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LMI-based Control Synthesis of Constrained Takagi-Sugeno Fuzzy Systems Subject to \mathcal{L}_2 or \mathcal{L}_∞ Disturbances

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Abstract

This paper is devoted to the development of a new saturated non-parallel distributed compensation control law for disturbed Takagi-Sugeno fuzzy systems subject to both control input and state constraints. In order to cover a large range of real-world applications, both \mathcal{L}_2 and \mathcal{L}_∞ disturbances are considered which result in two different control design procedures. A parameter-dependent version of the generalized sector condition is effectively exploited in a fuzzy Lyapunov control framework to handle the control input saturation. Moreover, the proposed control method is based on the concept of robust invariant set which is able to provide an *explicit* characterization of the estimated domain of attraction of the closed-loop system. Different optimization algorithms are also proposed to deal with the trade-off between different closed-loop requirements in a local control context. The design conditions are expressed in terms of linear matrix inequalities which can be solved efficiently with available solvers. The numerical examples illustrate how the proposed methodology leads to less conservative results as well as less computational complexity when compared to very recent works in the literature.

Keywords: Takagi-Sugeno fuzzy systems, control input saturation, fuzzy Lyapunov functions, domain of attraction, non-parallel distributed compensation control scheme, linear matrix inequality.

1. Introduction

Nowadays, the stability analysis and control design based on Takagi-Sugeno (T-S) fuzzy models [1] have become the most popular research platform in fuzzy model-based control [2]. Indeed, over the past two decades tremendous investigations have been devoted to the study of T-S control systems [2–8]. This fact is due to many outstanding features of T-S fuzzy models for control purposes [3]. First, they can be used to approximate any smooth nonlinear system with any given accuracy. In particular, the sector nonlinearity approach provides an exact T-S representation of a given nonlinear model in a compact set of the state variables. Second, thanks to its polytopic structure with linear systems in consequent parts, T-S representation allows for some possible extensions of linear control techniques to nonlinear systems.

The direct Lyapunov method has been efficiently exploited to study the stability and control synthesis of T-S fuzzy systems [2, 6–10]. The derived conditions are expressed in terms of linear matrix inequalities (LMIs) [11] such that they are efficiently solvable with available numerical solvers. It is noteworthy that depending on the choice of the Lyapunov function, the derived conditions have different degrees of conservativeness. The following three types of Lyapunov functions has been mainly investigated in the T-S fuzzy control framework, namely quadratic, piecewise and parameter-dependent Lyapunov functions [2]. Despite the low-complexity of the derived conditions [3], quadratic Lyapunov functions lead generally to conservative results [12]. Piecewise Lyapunov functions [13] could be applied to overcome this major drawback. However, this type of Lyapunov functions requires the membership functions to induce a polyhedral partition of the state space. This fact is not compatible with T-S fuzzy models obtained from original nonlinear systems by using the sector nonlinearity approach [14]. As a consequent, piecewise Lyapunov functions can only be used to deal with nonlinear systems in the sense of approximation. Recent LMI alternative methodology to stability conditions considering piecewise Lyapunov functions

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is presented in [15]. However, control design conditions based on piecewise Lyapunov functions are in general expressed in terms of bilinear matrix inequalities [16, 17] which are hardly tractable with available numerical solvers. The effectiveness of parameter-dependent Lyapunov functions for stability analysis and control design has been demonstrated in [9, 18–21]. This type of Lyapunov functions seems to be the best alternative to solve all drawbacks of both previous ones, especially for discrete-time T-S fuzzy systems [8, 12, 19, 22].

Physical constraints such as control input saturation and system state constraints are ubiquitous in real-world applications due to safety and/or economic reasons. The presence of input saturation seriously degrades the closed-loop performance, in the extreme case, the stability may be lost [23]. However, this practical control issue has not been completely addressed for T-S fuzzy control systems [10, 24–26]. Some notable works can be cited as follows. In [27–29], a norm-bounded approach was used to deal with the actuator saturation. The resulting non-saturated controllers are generally very conservative and often lead to poor closed-loop performance [23, 30]. Descriptor representation approach [17] has been recently employed to deal with continuous-time input-saturated T-S fuzzy systems in [31]. It should be noted that this result is only applied to a restrictive class of T-S fuzzy systems with all linear subsystems being open-loop stable. The saturation function was represented in polytopic form to deal with input nonlinearity of continuous-time T-S systems [30, 32, 33], then extended to time-delay T-S systems [34] and a class of switching T-S systems [5, 25]. However, based on quadratic Lyapunov functions these results could be conservative. It should be stressed that state constraints were not considered in most of these works (except for [25]). Such type of constraints appears naturally when the sector nonlinearity approach is used to obtain T-S representation of nonlinear systems [35]. Explicit consideration of these limitations allows to prevent destabilizing initial conditions of the closed-loop systems [26]. Especially, this becomes crucial when disturbance signals are actively involved in the systems [10]. Recently, interesting non-quadratic boundedness approach has been also proposed in [24] to deal with T-S fuzzy systems subject to both control input and state constraints. Notice that the results proposed in [24] require several line searches to solve the design conditions which is costly in terms of computation. Moreover, slack decision variables have been intensively introduced in [24] to reduce the conservatism of the results. Therefore, the resulting design conditions are of high complexity and not suitable for high dimensional T-S systems or T-S systems with important number of subsystems. These facts will be clearly shown in Section 5 by means of a numerical example. It is also important to highlight that the method in [24] cannot deal with the case where T-S systems are subject to \mathcal{L}_2 disturbances.

Motivated by the above control issues, this paper is devoted to the development of a new input-saturated control law for disturbed T-S fuzzy systems subject to both control input and state constraints. Differently from [24], the proposed method is based on the concept of robust invariant set [36]. The main contributions of the new method can be summarized as follows:

- A parameter-dependent version of the generalized sector condition has been effectively exploited in the framework of fuzzy Lyapunov function based control design to handle the actuator saturation. This fact leads to less conservative design conditions with low computational complexity compared to existing works dealing with the same class of problem.
- The new method can provide an *explicit characterization* of the estimated domain of attraction of the closed-loop system which is not the case of [24].
- The proposed results can be applied to T-S systems subject to \mathcal{L}_∞ or \mathcal{L}_2 disturbances. Numerical examples illustrate that the proposed methodology can be applied to a large class of nonlinear systems and suitable for real-world-applications.

The paper is organized as follows. Section 2 formulates the control problem and some useful preliminaries are also presented. In Section 3, we develop new non-quadratic design conditions for two different cases corresponding to two types of disturbances affecting the constrained T-S systems. Optimization algorithms for different control design purposes are presented in Section 4. The interests of the proposed method are clearly demonstrated by means of examples in Section 5. Finally, Section 6 provides some concluding remarks.

Notation. For an integer number r , Ω_r denotes the set $\{1, 2, \dots, r\}$. I denotes the identity matrix of appropriate dimension. For any square matrix X , $\text{He}(X) = X + X^\top$. $X > 0$ means that the matrix X is positive definite. The i th element of a vector u is denoted $u_{(i)}$ and $X_{(i)}$ denotes the i th row of matrix X . (*) stands for matrix blocks that can be deduced by symmetry. For a positive definite function $\mathbb{V}(x)$ defined on \mathbb{R}^{n_x} , we denote $\mathcal{E}_{\mathbb{V},\rho} = \{x \in \mathbb{R}^{n_x} : \mathbb{V}(x) \leq \rho\}$ and $\mathcal{E}_{\mathbb{V}} \equiv \mathcal{E}_{\mathbb{V},1}$. The scalar functions η_i , $i \in \Omega_r$, are said to verify the convex sum property on a set \mathcal{D} , if $\eta_i(\theta) \geq 0$ and $\sum_{i=1}^r \eta_i(\theta) = 1$ for $\forall \theta \in \mathcal{D}$. For such functions and for matrices Y_i and Z_{ij} of appropriate dimensions, we denote

$$\begin{aligned} Y_\theta &= \sum_{i=1}^r \eta_i(\theta(t)) Y_i, & Z_{\theta\theta} &= \sum_{i=1}^r \sum_{j=1}^r \eta_i(\theta(t)) \eta_j(\theta(t)) Z_{ij} \\ Y_\theta^{-1} &= \left(\sum_{i=1}^r \eta_i(\theta(t)) Y_i \right)^{-1}, & Y_{\theta+} &= \sum_{i=1}^r \eta_i(\theta(t+1)) Y_i \end{aligned} \quad (1)$$

Throughout this paper, the time argument will be dropped when convenient.

