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A Multiple-Parameterization Approach for Local Stabilization of Constrained Takagi-Sugeno Fuzzy Systems with Nonlinear Consequents

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Abstract

This paper addresses the local stabilization of constrained nonlinear systems with input saturation described by Takagi-Sugeno fuzzy models with nonlinear consequents. To reduce the design conservativeness, we propose a new delayed multiple-parameterization control approach based on a nonquadratic Lyapunov function with multiple delayed fuzzy summations. Both input saturation and state constraints are explicitly taken into account in the control design procedure. This multiple-parameterization condition is given in terms of linear matrix inequalities. Compared to existing results, the new approach offers a unified and concise control framework to design both non-delayed and delayed multidimensional nonlinear fuzzy controllers. Numerical examples are provided to demonstrate the effectiveness of the proposed multiple-parameterization approach in both reducing the design conservativeness and enlarging the estimation of the domain of attraction.

Keywords: Takagi-Sugeno fuzzy systems, input saturation, local stabilization, delayed control, nonquadratic Lyapunov functions.

1. Introduction

2 In practical control applications, input and state constraints frequently arise not
3 only from physical and technical restrictions but also from safety reasons [1]. Input
4 constraints emerge from actuator saturation [2] whereas state constraints can be related
5 to both safety/comfort specifications and modeling validity region. If these constraints
6 are not properly taken into account in the control design, the closed-loop performance
7 may be seriously degraded, possibly leading to instability in extreme cases [3]. Thus,
8 performing the control design taking into account these constraints is essential to en-
9 sure closed-loop stability and performance. For instance, this control issue has been

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10 addressed for linear [2] and linear parameter varying [4] systems with input saturation,
11 nonlinear systems such as vibrating flexible string systems with dead-zone nonlinearity
12 and input constraint [5–7], vibrating flexible riser systems with input saturation [8, 9],
13 and nonlinear systems with input saturation described by Takagi-Sugeno fuzzy models
14 [1, 3].

15 Using Takagi-Sugeno (TS) fuzzy modeling [10, 11], nonlinear systems can be rep-
16 resented by convex relations of simpler local models, which are usually valid within
17 a given domain of validity [12]. A well-known method to obtain an *exact* TS fuzzy
18 representation for nonlinear systems is the sector nonlinearity approach [12, Chapter
19 2]. However, it should be stressed that for this approach, the required number of fuzzy
20 rules *exponentially* grows according to the number of system nonlinearities. This may
21 lead to some major drawbacks on both the control design (conservativeness of results)
22 and the real-time implementation (numerical complexity) of TS fuzzy model-based
23 control technique. To avoid an excessive number of fuzzy rules while preserving the
24 convexity property of classical TS fuzzy systems, a new class of TS fuzzy systems with
25 local nonlinear models (N-TS) was proposed in [13, 14] for continuous-time nonlinear
26 systems and employed in [15] for the discrete-time counterparts. As aforementioned,
27 both TS and N-TS fuzzy representations are generally valid within a given subset of the
28 state-space. Hence, taking into account these state constraints is necessary to ensure
29 the fuzzy controller performance [16]. However, these constraints are usually neglected
30 in the fuzzy control literature, especially for N-TS fuzzy models. Therefore, the first
31 motivation of this work is (i) to perform the fuzzy control design taking into account
32 both state and input constraints.

33 By means of the direct Lyapunov method, it is possible to obtain sufficient condi-
34 tions to design fuzzy controllers for both TS and N-TS fuzzy systems. In addition,
35 taking advantage of the convexity of these models, the stabilization conditions can be
36 formulated in terms of linear matrix inequalities (LMIs). It is known that control ap-
37 proaches based on common quadratic Lyapunov functions and the parallel distributed
38 compensation (PDC) control law [12] can lead to conservative design results, espe-
39 cially for complex nonlinear systems requiring a large number of fuzzy rules when
40 using TS modeling [17]. Introducing LMI slack decision variables [12, 18, 19] and/or
41 taking into account the information of the membership functions in continuous-time
42 case [20] have been demonstrated as effective solutions to reduce the design conserva-
43 tiveness. Another control approach capable to progressively reduce conservativeness
44 is based on asymptotically necessary and sufficient (ANS) conditions [21–23], which,
45 however, may quickly lead to an excessive number of LMIs and intractable numerical
46 complexity [19].

