}_{\theta}\mathbb{H}_{\theta\theta} + \mathbb{B}_{\theta}^{u}\mathbb{K}_{\theta}\mathbb{X}_{\theta\theta}\right) & -\mathbb{B}_{\theta}^{\psi}S_{\theta} + \mathbb{X}_{\theta\theta}^{\top}\mathbb{G}_{\theta}^{\top}S_{\theta} & \mathbb{B}_{\theta}^{w} & \mathbb{X}_{\theta\theta}^{\top}\mathbb{C}_{\theta}^{\top} & \mathbb{X}_{\theta\theta}^{\top} - \mathbb{H}_{\theta\theta}^{\top} + \mathbb{A}_{\theta}\mathbb{R}_{\theta\theta} \\ * & -2S_{\theta} & 0 & 0 & 0 \\ * & * & -\gamma I & D_{\theta}^{w\top} & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\mathbb{R}_{\theta\theta} - \mathbb{R}_{\theta\theta}^{\top} \end{bmatrix} < 0. \quad (24)$$

$$\mathrm{He}\left(\begin{bmatrix} H_{21}^{\theta\theta} & H_{22}^{\theta\theta} & 0 & 0 & 0 & X+R_{21}^{\theta\theta} & R_{22}^{\theta\theta} \\ A_\theta H_{11}^\theta + B_\theta^u V_\theta - H_{21}^{\theta\theta} & A_\theta H_{12}^\theta - H_{22}^{\theta\theta} & -B_\theta^u S_\theta & B_\theta^w & 0 & A_\theta R_{11}^\theta - R_{21}^{\theta\theta} & A_\theta R_{12}^\theta - R_{22}^{\theta\theta} \\ W_j & 0 & -S_\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma I/2 & 0 & 0 & 0 \\ C_\theta X & 0 & 0 & D_\theta^w & -I/2 & 0 & 0 \\ -H_{11}^\theta & -H_{12}^\theta & 0 & 0 & 0 & -R_{11}^\theta & -R_{12}^\theta \\ X_{21}^{\theta\theta} - H_{21}^{\theta\theta} & X_{22}^{\theta\theta} - H_{22}^{\theta\theta} & 0 & 0 & 0 & -R_{21}^{\theta\theta} & -R_{22}^{\theta\theta} \end{bmatrix}\right) < 0. \tag{26}$$

Remark 1 From the results of Theorem 1, it is easily to formulate the LMI optimization that can search for a PDC controller solving the control problem defined in Section 2 while minimizing the estimated bound of the system $\mathcal{L}_2-$gain. Such an LMI optimization can be stated as follows:

$$\min \quad \gamma,$$

subject to: $\gamma > 0$ and the LMI conditions in Theorem 1.

Remark 2 Since local control context is considered here, in many cases it is desirable to achieve the largest estimate of the domain of attraction of the closed-loop system. Notice that the ellipsoid $\mathcal{E}(P)$ represents the domain of attraction of (2) in the absence of system disturbances, i.e. $w = 0$. Note also that the size of $\mathcal{E}(P)$ can be characterized by $\mathrm{trace}(P)$ [12]. Then, the maximization of $\mathcal{E}(P)$ can be formulated as follows:

$$\min \quad \mathrm{trace}(R), \tag{30}$$

subject to LMI conditions in Theorem 1 and

$$\begin{bmatrix} R & I \\ I & X \end{bmatrix} > 0, \tag{31}$$

where $R > 0$ is a slack variable. By Schur complement lemma, it follows from (31) that $R > P = X^{-1}$, thus $\mathrm{trace}(R) > \mathrm{trace}(P)$. Then, the minimization of $\mathrm{trace}(R)$ implies clearly the minimization of $\mathrm{trace}(P)$.

Remark 3 Notice that all matrices $\mathbb{X}_{\theta\theta}$, $\mathbb{H}_{\theta\theta}$, $\mathbb{R}_{\theta\theta}$, used in the theoretical development of Theorem 1 depend on the MFs $\eta_i(\theta)$, $i \in \Omega_r$, in a *polynomial* form. These matrices can be chosen in the following general forms defined in (32). In the expressions (32), N can be viewed as the degree of the polynomials in function of MFs of the each parameter-dependent matrix. Then, less conservative results can be achieved by increasing N. For a trade-off between relaxation and computational cost, the design conditions of Theorem 1 are obtained with $N = 1$. However the results for $N \geq 2$ can be easily generalized. Notice also that in the case of high-dimensional parameter-dependent summations (i.e. with a large value of N), the relaxation result in [36] would be more interesting than that of Lemma 3.