2. Problem Formulation and Preliminaries

2.1. Problem Formulation

In this paper, the fuzzy model proposed in [1] is used to approximate and/or represent a given nonlinear system. This type of model is described by fuzzy IF-THEN rules which represent local linear input-output relations of a nonlinear system. The i th rules of the discrete-time T-S fuzzy system subject to control input saturation can be represented in the following form

$$\begin{aligned} &\text{Model rule } i : \\ &\text{IF} \quad \theta_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } \theta_p \text{ is } M_{ip} \\ &\text{THEN} \quad \begin{cases} x(t+1) = A_i x(t) + B_i^u \text{sat}(u(t)) + B_i^w w(t) \\ z(t) = C_i x(t) \end{cases} \end{aligned} \quad (2)$$

where $\text{sat}(u_{(l)}) = \text{sign}(u_{(l)}) \min(|u_{(l)}|, u_{\max(l)})$, $l \in \Omega_{n_u}$, and M_{ij} , $i \in \Omega_r$, $j \in \Omega_p$, is the fuzzy set and r is the number of model rules; $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the system disturbance, $z \in \mathbb{R}^{n_z}$ is the performance output, and $\theta = [\theta_1, \dots, \theta_p]^\top \in \mathbb{R}^p$ is the vector of premise variables. The real matrices A_i , B_i^u , B_i^w , C_i , $i \in \Omega_r$, are constant and of adequate dimensions. Then, the T-S fuzzy system is defined as follows

$$\begin{cases} x(t+1) = \sum_{i=1}^r \eta_i(\theta) (A_i x(t) + B_i^u \text{sat}(u(t)) + B_i^w w(t)) \\ z(t) = \sum_{i=1}^r \eta_i(\theta) C_i x(t) \end{cases} \quad (3)$$

where the normalized membership functions $\eta_i(\theta)$, $i \in \Omega_r$, are defined as

$$\eta_i(\theta) = \frac{\lambda_i(\theta)}{\sum_{j=1}^r \lambda_j(\theta)}, \quad \lambda_j(\theta) = \prod_{l=1}^p M_{lj}(\theta) \quad (4)$$

In (4), $M_{lj}(\theta)$ denotes the membership function of fuzzy set M_{lj} . It is worth noting that the normalized membership functions $\eta_i(\theta)$, $i \in \Omega_r$, satisfy the convex sum property.

Remark 1. T-S fuzzy system is a class of fuzzy systems where the consequent parts are *functions* of premise variables [1]. These functions can be linear or affine as most of the cases in fuzzy control framework [2, 3]. However, T-S fuzzy systems with local *nonlinear* models have been also studied in the literature, see for instance [26, 37]. It can be clearly observed that the input-saturated system (2) is

a special class of T-S fuzzy systems. This class of T-S systems has been widely investigated in different control contexts, see [5, 10, 24, 29, 32, 33] and references therein for more details. We note that using the sector nonlinearity approach [3], an *exact* T-S fuzzy representation (3) can be straightforwardly obtained from an input-saturated nonlinear system as will be illustrated later.

In this paper, the following assumptions will be considered for the control design of system (3).

Assumption 1. All components of the state and the scheduling vectors are measured. Notice that the scheduling variable vector may be a function of the state variables.

Assumption 2. The control input vector u is bounded in amplitude, that is

$$-u_{\max(l)} \leq u(l) \leq u_{\max(l)}, \quad l \in \Omega_{n_u} \quad (5)$$

where the control bound $u_{\max(l)} > 0$ is given.

Assumption 3. The domain of validity \mathcal{P}_x of the T-S fuzzy system (3) is defined as

$$\mathcal{P}_x = \left\{ x \in \mathbb{R}^{n_x} : h_m^\top x \leq 1, \quad m \in \Omega_q \right\} \quad (6)$$

where the vectors $h_m \in \mathbb{R}^{n_x}$ are given and can be computed from the state constraints of (3).

Assumption 4. The disturbance signal w belongs to one of the following classes of function

$$\mathcal{W}_\delta^2 = \left\{ w : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_w}, \quad \sum_{t=1}^{\infty} w^\top(t) w(t) \leq \delta \right\} \quad (7)$$

$$\mathcal{W}_\delta^\infty = \left\{ w : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_w}, \quad w^\top(t) w(t) \leq \delta, \quad t \geq 0 \right\} \quad (8)$$

where the bound $\delta > 0$ is given.

Let us consider the following non-parallel distributed compensation (non-PDC) control law:

$$u(t) = \left(\sum_{i=1}^r \eta_i(\theta) G_i \right) \left(\sum_{i=1}^r \eta_i(\theta) H_i \right)^{-1} x(t) \quad (9)$$

where $H_i, i \in \Omega_r$, are regular matrices. Using the notations defined in (1), the closed-loop system can be rewritten from (3) and (9) as follows

$$\begin{cases} x(t+1) = (A_\theta + B_\theta^u G_\theta H_\theta^{-1}) x(t) - B_\theta^u \psi(t) + B_\theta^w w(t) \\ z(t) = C_\theta x(t) \end{cases} \quad (10)$$

where $\psi(u) = u - \text{sat}(u)$.

This paper aims at proposing a systematic method to design non-PDC controllers (9) such that the closed-loop system (10) satisfies the following properties:

- **Property 1.** [Regional internal stability] There exists a positive definite function $\mathbb{V}(x)$ such that all closed-loop trajectories starting from the set $\mathcal{E}_\mathbb{V}$ converge exponentially to the origin in the absence of disturbances or the disturbances are vanishing ($w = 0$).
- **Property 2** [Input-to-state stability and disturbance attenuation] Given vectors $h_m \in \mathbb{R}^{n_x}, m \in \Omega_q$, as in Assumption 3 and a positive scalar $\delta > 0$. Depending on the type of disturbances involved in the dynamics of system (10), see Assumption 4, we distinguish two following control problems.

Control Problem 1. When $w \neq 0$ and $w \in \mathcal{W}_\delta^2$. There exist positive scalars ρ and γ such that, for $\forall x(0) \in \mathcal{E}_{\mathbb{V},\rho}$, the corresponding closed-loop trajectory of (10) remains inside the domain of validity \mathcal{P}_x defined in (6). Moreover, the \mathcal{L}_2 -norm of the output signal z is bounded as follows

$$\|z(t)\|_2 \leq \gamma \|w(t)\|_2 + \rho, \quad \forall t \geq 0.$$

Control Problem 2. When $w \neq 0$ and $w \in \mathcal{W}_\delta^\infty$. All closed-loop trajectories of (10) initialized inside the set \mathcal{E}_\forall will be confined in the domain of validity \mathcal{P}_x defined in (6). Moreover, there exists a positive scalar number γ such that the \mathcal{L}_∞ -norm of the output signal z is bounded by

$$\|z(t)\|_\infty \leq \gamma, \quad x(0) = 0, \quad \forall t \geq 0.$$

2.2. Preliminary Results

Hereafter, some useful preliminary results for the theoretical developments in Section 3 are presented.

Fact 1. [38] Given positive definite matrix Φ and any matrix M of appropriate dimensions, the following matrix inequality holds

$$M^\top \Phi M \geq M + M^\top - \Phi^{-1} \quad (11)$$

Lemma 1. Given matrices $G_i \in \mathbb{R}^{n_u \times n_x}$, $H_i \in \mathbb{R}^{n_x \times n_x}$ and $W_i \in \mathbb{R}^{n_u \times n_x}$, for $i \in \Omega_r$, let us define the following set:

$$\mathcal{P}_u = \left\{ x \in \mathbb{R}^{n_x} : \left| (G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x \right| \leq u_{\max(l)}, \quad l \in \Omega_{n_u} \right\} \quad (12)$$

If $x \in \mathcal{P}_u$, then the inequality on the dead-zone nonlinearity $\psi(u)$, where u is defined in (9):

$$\psi^\top(u) S_\theta^{-1} [\psi(u) - W_\theta H_\theta^{-1} x] \leq 0 \quad (13)$$

holds for any positive diagonal matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, and for any scalar functions $\eta_i(\theta)$, $i \in \Omega_r$, satisfying the convex sum property.