47 An alternative solution to reduce design conservativeness is based on the choice
48 of different Lyapunov function candidates. In this context, various classes of Lya-
49 punov functions have been proposed to deal with the conservativeness issue, such as
50 piecewise Lyapunov functions [24], line integral Lyapunov functions [25], and fuzzy
51 (or nonquadratic) Lyapunov functions [26–28]. The effectiveness of fuzzy Lyapunov
52 functions for conservativeness reduction has been mainly demonstrated in the discrete-
53 time case, especially through multiple-parameterization approach [29]. Nevertheless,
54 the latter approach presents similar drawbacks as ANS conditions, *i.e.*, possibly leading
55 to numerical intractability. More recently, delayed multiple-parameterization approach

56 [17, 30, 31] has been proven to be an efficient framework to provide less conservative
57 design conditions for TS fuzzy systems regarding delayed fuzzy membership functions
58 in fuzzy controllers and nonquadratic Lyapunov functions. With the delayed control ap-
59 proach, the design conservativeness can be reduced without increasing substantially the
60 multiple-parameterization dimension, thus avoiding excessive numerical complexity.
61 However, both multiple-parameterization and delayed approaches have not been pro-
62 posed for the class of N-TS fuzzy systems. Then, the second motivation for this work is
63 (ii) to derive the control design conditions based on delayed multiple-parameterization
64 approach for N-TS fuzzy systems.

65 Since the state and the input constraints are explicitly taken into account in the
66 proposed control design, the local stability analysis is considered for the closed-loop
67 N-TS fuzzy systems. Within the local stabilization context, it is of particular interest to
68 characterize the closed-loop domain of attraction (DoA) [32]. In the case of TS or N-
69 TS fuzzy models, finding a DoA estimation is useful to provide the state-space region
70 where the fuzzy controllers can guarantee the closed-loop asymptotic stability. Then,
71 the third motivation of this paper is (iii) to determine an estimation of the domain of
72 attraction, as large as possible, for the closed-loop N-TS fuzzy systems.

73 Based on the aforementioned discussion, this paper investigates the local stabiliza-
74 tion of constrained discrete-time nonlinear systems with input saturation represented
75 by N-TS fuzzy models. This control problem was recently addressed in [16] using a
76 fuzzy dynamic output feedback controller whose design is based on a standard fuzzy
77 Lyapunov function, *i.e.*, without delayed membership functions. The authors in [16]
78 have tackled only the control problems (i) and (iii) mentioned above. However, solv-
79 ing these problems regarding delayed multiple-parameterization Lyapunov functions is
80 more involved than the case of a standard fuzzy Lyapunov function, which is a par-
81 ticular case of the previous one. Therefore, our main contributions are summarized as
82 follows:

- 83 • Propose a new multiple-parameterization framework for local stabilization of
84 constrained N-TS fuzzy systems with input saturation;
- 85 • Provide a *unified* control approach to design both *non-delayed* and *delayed* non-
86 linear fuzzy controllers for the studied class of fuzzy systems;
- 87 • Estimate the DoA of the closed-loop fuzzy systems taking into account both state
88 and input constraints.

89 Furthermore, numerical examples are appropriately given to illustrate the effective-
90 ness of the proposed control approach on reducing the design conservativeness and
91 enlarging the DoA estimation.