Remark 4 The conservatism of the proposed method comes mainly from the fact that matrix X in (32) is not parameter dependent. This is done on purpose to avoid the *unknown* time-derivatives of the MFs $\dot{\eta}_i(\theta)$, $i \in \Omega_r$, involved in the theoretical developments. Up two now, this inherent control issue is still very challenging in the framework of continuous-time T-S fuzzy systems, see [17, 18, 20, 37] and references therein. In [20], local design conditions based on fuzzy Lyapunov functions for continuous-time T-S systems have been proposed. An interesting feature of this method consists in the fact that implicit bounds on $\dot{\eta}_i(\theta)$, $i \in \Omega_r$, are not required for the design conditions as in many other works, for instance [17, 37]. It is noteworthy that although this requirement is understandable for stability results, it is more questionable for stabilization due to the explicit presence of the to-be-designed control actions u in the expressions of $\dot{\eta}_i = \frac{\partial \eta_i}{\partial x}\dot{x}$ and $\dot{x}$. As a consequence, the validity region of the obtained controller must be checked *a posteriori*. In this context, the method in [20] requires the knowledge on the control bounds to obtain LMI formulation. Therefore, even the goals pursued in the present work and the one in [20] are definitively different, they can be however compared objectively which is not the case for other non-quadratic methods previously cited. Note also that the control method in [20] already allows outperforming many well-known results existing in the literature of T-S control systems, see for instance [17, 18].

Remark 5 We note that non-quadratic Lyapunov based methods in the *local* control framework of continuous-time T-S systems usually involve complex design conditions [20]. The computational complexity of LMI-based optimization problems is related to the number of scalar decision variables $\mathcal{N}_{\mathrm{var}}$ and the number of rows $\mathcal{N}_{\mathrm{row}}$ of all considered LMIs. These numbers corresponding to the LMIs of different results are respectively given as follows.

(i) For the design conditions of Theorem 3 in [20]:

$$\begin{cases} \mathcal{N}_{\mathrm{var}} = 1 + r\left(n_x\left(n_x+1\right)/2 + n_u n_x + n_p n_u n_x\right) \\ \quad + n_p n_x^2 r^2 \left(n_x + n_u\right)\left(n_x + n_u + 1 + \left(n_x+1\right) r\right) \\ \mathcal{N}_{\mathrm{row}} = 1 + r n_x + r^3\left(n_x + n_z + n_w + n_p n_x^2 n_u\right) \\ \quad + r^2\left(n_u n_x + \left(n_x + n_u\right) n_p n_x^2\right) \end{cases}$$

$$\mathbb{X}_{(\theta)} = \begin{bmatrix} X & 0 \\ X_{21}(\theta) & X_{22}(\theta) \end{bmatrix} = \sum_{i_0=1}^{r}\sum_{i_1=1}^{r}\cdots\sum_{i_N=1}^{r}\left(\prod_{j=0}^{N}\eta_{i_j}(\theta)\right)\begin{bmatrix} X & 0 \\ X_{21}^{i_0 i_1 \ldots i_N} & X_{22}^{i_0 i_1 \ldots i_N} \end{bmatrix},$$
$$\mathbb{H}_{(\theta)} = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) \\ H_{21}(\theta) & H_{22}(\theta) \end{bmatrix} = \sum_{i_0=1}^{r}\sum_{i_1=1}^{r}\cdots\sum_{i_N=1}^{r}\left(\prod_{j=0}^{N}\eta_{i_j}(\theta)\right)\begin{bmatrix} H_{11}^{i_1 \ldots i_N} & H_{12}^{i_1 \ldots i_N} \\ H_{21}^{i_0 i_1 \ldots i_N} & H_{22}^{i_0 i_1 \ldots i_N} \end{bmatrix}, \quad (32)$$
$$\mathbb{R}_{(\theta)} = \begin{bmatrix} R_{11}(\theta) & R_{12}(\theta) \\ R_{21}(\theta) & R_{22}(\theta) \end{bmatrix} = \sum_{i_0=1}^{r}\sum_{i_1=1}^{r}\cdots\sum_{i_N=1}^{r}\left(\prod_{j=0}^{N}\eta_{i_j}(\theta)\right)\begin{bmatrix} R_{11}^{i_1 \ldots i_N} & R_{12}^{i_1 \ldots i_N} \\ R_{21}^{i_0 i_1 \ldots i_N} & R_{22}^{i_0 i_1 \ldots i_N} \end{bmatrix}.$$

(ii) For the design conditions in [15, 29]:

$$\begin{cases} \mathcal{N}_{\text{var}} = 3 + n_x(n_x+1)/2 + 2rsn_u n_x \\ \mathcal{N}_{\text{row}} = 4 + q(n_x+1) + 3n_x + rsn_u(n_x+1) \\ \quad + 2^{n_u} r^2 s(n_x + n_z + n_w) \end{cases}$$

with $s = 1$ (i.e. one switching region).