Proof. If $x \in \mathcal{P}_u$, then it can be deduced

$$-u_{\max(l)} \leq (G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x \leq u_{\max(l)}, \quad l \in \Omega_{n_u} \quad (14)$$

Notice that we have to show that

$$\psi^\top(u_{(l)}) \left(\sum_{i=1}^r \eta_i S_{i(l,l)} \right)^{-1} [\psi(u) - (W_\theta H_\theta^{-1}) x]_{(l)} \leq 0, \quad l \in \Omega_{n_u} \quad (15)$$

where $S_{i(l,l)}$, $i \in \Omega_r$, $l \in \Omega_{n_u}$, denotes the element of the l th row and l th column of matrix S_i . To this end, three possible cases according to the value of $u_{(l)}$ that may occur:

- **Case 1:** $-u_{\max(l)} \leq u_{(l)} \leq u_{\max(l)}$. It follows that $\psi(u_{(l)}) = 0$ and thus, (13) holds obviously.
- **Case 2:** $u_{(l)} > u_{\max(l)}$. Then,

$$\psi(u_{(l)}) = u_{(l)} - u_{\max(l)} = (G_\theta H_\theta^{-1})_{(l)} x - u_{\max(l)} > 0 \quad (16)$$

It follows from (14) that $(G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x \leq u_{\max(l)}$. Hence

$$\psi(u_{(l)}) - (W_\theta H_\theta^{-1})_{(l)} x = (G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x - u_{\max(l)} \leq 0 \quad (17)$$

Since $\psi(u_{(l)}) > 0$ in this case, then inequality (15) holds.

- **Case 3:** $u_{(l)} < -u_{\max(l)}$. Then,

$$\psi(u_{(l)}) = u_{(l)} + u_{\max(l)} = (G_\theta H_\theta^{-1})_{(l)} x + u_{\max(l)} < 0 \quad (18)$$

It follows from (14) that $(G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x \geq -u_{\max(l)}$. Hence,

$$\psi(u_{(l)}) - (W_\theta H_\theta^{-1})_{(l)} x = (G_\theta H_\theta^{-1} - W_\theta H_\theta^{-1})_{(l)} x + u_{\max(l)} \geq 0 \quad (19)$$

Combining the fact that $\psi(u_{(l)}) < 0$ in this case with inequality (19), it follows that (15) holds.

From the results of these three cases, the proof of Lemma 1 can be concluded. \square

Lemma 2. Let Υ_{ij}^k , $i, j, k \in \Omega_r$, be symmetric matrices of appropriate dimensions and $\{\nu_k\}_{k \in \Omega_r}$, $\{\omega_i\}_{i \in \Omega_r}$, be any families of functions satisfying the property of convex sum. The condition

$$\sum_{k=1}^r \sum_{i=1}^r \sum_{j=1}^r \nu_k \omega_i \omega_j \Upsilon_{ij}^k < 0$$

holds if

$$\begin{cases} \Upsilon_{ii}^k < 0, & i, k \in \Omega_r \\ \frac{2}{r-1} \Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, & i, j, k \in \Omega_r, \text{ and } i \neq j \end{cases} \quad (20)$$

The relaxation result in Lemma 2 is extended from the one in [39]. Other more efficient relaxation techniques can be found in [24, 40] at the expense of high computational costs.

3. Main Results

In this section, we derive LMI-based methods for designing controllers that can solve the control problems defined in Section 2. Two types of design conditions according to two classes of disturbances are provided.

3.1. Constrained T-S Fuzzy Systems Subject to \mathcal{L}_2 Disturbances

The following theorem provides sufficient conditions for designing a non-PDC controller (9) when T-S system (3) is subject to \mathcal{L}_2 disturbances.

Theorem 1. Given T-S fuzzy system (3) and a positive scalar δ where $w \in \mathcal{W}_\delta^2$. If there exist positive definite matrices $X_i \in \mathbb{R}^{n_x \times n_x}$, positive diagonal matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, matrices $H_i \in \mathbb{R}^{n_x \times n_x}$, $G_i \in \mathbb{R}^{n_u \times n_x}$, $W_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \Omega_r$, and positive scalars ρ, γ such that

$$\rho + \gamma\delta < 1 \quad (21)$$

$$\begin{bmatrix} H_i + H_i^\top - X_i & * \\ G_{i(l)} - W_{i(l)} & u_{\max(l)}^2 \end{bmatrix} > 0, \quad i \in \Omega_r, l \in \Omega_{n_u} \quad (22)$$

$$\begin{bmatrix} X_i & * \\ h_m X_i & 1 \end{bmatrix} \geq 0, \quad i \in \Omega_r, m \in \Omega_q \quad (23)$$

$$\Gamma_{ii}^k < 0, \quad i, k \in \Omega_r \quad (24)$$

$$\frac{2}{r-1} \Gamma_{ii}^k + \Gamma_{ij}^k + \Gamma_{ji}^k < 0, \quad i, j, k \in \Omega_r, \text{ and } i \neq j \quad (25)$$

where

$$\Gamma_{ij}^k = \begin{bmatrix} -H_i - H_i^\top + X_i & * & * & * & * \\ C_j H_i & -I & * & * & * \\ W_i & 0 & -2S_i & * & * \\ 0 & 0 & 0 & -\gamma I & * \\ A_j H_i + B_j^u G_i & 0 & -B_j^u S_i & B_j^w & -X_k \end{bmatrix} \quad (26)$$

Then, the non-PDC controller (9) solves Control Problem 1 stated in Section 2.

Proof. Using Lemma 2 with $\omega_i = \eta_i(\theta(t))$ and $\nu_k = \eta_k(\theta(t+1))$, $i, k \in \Omega_r$, it can be deduced from (23)-(24) that

$$\begin{bmatrix} -H_\theta - H_\theta^\top + X_\theta & * & * & * & * \\ C_\theta H_\theta & -I & * & * & * \\ W_\theta & 0 & -2S_\theta & * & * \\ 0 & 0 & 0 & -\gamma I & * \\ A_\theta H_\theta + B_\theta^u G_\theta & 0 & -B_\theta^u S_\theta & B_\theta^w & -X_{\theta+} \end{bmatrix} < 0 \quad (27)$$

Using the matrix property (11) of Fact 1 with $\mathbf{M} = H_\theta$ and $\Phi = X_\theta^{-1} = P_\theta$, and the well-known Schur complement lemma [11], inequality (27) implies clearly that

$$\begin{bmatrix} -H_\theta^\top P_\theta H_\theta + H_\theta^\top C_\theta^\top C_\theta H_\theta & * & * & * \\ W_\theta & -2S_\theta & * & * \\ 0 & 0 & -\gamma I & * \\ A_\theta H_\theta + B_\theta^u G_\theta & -B_\theta^u S_\theta & B_\theta^w & -X_{\theta+} \end{bmatrix} < 0 \quad (28)$$

By the congruence transformation with $\text{diag}(H_\theta^{-1}, S_\theta^{-1}, I, I)$, (28) is found to be equivalent to

$$\begin{bmatrix} -P_\theta + C_\theta^\top C_\theta & * & * & * \\ S_\theta^{-1} W_\theta H_\theta^{-1} & -2S_\theta^{-1} & * & * \\ 0 & 0 & -\gamma I & * \\ A_\theta + B_\theta^u G_\theta H_\theta^{-1} & -B_\theta^u & B_\theta^w & -X_{\theta+} \end{bmatrix} < 0 \quad (29)$$