92 *Notation.* The symbol ‘ \star ’ denotes matrix blocks deduced by symmetry. For a matrix
93 X , X^\top is the transpose matrix, $X \succ 0$ means that X is a positive definite matrix and
94 $X_{(l)}$ is the l th row. $\text{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1
95 and X_2 . For an integer $p > 1$, we denote $\mathcal{I}_p = \{1, \dots, p\} \subset \mathbb{N}$. $\text{co}\{\mathcal{S}\}$ denotes
96 the convex hull of the set \mathcal{S} . Function arguments are omitted when their meaning is
97 straightforward.

98 **2. Takagi-Sugeno Fuzzy Systems with Nonlinear Consequents**

99 This section describes the considered class of N-TS fuzzy models and their advan-
100 tages for the control design of nonlinear systems.

101 *2.1. System Description*

Consider the following class of N-TS fuzzy systems [15]:

$$x_{k+1} = \sum_{i=1}^r h_i(z_k)(A_i x_k + G_i \varphi(x_k) + B_i \text{sat}(u_k)) \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the input vector, $z_k \in \mathbb{R}^{n_z}$ is the vector of measurable premise variables, and r is the number of fuzzy rules. The state-space matrices (A_i, B_i, G_i) of the i th local fuzzy rule are of appropriate dimensions. The normalized membership functions $h_i(z)$, $i \in \mathcal{I}_r$, verify the convex sum property, i.e., $\sum_{i=1}^r h_i(z) = 1$, and $h_i(z) \geq 0$, $i \in \mathcal{I}_r$. Moreover, the control input is subject to a componentwise saturation map $\text{sat}(\cdot) : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_u}$ defined as

$$\text{sat}(u_{(l)}) = \text{sign}(u_{(l)}) \min(|u_{(l)}|, \bar{u}_{(l)}),$$

102 for all $l \in \mathcal{I}_{n_u}$, where $\bar{u}_{(l)} > 0$ is the saturation bound of the l th control input compo-
103 nent. The following assumptions are considered for system (1).

Assumption 1. *The vector of nonlinearities $\varphi(x) \in \mathbb{R}^{n_\varphi}$ satisfies the sector-boundedness condition $\varphi_{(i)}(x) \in \text{co}\{0, E_i x\}$, $\forall i \in \mathcal{I}_{n_\varphi}$, which can be equivalently rewritten as follows [32]:*

$$\varphi(x)^\top \mathcal{L}^{-1} (Ex - \varphi(x)) \geq 0 \quad (2)$$

104 where the matrix $E = [E_1^\top, \dots, E_{n_\varphi}^\top]^\top \in \mathbb{R}^{n_\varphi \times n_x}$ is given, and $\mathcal{L} \in \mathbb{R}^{n_\varphi \times n_\varphi}$ is a
105 positive definite diagonal matrix.

106 Inequality (2) in Assumption 1 is especially useful to derive convex control design
107 conditions for N-TS fuzzy systems.

Assumption 2. *The state trajectories of system (1) is constrained into the following polyhedral set:*

$$\mathcal{D}_x = \{x \in \mathbb{R}^{n_x} : S_{(j)} x \leq 1, j \in \mathcal{I}_{n_e}\}, \quad (3)$$

108 where the given matrix $S \in \mathbb{R}^{n_e \times n_x}$ characterizes the domain of validity \mathcal{D}_x .

109 The domain \mathcal{D}_x represents the state-space region where trajectories are constrained
110 to evolve due to both physical limitations and validity region of the N-TS fuzzy model (1).
111 Taking this domain into account in the control design is essential to ensure both suitable
112 closed-loop performance and stability [3, 16].