(iii) For the design conditions of Theorem 1 in this work:

$$\begin{cases} \mathcal{N}_{\text{var}} = n_x(n_x+1)/2 + r\left(n_u + 4n_x^2 + 2n_u n_x\right) \\ \quad + 6r^2 n_x^2 + 2 \\ \mathcal{N}_{\text{row}} = 1 + q(n_x+1) + rn_u(n_x+1) \\ \quad + r^2(4n_x + n_u + n_z + n_w) \end{cases}$$

In order to illustrate the computational burden of different control methods, the numbers of decision variables and LMI rows corresponding to the design conditions of Theorem 1 in [15], Theorem 3 in [20], and Theorem 1 in this work for different values of system dimensions are given in Table 1. We can observe that the numerical complexity of the new control method is reasonable compared to available results in the literature. Notice also that the number of LMI rows of conditions in [15] depends strongly on the numbers of control inputs (n_u) and switching regions (s).

Another advantage of the proposed method with respect to the non-quadratic one in [20] is that the controller structure (4) is very simple compared to the non-PDC control law in [20] since it does not require online parameter-dependent matrix inversion for real-time implementation. Therefore, the new control method seems to be more appropriate for real-world applications, especially for T-S fuzzy systems of high dimension and/or with an important number of subsystems. At last, from the viewpoint of input-saturated control systems, in many cases the proposed method can lead to less conservative results than those of [20]. These facts will be clearly illustrated in Section 4.

4 Numerical Examples

In this section, the effectiveness of the proposed method will be demonstrated by means of two numerical examples. The first one deals with a physically motivated nonlinear system. The second example aims for a study on the conservatism of the proposed method compared to several related works in the literature. All LMI problems are solved with the help of Matlab R2013b using YALMIP toolbox [38].

Example 1 Consider the well-known chaotic Lorenz system described by

$$\begin{cases} \dot{x}_1 = -\mu_1 x_1 + \mu_1 x_2 \\ \dot{x}_2 = \mu_2 x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 - \mu_3 x_3 \end{cases} \quad (33)$$

The T-S fuzzy representation of (33) is given as follows [39]:

Plant rules:
Rule 1: If x_1 is about η_1, then $x = A_1 x$ (34)
Rule 2: If x_1 is about η_2, then $x = A_2 x$

where

$$A_1 = \begin{bmatrix} -\mu_1 & \mu_1 & 0 \\ \mu_2 & -1 & 20 \\ 0 & -20 & -\mu_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\mu_1 & \mu_1 & 0 \\ \mu_2 & -1 & -30 \\ 0 & 30 & -\mu_3 \end{bmatrix}, \quad (35)$$
$$\eta_1(x_1) = (30 - x_1)/50, \quad \eta_2(x_1) = 1 - \eta_1(x_1).$$

Table 1 Illustration of the computational complexity for different control methods (green-good, black-intermediate, red-worst)

System parameters	Method in [15]		Method in [20]		Theorem 1	
	$\mathcal{N}_{\text{var}}$	$\mathcal{N}_{\text{row}}$	$\mathcal{N}_{\text{var}}$	$\mathcal{N}_{\text{row}}$	$\mathcal{N}_{\text{var}}$	$\mathcal{N}_{\text{row}}$
$r = n_x = 2, q = 4, n_u = n_w = n_z = s = n_p = 1$	14	60	255	125	143	65
$r = q = 4, n_x = n_u = n_p = 2, n_w = n_z = s = 1$	22	162	1565	745	473	203
$r = q = 4, n_x = s = 2, n_u = n_w = n_z = n_p = 1$	22	302	1565	745	473	203
$r = q = 4, n_x = s = 2, n_u = 3, n_w = n_z = n_p = 1$	54	1118	2941	1449	513	259

After introducing the control inputs, the system disturbance and the performance outputs into (34)-(35), we obtain the T-S system (2) with the following data:

$$B_1^u = \begin{bmatrix} \mu_1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2^u = \begin{bmatrix} 0 \\ \mu_2 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$B_1^w = \begin{bmatrix} 0.1\mu_1 \\ 0.1\mu_2 \\ 0.5 \end{bmatrix}, \quad B_2^w = \begin{bmatrix} -0.1\mu_1 \\ 0.1\mu_2 \\ 0.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$D_1^w = 0.1,\ D_2^w = -0.1,\ \mu_1 = 10,\ \mu_2 = 28,\ \mu_3 = 2.67.$$

It should be stressed that in order to guarantee the validity of the membership functions $\eta_1(x_1)$ and $\eta_2(x_1)$ in (35), the state constraints $\mathcal{P}_x = \{x \in \mathbb{R}^2 : -20 \leq x_1 \leq 30\}$ must be *explicitly* considered into the design procedure. Moreover, it is assumed that the control input is limited in amplitude with $u_{\max} = 3$ and the disturbance belongs to a class of $\mathcal{L}_2$ functions, i.e. $w \in \mathcal{W}_\delta$ with $\delta = 2.8$. Notice that all these system constraints have not been considered in previous works. Now, solving LMI conditions of Theorem 1 leads to the following results:

$$P = \begin{bmatrix} 0.47 & 0.11 & 0.07 \\ 0.11 & 0.12 & 0.02 \\ 0.07 & 0.02 & 0.10 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -7.16 & -2.65 & -1.30 \end{bmatrix},$$
$$\gamma = 0.33, \quad \rho = 0.04, \quad K_2 = \begin{bmatrix} -4.08 & -2.76 & -0.19 \end{bmatrix}. \tag{36}$$