Using again Schur complement lemma, inequality (29) can be proved to be equivalent to

$$\Xi^\top P_{\theta+} \Xi + \begin{bmatrix} -P_\theta + C_\theta^\top C_\theta & * & * \\ S_\theta^{-1} W_\theta H_\theta^{-1} & -2S_\theta^{-1} & * \\ 0 & 0 & -\gamma I \end{bmatrix} < 0 \quad (30)$$

where $\Xi = [A_\theta + B_\theta^u G_\theta H_\theta^{-1} \quad -B_\theta^u \quad B_\theta^w]$ and $P_{\theta+} = X_{\theta+}^{-1}$. Notice that if one pre- and post-multiplies (30) by vector $[x^\top \quad \psi^\top(u) \quad w^\top]$ and its transpose, the following inequality can be obtained after some algebraic manipulations

$$\begin{aligned} \mathbb{V}(x(t+1)) - \mathbb{V}(x(t)) + z^\top(t) z(t) - \gamma w^\top(t) w(t) \dots \\ - 2\psi^\top(u) S_\theta^{-1} [\psi(u) - W_\theta H_\theta^{-1} x(t)] < 0 \end{aligned} \quad (31)$$

where the positive definite function $\mathbb{V}(x(t))$ is defined as

$$\mathbb{V}(x(t)) = x^\top(t) \sum_{i=1}^n \eta_i(\theta) P_i x(t) = x^\top(t) P_\theta x(t) \quad (32)$$

Moreover, if condition (22) is verified, then it follows clearly that matrices $H_i, i \in \Omega_r$, are regular since $X_i > 0$. By Schur complement lemma and matrix property (11), it can be deduced from (22) that

$$H_\theta^\top P_\theta H_\theta - \frac{(G_{\theta(l)} - W_{\theta(l)})^\top (G_{\theta(l)} - W_{\theta(l)})}{u_{\max(l)}^2} \geq 0 \quad (33)$$

Pre- and post- multiplying (33) with the regular matrix $H_\theta^{-\top}$ yields

$$P_\theta - \frac{(G_{\theta(l)} H_\theta^{-1} - W_{\theta(l)} H_\theta^{-1})^\top (G_{\theta(l)} H_\theta^{-1} - W_{\theta(l)} H_\theta^{-1})}{u_{\max(l)}^2} \geq 0 \quad (34)$$

Then, it is easily observed that condition (34) implies the inclusion $\mathcal{E}_\mathbb{V} \subseteq \mathcal{P}_u$. Similarly, condition (22) implies $\mathcal{E}_\mathbb{V} \subseteq \mathcal{P}_x$. Since $\mathcal{E}_\mathbb{V} \subseteq \mathcal{P}_u \cap \mathcal{P}_x$, by Lemma 1 it follows from (31) that

$$\mathbb{V}(x(t+1)) - \mathbb{V}(x(t)) + z^\top(t) z(t) - \gamma w^\top(t) w(t) < 0 \quad (35)$$

From now, two cases can be distinguished.

1. If $w = 0$, it follows from (35) that

$$\Delta \mathbb{V} < 0, \quad \forall x \in \mathcal{E}_\mathbb{V} \quad (36)$$

where $\Delta \mathbb{V} = \mathbb{V}(x(t+1)) - \mathbb{V}(x(t))$. Condition (36) ensures that for $\forall x(0) \in \mathcal{E}_\mathbb{V}$, the corresponding trajectory converges asymptotically to the origin.

2. If $w \neq 0$ and $w \in \mathcal{W}_\delta^2$, summing both sides of inequality (35) from 0 to the T_f -th instant yields

$$\mathbb{V}(x(T_f)) < \mathbb{V}(x(0)) + \gamma \sum_{t=1}^{T_f} w^\top(t) w(t) - \sum_{t=1}^{T_f} z^\top(t) z(t), \quad \forall x \in \mathcal{E}_\mathbb{V} \quad (37)$$

From (21) and (37), it follows that $\mathbb{V}(x(T_f)) < \rho + \gamma\delta < 1$. Thus, we can conclude that every closed-loop trajectory starting from $\mathcal{E}_{\mathbb{V}\rho}$ will be confined in the set $\mathcal{E}_\mathbb{V}$. In addition, from (37) and considering the limit case $T_f \rightarrow \infty$, we obtain $\sum_{t=1}^{\infty} z^\top(t) z(t) < \mathbb{V}(x(0)) + \gamma \sum_{t=1}^{\infty} w^\top(t) w(t)$, which means that the \mathcal{L}_2 -norm of the output signal is bounded: $\|z\|_2 < \gamma \|w\|_2 + \rho$.

Then, the desired results of Theorem 1 follow. \square

Remark 2. It should be noticed from Theorem 1 that there is a trade-off between the size of the set of admissible initial conditions $\mathcal{E}_{\mathbb{V}\rho}$ and the maximal energy level δ of the disturbances when $x(0) \neq 0$ [25]. Concretely, the lower is the admissible δ , the larger is the estimate of the domain of attraction $\mathcal{E}_\mathbb{V}$.

3.2. Constrained T-S Fuzzy Systems Subject to \mathcal{L}_∞ Disturbances

The following theorem provides conditions for designing a non-PDC controller (9) when T-S fuzzy system (3) is subject to \mathcal{L}_∞ disturbances.

Theorem 2. Given T-S fuzzy system (3) and positive scalars $\delta, \tau_1 < 1$, where $w \in \mathcal{W}_\delta^\infty$. If there exist positive definite matrices $X_i \in \mathbb{R}^{n_x \times n_x}$, positive diagonal matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, matrices $H_i \in \mathbb{R}^{n_x \times n_x}$, $G_i \in \mathbb{R}^{n_u \times n_x}$, $W_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \Omega_r$, and positive scalars γ, τ_2 such that conditions (22)-(23) hold and

$$\tau_1 - \tau_2\delta > 0 \quad (38)$$

$$\begin{bmatrix} H_i + H_i^\top - X_i & * \\ C_j H_i & \gamma I \end{bmatrix} \geq 0, \quad i, j \in \Omega_r \quad (39)$$

$$\Psi_{ii}^k < 0, \quad i, k \in \Omega_r \quad (40)$$

$$\frac{2}{r-1} \Psi_{ii}^k + \Psi_{ij}^k + \Psi_{ji}^k < 0, \quad i, j, k \in \Omega_r, \text{ and } i \neq j \quad (41)$$

where

$$\Psi_{ij}^k = \begin{bmatrix} (\tau_1 - 1)(H_i + H_i^\top - X_i) & * & * & * \\ W_i & -2S_i & * & * \\ 0 & 0 & -\tau_2 I & * \\ A_j H_i + B_j^u G_i & -B_j^u S_i & B_j^w & -X_k \end{bmatrix} \quad (42)$$

Then, the non-PDC controller (9) solves Control Problem 2 stated in Section 2.

Proof. Following the same line as in the proof of Theorem 1, conditions (22)-(23) and (40)-(41) imply the satisfaction of the following inequality:

$$\begin{aligned} \mathbb{V}(x(t+1)) + (\tau_1 - 1)\mathbb{V}(x(t)) - \tau_2 w^\top(t) w(t) \dots \\ - 2\psi^\top(u) S_\theta^{-1} [\psi(u) - W_\theta H_\theta^{-1} x(t)] < 0, \quad \forall x \in \mathcal{E}_\mathbb{V} \end{aligned} \quad (43)$$

where the positive definite function $\mathbb{V}(x(t))$ is defined in (32). By Lemma 1, it follows from (43) that

$$\mathbb{V}(x(t+1)) + (\tau_1 - 1)\mathbb{V}(x(t)) - \tau_2 w^\top(t) w(t) < 0, \quad \forall x \in \mathcal{E}_\mathbb{V} \subseteq \mathcal{P}_u \quad (44)$$

We examine the two following cases.

1. If $w = 0$, it can be deduced from (44) that

$$\Delta\mathbb{V} < -\tau_1 \mathbb{V}(x(t)), \quad \forall x \in \mathcal{E}_\mathbb{V} \quad (45)$$

which means that all closed-loop trajectories starting from the set $\mathcal{E}_\mathbb{V}$ converge asymptotically to the origin with a decay rate τ_1 .

2. If $w \neq 0$ and $w \in \mathcal{W}_\delta^\infty$, the satisfaction of (38) and (44) implies clearly that

$$\Delta \mathbb{V} + \tau_1 (\mathbb{V}(x(t)) - 1) - \tau_2 \left(\delta - w^\top(t) w(t) \right) < 0, \quad \forall x \in \mathcal{E}_\mathbb{V} \quad (46)$$

Condition (46) guarantees that the set $\mathcal{E}_\mathbb{V}$ is robustly positively invariant [36] with respect to the closed-loop system (10). Moreover, the satisfaction of condition (39) implies

$$\begin{bmatrix} P_\theta & * \\ C_\theta & \gamma I \end{bmatrix} \geq 0 \quad (47)$$

From (47), it is easily deduced that

$$z^\top z = x^\top C_\theta^\top C_\theta x \leq \gamma x^\top P_\theta x \leq \gamma, \quad \forall x \in \mathcal{E}_\mathbb{V} \quad (48)$$

which means that the \mathcal{L}_∞ -norm of the output signal is bounded: $\|z\|_\infty \leq \gamma$.