113 **2.2. System Motivation**

To highlight the interests of using N-TS fuzzy models, we consider an input-saturated nonlinear system constructed by N inverted pendulums series connected via linear springs. Using the Euler's discretization method, the system dynamics can be described by the following state-space representation:

$$x_{k+1(j)} = x_{k(j)} + Tx_{k(n)} \quad (4a)$$

$$\begin{aligned} x_{k+1(n)} = & x_{k(n)} + \frac{gT}{l_\kappa} \sin x_{k(j)} + \frac{T}{m_\kappa l_\kappa^2} \text{sat}(u_{k(\kappa)}) \\ & + (1 - \delta_1^\kappa) \frac{k_{\kappa-1} a^2 T}{m_\kappa l_\kappa^2} \left(\sin x_{k(j-2)} \cos x_{k(j-2)} - \sin x_{k(j)} \cos x_{k(j)} \right) \\ & + (1 - \delta_N^\kappa) \frac{k_\kappa a^2 T}{m_\kappa l_\kappa^2} \left(\sin x_{k(j+2)} \cos x_{k(j+2)} - \sin x_{k(j)} \cos x_{k(j)} \right) \end{aligned} \quad (4b)$$

114 where $\kappa \in \mathcal{I}_N$, $j = 2\kappa - 1$, $n = 2\kappa$, and δ_p^q , $p, q \in \mathbb{Z}$, denotes the Kronecker delta,
 115 $x_{k(\kappa)}$ is the rod angle with respect to the vertical axis, m_κ is the mass, and l_κ is the rod
 116 length of the κ th pendulum. $u_{k(\kappa)}$ is the torque applied to the base of the κ th pendulum.
 117 T is the sample time, g is the gravitational acceleration, k_s , $s \in \mathcal{I}_{N-1}$, are the spring
 118 elastic constants, and a is their connection heights.

119 By the sector nonlinearity approach [12, Chapter 2], an *exact* TS fuzzy representa-
 120 tion can be obtained for system (4) choosing the premise variables as $z_{k(\kappa)} = \sin x_{k(j)}$,
 121 and $z_{k(N+\kappa)} = \sin x_{k(j)} \cos x_{k(j)}$, for all $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_N$. This leads to a clas-
 122 sical TS fuzzy model with 2^{2N} fuzzy rules. However, note that $\sin x_{k(\kappa)} \cos x_{k(\kappa)} \in$
 123 $\text{co}\{0, x_{k(\kappa)}\}$. Then, a N-TS fuzzy model can be constructed with N premise vari-
 124 ables $z_{k(\kappa)} = \sin x_{k(j)}$ and N local nonlinearities $\varphi_\kappa(x_k) = \sin x_{k(j)} \cos x_{k(j)}$, for
 125 $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_N$. In this case, the number of fuzzy rules is reduced to only 2^N . As
 126 a direct consequence, not only the numerical complexity for control design and imple-
 127 mentation but also the conservativeness of the results can be drastically reduced when
 128 N-TS fuzzy models are used, see also [13, 15].

129 **3. Problem Formulation**

130 The considered control problem is formulated in this section. First, we provide
 131 technical definitions related to the theory of multisets to represent compactly multiple
 132 fuzzy summations with arbitrary delays. Then, the control problem is stated.

133 **3.1. Technical Definitions**

134 Hereafter, technical definitions related to the theory of multisets [33] and multiple
 135 fuzzy summations are presented. Similar notations on multisets of delays and multi-
 136 indexes can be found in [17, 22].

137 **Definition 1** (Multisets, see [33]). *Let $D = \{d_1, d_2, \dots, d_n\}$ be a set. A multiset G_D*
 138 *over D is a cardinal-valued function $G_D : D \mapsto \mathbb{N}$ such that for $d \in \text{Dom}(G_D)$*
 139 *implies the cardinal $\mathbf{1}_{G_D}(d) > 0$. The value $\mathbf{1}_{G_D}(d)$ is called multiplicity of d and*
 140 *represents the number of occurrences of d in G_D . A multiset G_D is denoted here by the*

141 set of pairs $G_D = \{\langle \mathbf{1}_{G_D}(d_1), d_1 \rangle, \dots, \langle \mathbf{1}_{G_D}(d_n), d_n \rangle\}$. If the multiplicity of a given
 142 element $d \in D$ is 1, it is simply denoted $\langle 1, d \rangle = d$.