Fig. 1 shows the closed-loop behaviors corresponding to the initial condition $x_0^\top = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ for the energy-bounded disturbance defined as follows:

$$w(t) = \begin{cases} 9, & \text{if} \quad 1 \leq t \leq 1.1 \\ 0, & \text{otherwise} \end{cases} \tag{37}$$

The trajectory of the performance output z and the $\mathcal{L}_2-$gain ratio $\int_0^\infty z(t)^\top z(t)\,dt / \left(\int_0^\infty w(t)^\top w(t)\,dt + \rho\right)$ corresponding to the results given in Fig. 1 are depicted in Fig. 2. It can be clearly observed that the effect of the disturbance w is well attenuated with the proposed controller and the $\mathcal{L}_2-$gain performance of the closed-loop T-S system is respected under the considered simulation condition.

Fig. 3 shows the comparison between the closed-loop responses obtained with Theorem 1 and those obtained with the design conditions in [1, Chapter 3]. We observe that the *explicit* consideration of system constraints into the proposed design procedure can help to improve significantly the transient behaviors of the closed-loop T-S fuzzy system. Moreover, note also that the control method in [1] cannot provide any guarantee on the closed-loop stability in the presence of system constraints.

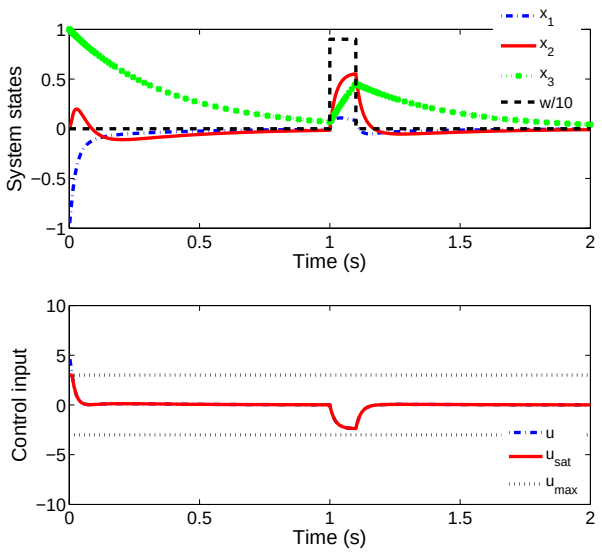

Fig. 1 Closed-loop behaviors corresponding to the initial condition $x_0^\top = [-1\ 0\ 1]$ and the energy-bounded disturbance in (37) with respect to the feedback gains given in (36).

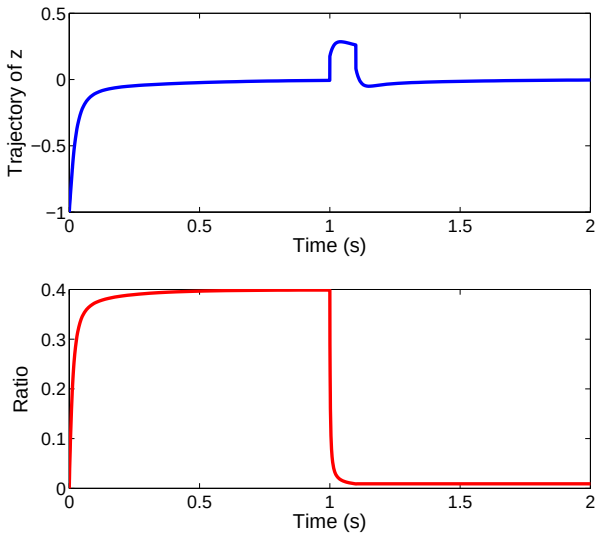

Fig. 2 Trajectory of the performance output z and the ratio $\int_0^\infty z(t)^\top z(t)\,dt / \left(\int_0^\infty w(t)^\top w(t)\,dt + \rho\right)$ corresponding to the results presented in Fig. 1.