The proof of Theorem 2 can be now concluded. \square

Remark 3. It is noteworthy that differently from Theorem 2, the set $\mathcal{E}_\mathbb{V}$ obtained with design conditions in Theorem 1 is not invariant with respect to the closed-loop system (10). Indeed, if $\forall x(0) \in \mathcal{E}_\mathbb{V}$ and $\forall x(0) \notin \mathcal{E}_{\mathbb{V}\rho}$, then it is not possible to guarantee that the corresponding closed-loop trajectory remains inside the set $\mathcal{E}_\mathbb{V}$ for $\forall w \in \mathcal{W}_\delta^2$ defined in (7).

Remark 4. The decay rate τ_1 is related to the time-domain performance of the closed-loop system [3]. Therefore, it can be regarded as a tuning parameter for the proposed controller. Due to the multiplication of τ_1 and other decision variables (such as X_i), the design conditions of Theorem 2 are a set of linear matrix inequalities with a line search over the scalar τ_1 . From numerical point of view, a line search for τ_1 can be effectively done with 100 points linearly gridded in the interval $[0, 1)$. Then, a finite set of LMI constraint problems with $\tau_1 \in \{0, 0.01, 0.02, \dots, 0.99\}$ will be solved. Therefore, τ_1 is simply a parameter for LMI optimization problem. Notice that similar constructive numerical procedure can be found in [41].

Remark 5. The control design conditions in Theorems 1 and 2 are based on the choice of parameter-dependent Lyapunov function (32). This type of Lyapunov functions allows for an LMI formulation. Moreover, it can help to reduce effectively the conservatism of the results compared to those obtained with common quadratic Lyapunov function $\mathbb{V}(x(t)) = x^\top(t) P x(t)$. Indeed, the latter is simply a special case of (32) by imposing $P_i = P$ with $i \in \Omega_r$. It should be also noticed that the proposed method can be easily generalized by using more complex parameter-dependent Lyapunov functions [12, 19, 20] to reduce further the conservatism at the expense of computational cost.

4. Optimization Algorithms for Control Design

Both Theorems 1 and 2 provide LMI conditions to check easily the feasibility of the control problems defined in Section 2. Since the results are developed for local stability and performance, there is a trade-off between different closed-loop requirements. According to the design purposes, two following optimization problems can be formulated.

Optimization Problem 1. *Find a controller (9) solving the control problems defined in Section 2 such that the estimate of the domain of attraction $\mathcal{E}_\mathbb{V}$ is maximized. \diamond*

It should be noticed that this control objective can be done easily in the quadratic Lyapunov framework by minimizing the trace of the Lyapunov matrix [11]. However, the estimated domain of attraction in this paper is *non-convex* due to its associate fuzzy Lyapunov function (32). To overcome this obstacle, we propose here to maximize the largest trace (X_k), $k \in \Omega_r$. This leads to the following optimization problems to solve Optimization Problem 1.

- For constrained T-S fuzzy systems (3) subject to \mathcal{L}_2 disturbances:

$$\max \quad \mu \quad (49)$$

subject to $\mu > 0$ and

$$\text{trace}(X_i) \leq \mu, \quad i \in \Omega_r, \quad \text{LMI conditions (21)-(25) in Theorem 1.}$$

- For constrained T-S fuzzy systems (3) subject to \mathcal{L}_∞ disturbances:

$$\max \quad \mu \quad (50)$$

subject to $\mu > 0$ and

$$\text{trace}(X_i) \leq \mu, \quad i \in \Omega_r, \quad \text{LMI conditions (22)-(23) and (38)-(41) in Theorem 2.}$$

Optimization Problem 2. Find a controller (9) solving the control problems defined in Section 2 such that for a given bound δ on the \mathcal{L}_2 -norm (respectively \mathcal{L}_∞ -norm) of admissible disturbances, the effect of the disturbances on closed-loop system (10) is minimized. \diamond

This problem can be straightforwardly solved with the following optimizations.

- For constrained T-S fuzzy systems (3) subject to \mathcal{L}_2 disturbances:

$$\min \quad \gamma \quad (51)$$

subject to

$$\text{LMI conditions (21)-(25) in Theorem 1.}$$

- For constrained T-S fuzzy systems (3) subject to \mathcal{L}_∞ disturbances:

$$\min \quad \gamma \quad (52)$$

subject to

$$\text{LMI conditions (22)-(23) and (38)-(41) in Theorem 2.}$$

Notice that the optimization problems in (49), (50), (51), and (52) are *convex* and can be effectively solved with available LMI numerical solvers.

Remark 6. It is noteworthy that if an LMI solver based on interior point methods [11], the computational cost of the LMI optimization problem can be estimated as being proportional to $\mathcal{N}_{\text{var}}^3 \mathcal{N}_{\text{row}}$, where \mathcal{N}_{var} is the total number of scalar decision variables and \mathcal{N}_{row} the total row size of the LMIs [42]. Concerning the proposed algorithms, these numbers are evaluated as follows

- For design conditions of Theorem 1

$$\begin{cases} \mathcal{N}_{\text{row}} = 2 + 2rqn_x + rn_u(n_x + 1) + r^3(2n_x + n_u + n_w) \\ \mathcal{N}_{\text{var}} = 1 + rn_x(n_x + 2n_u) + n_x(n_x + 1)/2 + n_u(n_u + 1)/2 \end{cases} \quad (53)$$

- For design conditions of Theorem 2

$$\begin{cases} \mathcal{N}_{\text{row}} = 2 + 2rqn_x + rn_u(n_x + 1) + r^3(2n_x + n_u + n_w + n_z) \\ \mathcal{N}_{\text{var}} = 2 + rn_x(n_x + 2n_u) + n_x(n_x + 1)/2 + n_u(n_u + 1)/2 \end{cases} \quad (54)$$

Computational complexity of design conditions plays an important role in practice for both design and implementation of controllers. In the next section, we will evaluate such complexity between different design methods.

5. Numerical Examples

In this section, the effectiveness of the proposed method is demonstrated with two examples. The first one is used to show the motivations for considering the system constraints in the control design for real-world applications. The second example aims at studying the conservativeness and also the complexity of the proposed method compared to existing works. The characterization of the estimated domain of attraction is also illustrated in this example. All LMI optimizations are done with LMI Control Toolbox [42] in Matlab R2011b with the vector `OPTIONS` equal to `[0 100 10e9 10 0]`.

Example 1. Consider the following truck-trailer control problem [43–45]

$$\begin{cases} x_1(t+1) = (1 - vT/L)x_1(t) + (vT/l)\text{sat}(u(t)) \\ x_2(t+1) = (vT/L)x_1(t) + x_2(t) + 0.2w(t) \\ x_3(t+1) = x_3(t) + vT\sin(\theta(t)) + 0.1w(t) \\ z(t) = 7x_1(t) - 2x_2(t) + 0.03x_3(t) \end{cases} \quad (55)$$

where $x_1(t)$ is the difference of angle between the truck and the trailer, $x_2(t)$ the angle of the trailer compare to the horizontal axe, $x_3(t)$ the position related to the vertical axe of the back of the trailer, $u(t)$ the steering angle that can be applied to the front wheels of the truck, T is the sampling time and $\theta(t) = x_2(t) + (vT/2L)x_1(t)$. The model parameters are given as: $l = 2.8m$, $L = 5.5m$, $v = -1.0m/s$, and $T = 2s$.