143 From the last definition, the cardinality of a multiset G_D is computed by $|G_D| =$
 144 $\sum_{d \in D} \mathbf{1}_{G_D}(d)$.

145 **Definition 2** (Operations on multisets). Consider two multisets G_A and G_B . Useful
 146 operations on multisets are defined as follows:

- 147 • Union: $G_{\cup} = G_A \cup G_B = \{d \in G_{\cup} : \mathbf{1}_{G_{\cup}}(d) = \max\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}$
- 148 • Intersection: $G_{\cap} = G_A \cap G_B = \{d \in G_{\cap} : \mathbf{1}_{G_{\cap}}(d) = \min\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}$
- 149 • Sum: $G_{\oplus} = \{d \in G_{\oplus} : \mathbf{1}_{G_{\oplus}}(d) = \mathbf{1}_{G_A}(d) + \mathbf{1}_{G_B}(d)\}$.

Definition 3 (Multiple fuzzy summation). Let

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^r \cdots \sum_{i_n=1}^r h_{i_1}(z_{k+d_1}) \cdots h_{i_n}(z_{k+d_n}) P_{(i_1, \dots, i_n)} \quad (5)$$

150 be a multiple fuzzy summation of matrices $P_{(i_1, \dots, i_n)}$. The multiset of delays involved
 151 in this summation is denoted by $G_0^P = \{d_1, \dots, d_n\}$, for $d_i \in \mathbb{Z}_{\leq 0}$, $i \in \mathcal{I}_n$. If the
 152 summation is evaluated at the instant $k + \alpha$, the summation is denoted by $\mathbb{P}_{G_\alpha^P}$ and the
 153 multiset of delays is $G_\alpha^P = \{\alpha + d_1, \dots, \alpha + d_n\}$.

154 The above notations and operations on multisets are illustrated in the following
 155 example.

156 **Example 1.** Consider the 4-dimensional fuzzy summation

$$\sum_{i_1=1}^r \sum_{i_2=1}^r \sum_{i_3=1}^r \sum_{i_4=1}^r h_{i_1}(z_k) h_{i_2}(z_k) h_{i_3}(z_k) h_{i_4}(z_{k-1}) P_{(i_1, i_2, i_3, i_4)}.$$

157 The multiset of delays associated to this fuzzy summation is $G_0^P = \{\langle 3, 0 \rangle, -1\}$ and its
 158 cardinality is $|G_0^P| = 4$. If this fuzzy summation is evaluated at the instant $k + \alpha$, then
 159 the corresponding multiset of delays is $G_\alpha^P = \{\langle 3, \alpha \rangle, \alpha - 1\}$.

160 To illustrate the operations on multisets, we consider the multiset of delays $G_0^X =$
 161 $\{0, \langle 2, -1 \rangle, -2\}$. The union of G_0^X with the multiset of delays G_0^P is $G_0^P \cup G_0^X =$
 162 $\{\langle 3, 0 \rangle, \langle 2, -1 \rangle, -2\}$; the intersection is $G_0^P \cap G_0^X = \{0, -1, -2\}$ and the sum is
 163 $G_0^P \oplus G_0^X = \{\langle 4, 0 \rangle, \langle 3, -1 \rangle, -2\}$.

164 **Definition 4** (Index set and multi-index). The index set of a multiple fuzzy summation
 165 with multiset of delays G_0^P is the set of all indexes in the sum. It is denoted here by
 166 $\mathbb{I}_{G_0^P} = \{\mathbf{i} = (i_1, \dots, i_{|G_0^P|}) : i_j \in \mathcal{I}_r, j \in \mathcal{I}_{|G_0^P|}\}$. An element $\mathbf{i} \in \mathbb{I}_{G_0^P}$ is called
 167 multi-index.