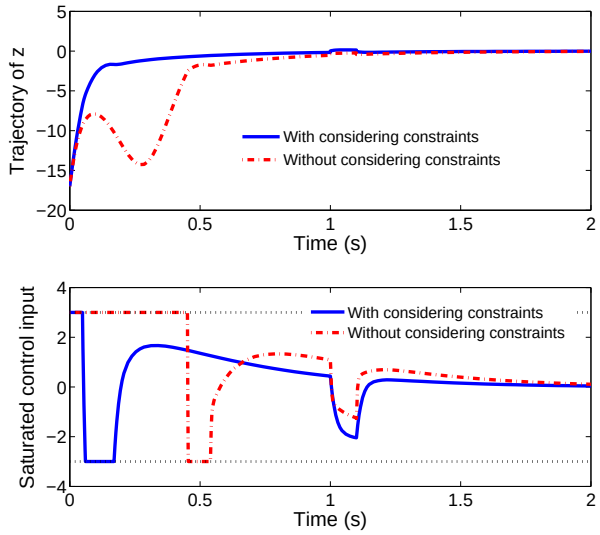

Fig. 3 Closed-loop behaviors corresponding to the initial condition $x_0^\top = [-17\ 7\ 30]$ and the energy-bounded disturbance in (37) obtained with Theorem 1 and the control method proposed in [1].

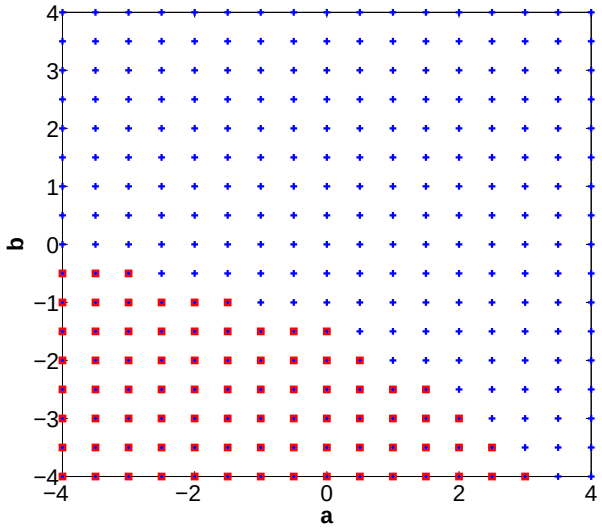

Fig. 4 Feasibility spaces obtained with Theorem 1 (+) and control methods proposed in [15, 29] (□).

Example 2 Consider a family of T-S systems (2) with the following system data:

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, && A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \\
&A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}, && A_4 = \begin{bmatrix} 0.89 & -5.29 \\ 0.1 & 0 \end{bmatrix}, \\
&B_1^u = B_4^u = \begin{bmatrix} 1\ 0 \end{bmatrix}^\top, && B_2^u = \begin{bmatrix} 8\ 0 \end{bmatrix}^\top, \\
&B_3^u = \begin{bmatrix} -b+6\ -1 \end{bmatrix}^\top, && B_i^w = \begin{bmatrix} 0.1\ 0.1 \end{bmatrix}^\top, \\
&C_1 = C_4 = \begin{bmatrix} -0.1\ -0.4 \end{bmatrix}, && C_2 = \begin{bmatrix} 0.1\ -0.1 \end{bmatrix}, \\
&D_i^w = 0.15,\ i \in \Omega_4, && C_3 = \begin{bmatrix} 0.2\ 0.1 \end{bmatrix},
\end{aligned}
\tag{38}
$$

where $(a, b) \in [-4, 4]$. The membership functions in (38) are given by

$$\eta_1(x_1) = \frac{1 - \sin(x_1)}{2}, \quad \eta_2(x_1) = \frac{1 + \sin(x_1)}{2}.$$

The constraints of the T-S system (38) are characterized by $\delta = 0.1$, $u_{\max} = 5$ and

$$\mathcal{P}_x = \left\{ x \in \mathbb{R}^2 :\ |x_1| \leq 1,\ |x_2| \leq 1 \right\}.$$

We note that all T-S systems given in (38) are open-loop *unstable*. Now, let us check the existence of a stabilizing controller of (38) for all the points of a 17-by-17 square grid on the parameter space (a, b) by using different approaches dealing with similar local control context. The feasibility spaces obtained with conditions of Theorem 1 and those of the quadratic approach using polytopic representation of the saturation function in [15, 29] are depicted in Fig. 4. It can be easily observed that the proposed method provides *significantly* less conservative results than those of [15, 29]. It is important to note also that the following design conditions based on:

(i) fuzzy Lyapunov functions using a non-parallel distributed control law proposed in [10, 20, 21], and
(ii) a descriptor redundancy representation of the closed-loop system using a polytopic representation of the saturation function proposed in [34],

are all *unfeasible* for any point of the considered parameter grid. We note that for an objective comparison, the system constraints and the induced $\mathcal{L}_2-$gain performance have been considered in the design procedure of [10, 21] as the same way in [20]. Notice also that the results proposed in [10, 20, 21] have already outperformed (in terms of conservatism reduction) many other outstanding results in non-quadratic T-S control framework, for instance [17, 18]. For this example, the numbers $\mathcal{N}_{\text{var}}$ and $\mathcal{N}_{\text{row}}$ characterizing the computational complexity of LMI conditions (see Remark 5) for different methods are shown in Table 2. We can see that both $\mathcal{N}_{\text{var}}$ and $\mathcal{N}_{\text{row}}$ of the new control method are more important than those of [15, 29]. This is due to the use of Lemma 2 to introduce slack variables for the purpose of conservatism reduction. However, the computational burden of the proposed method is *significantly* lower than that of the non-quadratic approach in [20] even though this latter cannot provide any solution for the considered example.