Following the same modeling procedure as in [43, 45] the nonlinearity $\sin(\theta)$ can be exactly represented as follows

$$\sin(\theta) = \theta\eta_1(\theta) + g\theta\eta_2(\theta)$$

for $-179.427^\circ < \theta < 179.427^\circ$ and $g = 10^{-2}/\pi$. The membership functions are given by

$$\eta_1(\theta) = 1 - \eta_2(\theta), \quad \eta_2(\theta) = \begin{cases} \frac{\theta - \sin(\theta)}{\theta(1-g)}, & \theta \neq 0 \\ 0, & \theta = 0 \end{cases} \quad (56)$$

Then, the nonlinear system in (55) can be exactly represented by T-S fuzzy model (3) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 - vT/L & 0 & 0 \\ vT/L & 1 & 0 \\ v^2T^2/(2L) & vT & 1 \end{bmatrix}, \quad B_1^u = \begin{bmatrix} vT/l \\ 0 \\ 0 \end{bmatrix}, \quad B_1^w = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad C_1^\top = \begin{bmatrix} 7 \\ -2 \\ 0.03 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 1 - vT/L & 0 & 0 \\ vT/L & 1 & 0 \\ gv^2T^2/(2L) & gvT & 1 \end{bmatrix}, \quad B_2^u = \begin{bmatrix} vT/l \\ 0 \\ 0 \end{bmatrix}, \quad B_2^w = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad C_2 = C_1 \end{aligned} \quad (57)$$

Notice that system (55) is open-loop unstable and in contrast to previous works [43–45], the following limitations on the energy of the disturbance and the control input are *explicitly* considered in the control design procedure

$$\delta \leq 6, \quad u_{\max} = 1.$$

Moreover, with the expression of $\theta(t)$ and the imposed condition $-179.427^\circ < \theta < 179.427^\circ$ to obtain the exact T-S representation (56)-(57) of the truck-trailer system (55), the state constraints of the form (3) can be easily deduced. Note that these inherent constraints were not considered in the control design in [43–45]. Solving Optimization Problem 2 with conditions in (51) provides feasible solution with $\gamma = 0.1629$ and

$$\begin{aligned} G_1 &= [0.531 \quad 1.663 \quad -3.053], \quad H_1 = \begin{bmatrix} 0.856 & 2.874 & -2.871 \\ 2.878 & 11.446 & 92.949 \\ -2.434 & 94.678 & 6.868 \times 10^3 \end{bmatrix} \\ G_2 &= [0.578 \quad 1.839 \quad -2.042], \quad H_2 = \begin{bmatrix} 0.877 & 2.945 & -3.019 \\ 2.898 & 11.481 & 91.128 \\ -6.122 & 80.314 & 6.779 \times 10^3 \end{bmatrix} \end{aligned} \quad (58)$$

For simulation purposes, the energy-bounded disturbance $w(t)$ is selected as a constant of 2.8 happening at $t = 15s$ with duration 2.1s. Applying the controller gains given in (58), the closed-loop responses corresponding to the initial condition $x_0^\top = [\pi/2 \ 3\pi/4 \ -10]$ are indicated in Figure 1. Two phases can be distinguished for this simulation result. For the first phase from $t = 0s$ to $t = 15s$, the disturbance is not involved in the dynamics of the truck-trailer system, and all three state variables converge to the origin. The second phase is from $t = 15s$ till the end of the simulation. It can be observed that at $t = 15s$, the considered \mathcal{L}_2 disturbance $w(t)$ begins to act on the system dynamics for a duration of 2.1s (see the dashdot red line in Figure 1), the system state variables are therefore perturbed. However, the proposed robust controller is able to effectively reject the disturbance effect and all states converge again at the end of the simulation.

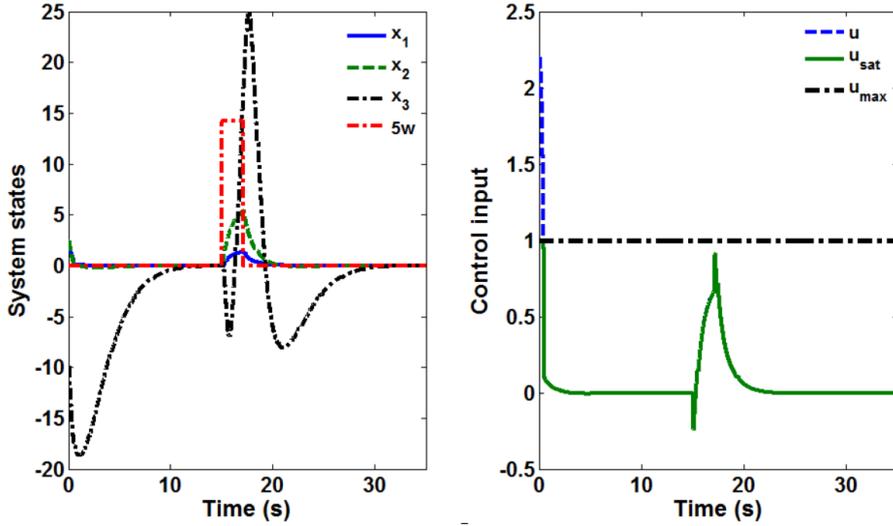


Figure 1: Closed-loop behaviors of (55) corresponding to the initial condition $x_0^\top = [\pi/2 \ 3\pi/4 \ -10]$ and the non-PDC controller with matrices given in (58).

It should be stressed that existing controllers provided in [12, 43, 45] are not able to stabilize the closed-loop system (55) in this case, see Figure 2. These facts confirm strongly the interests of considering *explicitly* the actuator saturation and the state constraints in the control design procedure. Notice also that the results proposed in [24] for constrained T-S systems subject to \mathcal{L}_∞ disturbances cannot be *rigorously* applied to this case where the involved disturbance $w(t)$ belongs to a class of \mathcal{L}_2 signal.

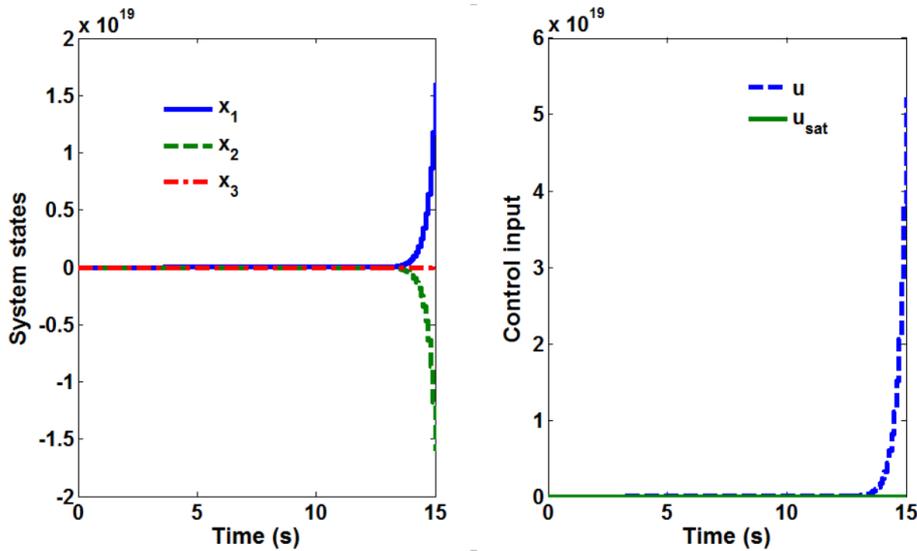


Figure 2: Closed-loop behaviors of system (55) corresponding to initial condition $x_0^\top = [\pi/2 \ 3\pi/4 \ -10]$ obtained with design conditions in [45].