168 **Remark 1.** For the formulations presented in this paper, the following two subsets of
 169 $\mathbb{I}_{G_0^P}$ will be useful. The set of upper-triangle indexes, which is defined by $\mathbb{I}_{G_0^P}^+ = \{\mathbf{i} \in$
 170 $\mathbb{I}_{G_0^P} : i_j \leq i_{j+1}, j \in \mathcal{I}_{|G_0^P|-1}\}$, and the set of multi-index permutations defined by
 171 $\mathcal{P}(\mathbf{i}) \subset \mathbb{I}_{G_0^P}$, for some $\mathbf{i} \in \mathbb{I}_{G_0^P}$. These subsets of $\mathbb{I}_{G_0^P}$ can also be found in [21, 22].

172 **Example 2.** Consider the index set $\mathbb{I}_G = \{\mathbf{i} = (i_1, i_2, i_3) : i_j \in \mathcal{I}_2, j \in \mathcal{I}_3\}$.
 173 The set of upper-triangle indexes is $\mathbb{I}_G^+ = \{(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2)\}$. In
 174 addition, consider the multi-index $\mathbf{i} = (1, 2, 2) \in \mathbb{I}_G^+$. The set of permutations is
 175 $\mathcal{P}(\mathbf{i}) = \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$.

176 **Definition 5** (Projection of a multi-index). The projection of a multi-index $\mathbf{i} \in \mathbb{I}_{G_A}$ to
 177 the multiset G_B , denoted by $\text{pr}_{G_B}^{\mathbf{i}}$, is the part of \mathbf{i} corresponding to delays in $G_A \cap G_B$.

178 It is important to note that the projection may be not unique. For instance, consider
 179 the multi-index $\mathbf{i} = (1, 2, 3, 4) \in \mathbb{I}_{G_0^P}$, where $G_0^P = \{\{3, 0\}, -1\}$. The projection to
 180 the multiset of delays $G_0^C = \{0, -1\}$ can be $\text{pr}_{G_0^C}^{\mathbf{i}} = (1, 4)$, or $\text{pr}_{G_0^C}^{\mathbf{i}} = (2, 4)$, or
 181 $\text{pr}_{G_0^C}^{\mathbf{i}} = (3, 4)$.

Remark 2. The multiple fuzzy summation (5) can be rewritten using the multi-index notation as follows:

$$\mathbb{P}_{G_0^P} = \prod_{j=1}^n \sum_{i_j=1}^r h_{i_j}(z_k+d_j) P_{(i_1, \dots, i_n)} = \sum_{\mathbf{i}_0^P \in \mathbb{I}_{G_0^P}} h_{\mathbf{i}_0^P} P_{\mathbf{i}_0^P},$$

with $\mathbb{I}_{G_0^P} = \{\mathbf{i}_0^P = (i_1, \dots, i_n) : i_j \in \mathcal{I}_r, j \in \mathcal{I}_n\}$. Similarly, the N-TS fuzzy system (1) can be rewritten in the form:

$$x_{k+1} = \mathbb{A}_{G_0^A} x_k + \mathbb{G}_{G_0^G} \varphi(x_k) + \mathbb{B}_{G_0^B} \text{sat}(u_k), \quad (6)$$

182 with $G_0^A = G_0^G = G_0^B = \{0\}$.

183 3.2. Problem Statement

For the stabilization of the N-TS fuzzy model (6), the following delayed nonlinear fuzzy control law is proposed:

$$u_k = \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} x_k + \mathbb{K}_{G_0^K} \mathbb{L}_{G_0^L}^{-1} \varphi(x_k), \quad (7)$$