Table 2 Comparison of numerical complexity between different control design methods

Complexity	Theorem 1	Method in [20]	Method in [15, 29]
$\mathcal{N}_{\text{row}}$	203	745	162
$\mathcal{N}_{\text{var}}$	473	1565	22

At a first glance, it seems very surprising that the new quadratic method could lead to much less conservative results than the fuzzy Lyapunov based methods proposed in [10,20,21]. However, from the viewpoint of *constrained* T-S fuzzy systems, the design conservatism comes not only from the choice of Lyapunov functions but also the methods to handle the system constraints. Indeed, the control method in [20] made use of an *unsaturated* control law (due to its way to deal with the time-derivative of the membership functions involved in the theoretical developments) which can lead to over-conservative results as shown in [15]. Moreover, that method depends strongly on the value of control bound limits, i.e. $u_{\max}$. Indeed, feasible solutions can be generally obtained with [10, 20, 21] only if the limitations on the control input amplitude become *arbitrarily* large, for example $u_{\max} \geq 24$ for the LMIs of [20], $u_{\max} \geq 19.1$ for the LMIs of [10], and $u_{\max} \geq 15.5$ (with $\epsilon = 5$) for the LMIs of [21], in the case of the constrained T-S system (38).

Now, let us consider the constrained T-S system (38) with $a = b = 0$. It is noteworthy that none of the results of existing works in [10, 15, 20, 21, 29, 34] can provide a stabilizing controller for the corresponding T-S system. Solving the LMI optimization problem in (30) leads to the following results:

$$P = \begin{bmatrix} 1.43 & 3.19 \\ 3.19 & 24.00 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -6.93 & -6.60 \end{bmatrix},$$
$$\gamma = 10.0, \quad K_2 = \begin{bmatrix} -3.97 & -9.80 \end{bmatrix}.$$

We can observe in Fig. 5 that the ellipsoid $\mathcal{S}_1 \equiv \mathcal{E}(P)$ is maximized along the polyhedral set $\mathcal{S}_2 \equiv \mathcal{P}_u \cap \mathcal{P}_x$ which is, in turn, included inside the following sector set:

$$\mathcal{S}_3 = \left\{ x \in \mathbb{R}^{n_x} : \left| \sum_{i=1}^{r} \eta_i \left(K_{i(l)} - G_{i(l)} \right) x \right| \leq u_{\max(l)} \right\},$$

for $l \in \Omega_{n_u}$. Moreover, simulated trajectories of the closed-loop T-S system starting from $\mathcal{S}_1$ always remain inside this set. Therefore, the set inclusion property stated in Section 2 is clearly verified.

The results obtained with the above examples confirm the strong interest of the proposed control method compared to existing results in the local control framework of constrained T-S fuzzy systems.

5 Conclusions

In this paper, a new control method is proposed to improve the quadratic design conditions of constrained T-S fuzzy systems. The controller design amounts to solving a set of LMIs with some available numerical tools. By exploiting the descriptor redundancy representation of the closed-loop system as well as a special matrix transformation, slack decision variables are judiciously introduced to reduce effectively the design conservatism. We also propose to use a parameter-dependent version of generalized sector condition for dealing with control input nonlinearity. By means of numerical experiments, it has been demonstrated that these facts allows for an outperformance in terms of relaxation of the proposed method compared to several recent works dealing with similar local control issue in both quadratic and non-quadratic T-S control frameworks. Moreover, the control structure of the PDC control law is simple and the computational complexity of the proposed design conditions is reasonable for real-time applications. Future works focus on the application of the new method to intelligent vehicle systems [40]. Moreover, extensions of the proposed approach to T-S output feedback control contexts [2, 8] would be also interesting.

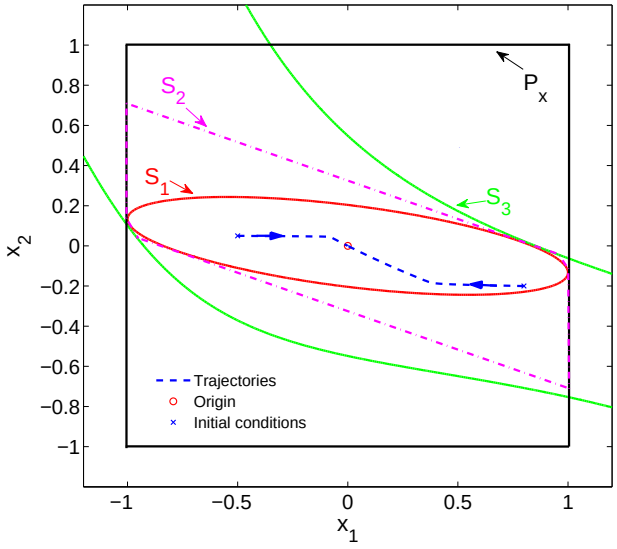

Fig. 5 Set inclusion property and simulated system trajectories.