Example 2. Let us consider the following constrained nonlinear system borrowed from [12, 24]:

$$\begin{cases} x_1(t+1) = x_1(t) - x_1(t)x_2(t) + (5 + x_1(t)) \text{sat}(u(t)) + 0.5x_1(t)w(t) \\ x_2(t+1) = -x_1(t) - 0.5x_2(t) + 2x_1(t) \text{sat}(u(t)) \\ z(t) = x_1(t) \end{cases} \quad (59)$$

for $-\beta \leq x_1(t) \leq \beta$ with $\beta > 0$ and $u_{\max} = 1$. By choosing the premise variable as $\theta(t) = x_1(t)$ and using the sector nonlinearity approach, one has

$$x_1(t) = \eta_1(x_1) \beta - \eta_2(x_1) \beta$$

where the membership functions $\eta_1(x_1)$ and $\eta_2(x_1)$ are given by

$$\eta_1(x_1) = \frac{x_1 + \beta}{2\beta}, \quad \eta_2(x_1) = 1 - \eta_1(x_1).$$

Then, the nonlinear system (59) can be *exactly* represented in the compact set $\mathcal{P}_x = [-\beta, \beta]$ of the variable $x_1(t)$ by the following two rules discrete-time T-S fuzzy model:

Model rule 1:

$$\begin{array}{l} \text{If } x_1(t) \text{ is } \eta_1(x_1) \\ \text{Then } \begin{cases} x(t+1) = A_1x(t) + B_1^u \text{sat}(u(t)) + B_1^w w(t) \\ z(t) = C_1x(t) \end{cases} \end{array} \quad (60)$$

Model rule 2:

$$\begin{array}{l} \text{If } x_1(t) \text{ is } \eta_2(x_1) \\ \text{Then } \begin{cases} x(t+1) = A_2x(t) + B_2^u \text{sat}(u(t)) + B_2^w w(t) \\ z(t) = C_2x(t) \end{cases} \end{array}$$

The system matrices of the open-loop unstable T-S fuzzy system (60) are given as follows

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix}, B_1^u = \begin{bmatrix} 5 + \beta \\ 2\beta \end{bmatrix}, B_1^w = \begin{bmatrix} \beta/2 \\ 0 \end{bmatrix}, C_1 = [1 \quad 0] \\ A_2 &= \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix}, B_2^u = \begin{bmatrix} 5 - \beta \\ -2\beta \end{bmatrix}, B_2^w = \begin{bmatrix} -\beta/2 \\ 0 \end{bmatrix}, C_2 = [1 \quad 0] \end{aligned} \quad (61)$$

Assume also that the T-S system (60) is subject to amplitude-bounded disturbance $w(t) = 0.5 \sin(t)$. This example aims at studying the conservatism together with the complexity of the proposed method compared to recent work [24] dealing with the same control context, namely constrained T-S systems subject to \mathcal{L}_∞ disturbances in Theorem 2. Table 1 shows the maximal β , denoted by β^* , for which stabilizing controller can be computed from different methods and also the numbers characterizing the complexity of these methods.

Table 1: Comparison between different design methods

Design conditions	Theorem 2	Theorem 2 in [24]
β^*	1.68	1.55
\mathcal{N}_{row}	72	168
\mathcal{N}_{var}	25	581

It can be observed from Table 1 that the new method provides not only less conservative results but also much less complex design conditions in terms of computational cost. The general formulas to compute the numbers characterizing complexity of the LMI optimization problem of the method in [24] are given by

$$\begin{cases} \mathcal{N}_{\text{row}} = r^2(2n_x + n_w) \left(2 + 3r(r-1)/2 + (r(r-1)/2)^2 \right) + \dots \\ \quad + 2r(3n_x + n_u + n_w) + r^2(r-1)(3n_x + n_u + n_w)/2 \\ \mathcal{N}_{\text{var}} = 1 + n_x(n_x + 1)/2 + n_u(n_u + 1)/2 + rn_x(n_x + n_u) + \dots \\ \quad + r^2 \left(r^2 + (n_x + n_u)^2 + n_x(n_x + 1)/2 + (2n_x + n_w)^2 \right) + r^4(2n_x + n_w)^2 \end{cases} \quad (62)$$

Compared to (54), we can see from (62) that the computational complexity of the method in [24] becomes *excessively* important with high dimension T-S fuzzy systems and/or T-S fuzzy systems with important number of sub-systems. This fact is also a major drawback of [24] for real-world applications.

Now, taking $\beta = 1.6$ (for which the design conditions in [24] are *infeasible*), solving LMI optimization problem in (50) with $\tau_1 = 0.15$ for system (61) yields $\mu = 0.5779$ and

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.512 & 0.148 \\ 0.148 & 0.182 \end{bmatrix}, & H_1 &= \begin{bmatrix} 2.675 & -1.636 \\ -2.036 & 8.809 \end{bmatrix}, & G_1 &= [-0.728 \quad 2.178] \\ P_2 &= \begin{bmatrix} 0.393 & -0.018 \\ -0.018 & 0.211 \end{bmatrix}, & H_2 &= \begin{bmatrix} 2.022 & 0.339 \\ -0.958 & 5.029 \end{bmatrix}, & G_2 &= [0.023 \quad -2.213] \end{aligned} \quad (63)$$

Figure 3 depicts the guaranteed domain of attraction and the phase portrait of the closed-loop system (61) obtained with the proposed method. It can be observed that this domain has clearly *non-quadratic* form which is maximized inside the intersection $\mathcal{P}_x \cap \mathcal{P}_u$. Moreover, it can be observed from the phase portrait that as expected the controller with matrices given in (63) provides stable behaviors for all trajectories initialized inside the guaranteed domain of attraction. It is worth noting that the method in [24] does not provide any characterization on the estimate of the domain of attraction.

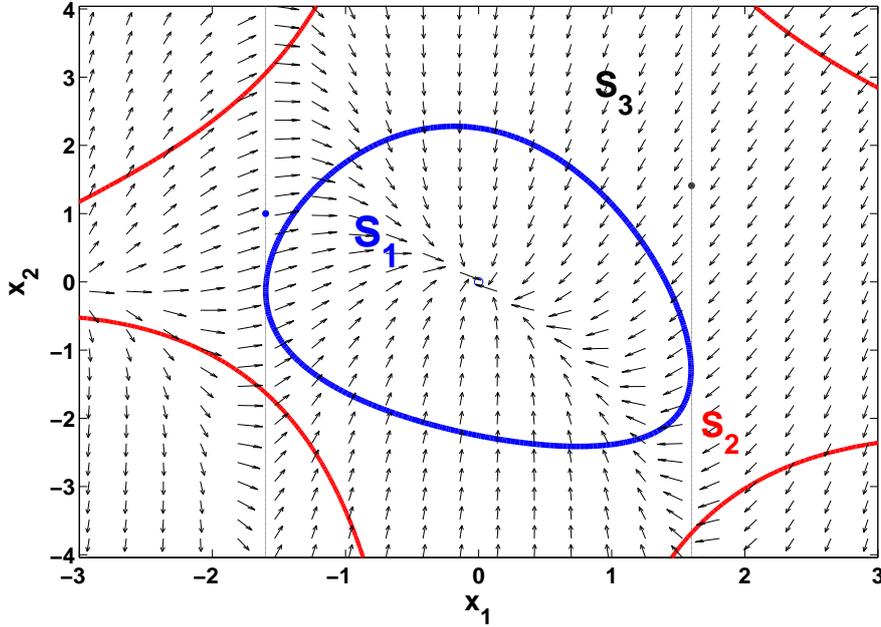


Figure 3: Results obtained with $\beta = 1.6$: guaranteed domain of attraction $S_1 = \mathcal{E}_v$, generalized sector condition for input saturation $S_2 = \mathcal{P}_u$, domain of validity $S_3 = \mathcal{P}_x$.

Now, let us illustrate further the effectiveness of the results obtained with Theorem 2 by examining two following *extreme* cases.

- **Case 1.** Constrained T-S system (61) with maximal $\beta_1^* = 1.55$ that can be obtained with [24].
- **Case 2.** Constrained T-S system (61) with maximal $\beta_2^* = 1.68$ that can be given by Theorem 2. Of course, conditions in [24] are *all* infeasible in this case.

It is worth noting that for Case 1 the maximal decay rate can be obtained with Theorem 2 in this paper (respectively Theorem 2 in [24]) is $\tau_1 = 0.47$ (respectively $\tau_1 = 0.11$). This fact confirms again that the proposed design conditions are less conservative than those in [24] and the resulting controller can provide faster closed-loop time convergence than that in [24], see Remark 4. Applying Theorem 2 for Case 1 with $\tau_1 = 0.47$, one has

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.803 & 0.386 \\ 0.386 & 0.232 \end{bmatrix}, & H_1 &= \begin{bmatrix} 0.677 & -0.033 \\ -0.897 & 7.510 \end{bmatrix}, & G_1 &= [-0.259 \quad 1.857] \\ P_2 &= \begin{bmatrix} 0.772 & -0.116 \\ -0.116 & 0.234 \end{bmatrix}, & H_2 &= \begin{bmatrix} 0.903 & 0.642 \\ -0.362 & 4.568 \end{bmatrix}, & G_2 &= [-0.065 \quad -1.943] \end{aligned} \quad (64)$$

The closed-loop behaviors of T-S system (61) obtained with matrices for the controller given in (64) and initial condition $x_0^\top = [\beta_1^* \quad -4]$ is depicted in Figure 4. It can be clearly observed that controller with matrices given in (64) induces stable closed-loop behaviors and the disturbance is effectively rejected in this case. Notice also that existing controllers in [12, 43, 45] lead to unstable behaviors under the same conditions since system constraints were not *explicitly* considered in the control design for these controllers, see Figure 5.