Acknowledgements This work was supported in part by the Program "Vehicle for the Future" via ADEME (French Environment and Energy Management Agency), the European Community (through the FEDER European Funds for Regional Development), the region Hauts-de-France in the framework of the CERVIFER project; in part by the International Campus on Safety and Intermodality in Transportation, the Délégation Régionale à la Recherche et à la Technologie, the Ministère de l'Enseignement Supérieur et de la Recherche and the Centre National de la Recherche Scientifique (CNRS); in part by the Mexican Agency PRODEP via Project No. DSA/103.5/16/10200. The authors gratefully acknowledge the support of these institutions.

References

1. K. Tanaka, H. Wang, Fuzzy Control Systems Design and Analysis: a Linear Matrix Inequality Approach, New York: Wiley, Wiley-Interscience, 2001.
2. X. Su, L. Wu, P. Shi, Y.-D. Song, A novel approach to output feedback control of fuzzy stochastic systems, Automatica 50 (12) (2014) 3268–3275.

3. A.-T. Nguyen, M. Dambrine, J. Lauber, Simultaneous design of parallel distributed output feedback and anti-windup compensators for constrained Takagi-Sugeno fuzzy systems, Asian Journal of Control 18 (5) (2016) 1641–1654.
4. H.-K. Lam, E. Chan, Stability analysis of sampled-data fuzzy-model-based control systems, International Journal of Fuzzy Systems 10 (2) (2008) 129.
5. M.-C. Chen, W.-Y. Wang, S.-F. Su, Y.-H. Chien, Robust T-S fuzzy-neural control of uncertain active suspension systems, International Journal of Fuzzy Systems 12 (4) (2010) 321–329.
6. A.-T. Nguyen, J. Lauber, M. Dambrine, Robust $\mathcal{H}_\infty$ control design for switching uncertain system: Application for turbocharged gasoline air system control, in: 51st IEEE Conference on Decision and Control (CDC), 2012, pp. 4265–4270.
7. T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, IEEE Transactions on Systems, Man and Cybernetics SMC-15 (1) (1985) 116–132.
8. X. Su, P. Shi, L. Wu, Y.-D. Song, A novel control design on discrete-time Takagi-Sugeno fuzzy systems with time-varying delays, IEEE Transactions on Fuzzy Systems 21 (4) (2013) 655–671.
9. A.-T. Nguyen, T. Laurain, R. Palhares, J. Lauber, C. Sentouh, J.-C. Popieul, LMI-based control synthesis of constrained Takagi-Sugeno fuzzy systems subject to $\mathcal{L}_2$ and $\mathcal{L}_\infty$ disturbances, Neurocomputing 207 (2016) 793–804.
10. J.-T. Pan, T.-M. Guerra, S.-M. Fei, A. Jaadari, Nonquadratic stabilization of continuous T-S fuzzy models: LMI solution for a local approach, IEEE Transactions on Fuzzy Systems 20 (3) (2012) 594–602.
11. L. Wu, X. Yang, H. K. Lam, Dissipativity analysis and synthesis for discrete-time T-S fuzzy stochastic systemswith time-varying delay, IEEE Transactions on Fuzzy Systems 22 (2) (2014) 380–394.
12. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Vol. 15 of Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
13. V. Montagner, R. Oliveira, P. Peres, Convergent LMI relaxations for quadratic stabilizability and control of Takagi-Sugeno fuzzy systems, IEEE Transactions on Fuzzy Systems 17 (4) (2009) 863–873.
14. H. Tuan, P. Apkarian, T. Narikiyo, Y. Yamamoto, Parameterized linear matrix inequality techniques in fuzzy control system design, IEEE Transactions on Fuzzy Systems 9 (2) (2001) 324–332.
15. A.-T. Nguyen, M. Dambrine, J. Lauber, Lyapunov-based robust control design for a class of switching non-linear systems subject to input saturation: application to engine control, IET Control Theory & Applications 8 (2014) 1789–1802.
16. M. Ksantini, M. Hammami, F. Delmotte, On the global exponential stabilization of Takagi-Sugeno fuzzy uncertain systems, International Journal of Innovative Computing, Information and Control 11 (1) (2015) 281–294.
17. K. Tanaka, T. Hori, H. O. Wang, A multiple Lyapunov function approach to stabilization of fuzzy control systems, IEEE Transactions on Fuzzy Systems 11 (4) (2003) 582–589.
18. B.-J. Rhee, S. Won, A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design, Fuzzy Sets and Systems 157 (9) (2006) 1211–1228.
19. H. Lam, F. H. Leung, LMI-based stability and performance conditions for continuous-time nonlinear systems in Takagi-Sugeno's form, IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics) 37 (5) (2007) 1396–1406.
20. T. M. Guerra, M. Bernal, K. Guelton, S. Labiod, Non-quadratic local stabilization for continuous-time Takagi-Sugeno models, Fuzzy Sets and Systems 201 (2012) 40–54.
21. R. Márquez, T. M. Guerra, M. Bernal, A. Kruszewski, Asymptotically necessary and sufficient conditions for Takagi-Sugeno models using generalized non-quadratic parameter-dependent controller design, Fuzzy Sets and Systems (2015) –doi:http://dx.doi.org/10.1016/j.fss.2015.12.012.
22. T.-M. Guerra, L. Vermeiren, LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form, Automatica 40 (5) (2004) 823–829.
23. D. Lee, J. Park, Y. Joo, Improvement on non-quadratic stabilization of discrete-time Takagi-Sugeno fuzzy systems: Multiple-parameterization approach, IEEE Transactions on Fuzzy Systems 18 (2) (2010) 425–429.
24. A.-T. Nguyen, A. Dequidt, M. Dambrine, Anti-windup based dynamic output feedback controller design with performance consideration for constrained Takagi-Sugeno systems, Engineering Applications of Artificial Intelligence 40 (2015) 76–83.
25. S. Tarbouriech, G. Garcia, J. Gomes da Silva Jr., I. Queinnec, Stability and Stabilization of Linear Systems with Saturating Actuators, London: Springer-Verlag, 2011.
26. H. Gassara, A. El Hajjaji, M. Chaabane, Control of time delay polynomial fuzzy model subject to actuator saturation, International Journal of Fuzzy Systems (2016) –doi:10.1007/s40815-015-0133-1.
27. H. Du, N. Zhang, Fuzzy control for nonlinear uncertain electro-hydraulic active suspensions with input constraint, IEEE Transactions on Fuzzy Systems 17 (2) (2009) 343–356.
28. C.-S. Tseng, B.-S. Chen, $\mathcal{H}_\infty$ fuzzy control design for nonlinear systems subject to actuator saturation, in: Fuzzy Systems, 2006 IEEE International Conference on, IEEE, 2006, pp. 783–788.
29. Y.-Y. Cao, Z. Lin, Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation, IEEE Transactions on Fuzzy Systems 11 (1) (2003) 57–67.
30. C.-S. Ting, A robust fuzzy control approach to stabilization of nonlinear time-delay systems with saturating inputs, International Journal of Fuzzy Systems 10 (1) (2008) 50–60.
31. S.-H. Kim, C. Lee, J. Park, $\mathcal{H}_\infty$ state-feedback control for fuzzy systems with input saturation via fuzzy weighting-dependent Lyapunov functions, Computers & Mathematics with Applications 57 (6) (2009) 981–990.
32. K. Tanaka, H. Ohtake, H. O. Wang, A descriptor system approach to fuzzy control system design via fuzzy Lyapunov functions, IEEE Transactions on Fuzzy Systems 15 (3) (2007) 333–341.
33. K. Guelton, T. Bouarar, N. Manamanni, Robust dynamic output feedback fuzzy lyapunov stabilization of Takagi-Sugeno systems - a descriptor redundancy approach, Fuzzy sets and systems 160 (19) (2009) 2796–2811.
34. S. Bezzaoucha, B. Marx, D. Maquin, J. Ragot, Stabilization of nonlinear systems subject to uncertainties and actuator saturation, in: American Control Conference, IEEE, 2013, pp. 2403–2408.
35. D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou, A new robust D-stability condition for real convex polytopic uncertainty, Systems & Control Letters 40 (1) (2000) 21–30.
36. A. Sala, Arino, Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem, Fuzzy Sets and Systems 158 (24) (2007) 2671–2686.
37. L. A. Mozelli, R. M. Palhares, F. Souza, E. M. Mendes, Reducing conservativeness in recent stability conditions of T-S fuzzy systems, Automatica 45 (6) (2009) 1580–1583.
38. J. Löfberg, YALMIP: A toolbox for modeling and optimization in MATLAB, in: IEEE International Symposium on Computer Aided Control Systems Design, 2004, pp. 284–289.
39. H. J. Lee, J. B. Park, G. Chen, Robust fuzzy control of nonlinear systems with parametric uncertainties, IEEE Transactions on Fuzzy Systems 9 (2) (2001) 369–379.
40. A.-T. Nguyen, C. Sentouh, J.-C. Popieul, B. Soualmi, Shared lateral control with online adaptation of the automation degree for driver steering assist system: A weighting design approach, in: IEEE 54th annual Conference on Decision and Control, Osaka, Japan, 2015, pp. 857–862.