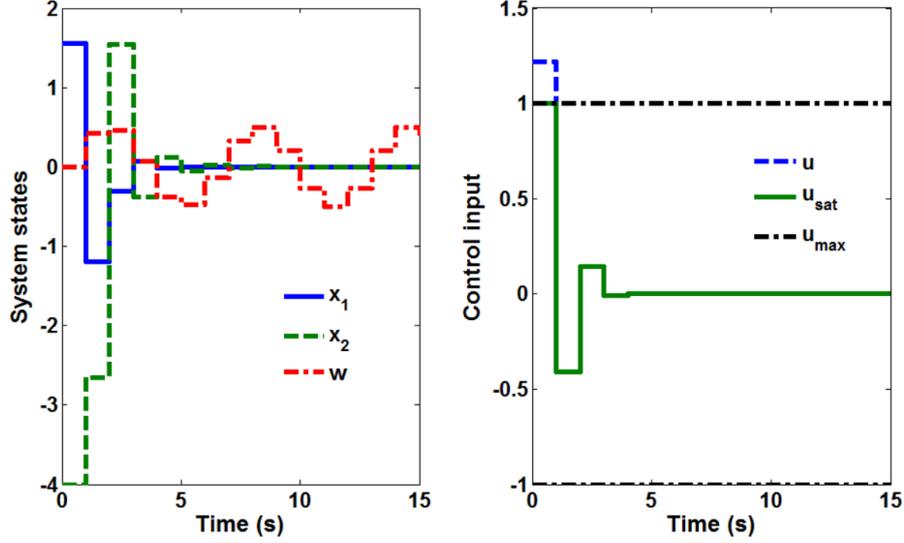


Figure 4: Case 1 with $\beta_1^* = 1.55$. Closed-loop behaviors of (61) corresponding to the initial condition $x_0^\top = [\beta_1^* \quad -4]$ obtained for the non-PDC controller with matrices given in (64).

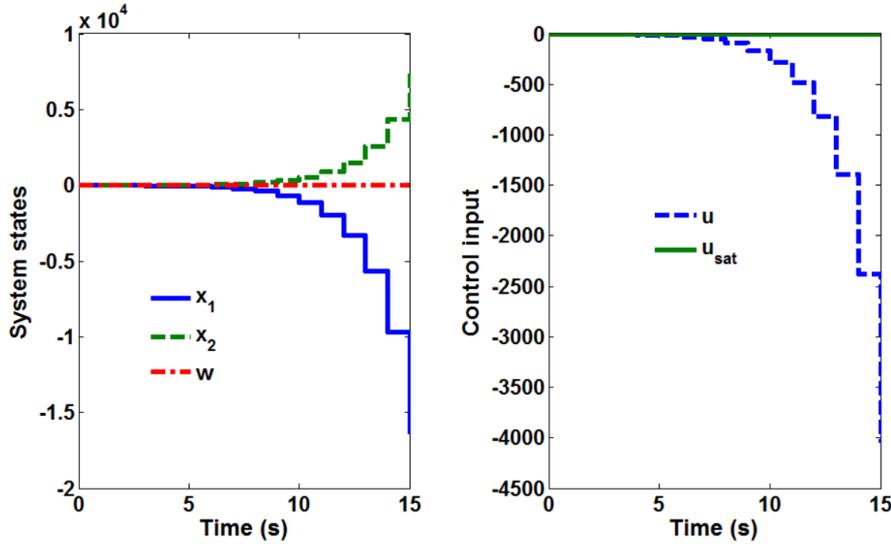


Figure 5: Case 1 with $\beta_1^* = 1.55$. Closed-loop behaviors of (61) corresponding to the initial condition $x_0^\top = [\beta_1^* \quad -4]$ obtained with design conditions in [12].

We now apply Theorem 2 for Case 2 with $\tau_1 = 0.21$, the following results are obtained:

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.067 & 0.288 \\ 0.288 & 0.207 \end{bmatrix}, & H_1 &= \begin{bmatrix} 1.424 & -0.016 \\ -1.246 & 7.668 \end{bmatrix}, & G_1 &= [-0.411 \quad 1.919] \\ P_2 &= \begin{bmatrix} 0.378 & -0.071 \\ -0.071 & 0.237 \end{bmatrix}, & H_2 &= \begin{bmatrix} 1.774 & 0.869 \\ -0.299 & 4.564 \end{bmatrix}, & G_2 &= [-0.204 \quad -1.995] \end{aligned} \quad (65)$$

Figure 6 shows the closed-loop responses corresponding to Case 2 with initial condition $x_0^\top = [\beta_2^* \quad 4]$. We can see that similarly to Case 1, the proposed controller can stabilize the closed-loop system in this *extreme* case despite the presence of disturbance $w(t)$. Again, controllers given in [12, 43, 45] cannot

provide such stable responses under the same conditions. The intensive simulation results given above have clearly pointed out the advantages of the proposed control method.

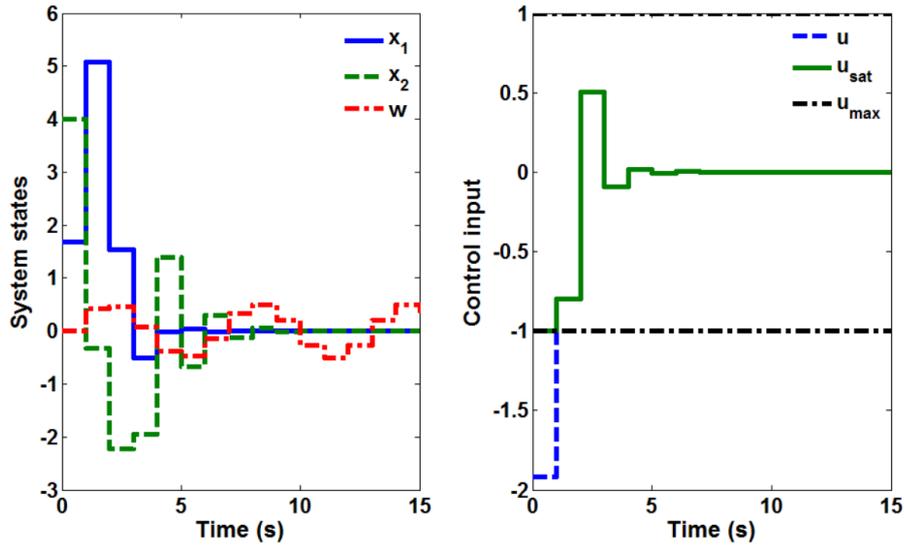


Figure 6: Case 2 with $\beta_2^* = 1.68$. Closed-loop behaviors of T-S system (61) corresponding to the initial condition $x_0^\top = [\beta_2^* \ 4]$ obtained for the non-PDC controller with matrices given in (65).

6. Concluding Remarks

In this paper, a new LMI-based method has been proposed to design saturated non-PDC control law for constrained T-S fuzzy systems subject to both control input and state constraints and also \mathcal{L}_2 or \mathcal{L}_∞ disturbances. This method relies on the use of a parameter-dependent Lyapunov function and a parameter-dependent version of generalized sector condition for control input nonlinearity. These facts allow reducing not only the conservativeness but also the computational complexity of the design conditions. Moreover, the concept of invariant set is effectively exploited to cope with local control context which provides a characterization of the closed-loop estimate of the domain of attraction. The interests of the proposed method have been clearly demonstrated by means of illustrative examples. The new method can be applied to a very large class of nonlinear systems. Moreover, the control structure is relatively simple with low-complexity design conditions. These facts are particularly interesting for real-world applications. Indeed, some challenging automotive control problems [4, 46] are currently studied with the proposed results. Moreover, theoretical extensions to T-S fuzzy systems subject to nested saturations [23] is also a topic for future research.

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