184 where $\mathbb{F}_{G_0^F}$, $\mathbb{H}_{G_0^H}$, $\mathbb{K}_{G_0^K}$ are respectively the multiple fuzzy summations of matrices
 185 $F_{\mathbf{i}_0^F} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{i}_0^F \in \mathbb{I}_{G_0^F}$, $H_{\mathbf{i}_0^H} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{i}_0^H \in \mathbb{I}_{G_0^H}$, and $K_{\mathbf{i}_0^K} \in \mathbb{R}^{n_u \times n_\varphi}$, $\mathbf{i}_0^K \in$
 186 $\mathbb{I}_{G_0^K}$, and $\mathbb{L}_{G_0^L}$ is composed of diagonal matrices $L_{\mathbf{i}_0^L} \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $\mathbf{i}_0^L \in \mathbb{I}_{G_0^L}$. For real-
 187 time implementation, the multisets of delays involved in (7) cannot be defined with
 188 positive delays, which corresponds to premise variables at future time samples, since it
 189 leads to a non-causal closed-loop dynamics.

190

191 **Remark 3.** The proposed control law (7) generalizes other fuzzy controllers existing
 192 in the literature. For instance, the PDC control law can be obtained by selecting $G_0^F =$
 193 $\{0\}$, $G_0^H = \{0\}$, and $\varphi(x_k) = 0$, and the classical non-PDC law can be constructed
 194 with $G_0^F = G_0^H = \{0\}$ and $\varphi(x_k) = 0$. Moreover, the nonlinear non-PDC control
 195 law used in [15] for the stabilization of N-TS fuzzy models can also be obtained by
 196 selecting $G_0^F = G_0^H = G_0^K = G_0^L = \{0\}$.

Substituting (7) into (6), the closed-loop dynamics is rewritten as

$$x_{k+1} = \mathbb{A}_{cl}x_k + \mathbb{G}_{cl}\varphi(x_k) - \mathbb{B}_{G_0^B}\psi(u_k) \quad (8)$$

197 where $\mathbb{A}_{cl} = \mathbb{A}_{G_0^A} + \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}$, $\mathbb{G}_{cl} = \mathbb{G}_{G_0^G} + \mathbb{B}_{G_0^B}\mathbb{K}_{G_0^K}\mathbb{L}_{G_0^L}^{-1}$, and $\psi(u_k) =$
 198 $u_k - \text{sat}(u_k)$ is the dead-zone nonlinearity. The following lemma is useful to deal with
 199 this nonlinearity.

Lemma 1. Consider the set

$$\mathcal{D}_u = \left\{ x_k \in \mathbb{R}^{n_x} : |\Pi| \leq \bar{u}_{(l)}, \forall l \in \mathcal{I}_{n_u} \right\}, \quad (9)$$

where

$$\Pi = \left(\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} - \mathbb{W}_{G_0^W}\mathbb{H}_{G_0^H}^{-1} \right)_{(l)} x_k + \left(\mathbb{K}_{G_0^K}\mathbb{L}_{G_0^L}^{-1} - \mathbb{J}_{G_0^J}\mathbb{L}_{G_0^L}^{-1} \right)_{(l)} \varphi(x_k),$$

and $\mathbb{W}_{G_0^W}$ and $\mathbb{J}_{G_0^J}$ are respectively the multiple fuzzy summations of matrices $W_{i_0^W} \in \mathbb{R}^{n_u \times n_x}$, $i_0^W \in \mathbb{I}_{G_0^W}$, and $J_{i_0^J} \in \mathbb{R}^{n_u \times n_\varphi}$, $i_0^J \in \mathbb{I}_{G_0^J}$. If $x_k \in \mathcal{D}_u$, then

$$\psi(u_k)^\top \mathbb{U}_{G_0^U}^{-1} \left(\psi(u_k) - \mathbb{W}_{G_0^W}\mathbb{H}_{G_0^H}^{-1}x_k - \mathbb{J}_{G_0^J}\mathbb{L}_{G_0^L}^{-1}\varphi(x_k) \right) \leq 0 \quad (10)$$

200 for positive definite diagonal matrices $U_{i_0^U} \in \mathbb{R}^{n_u \times n_u}$, $i_0^U \in \mathbb{I}_{G_0^U}$, and u_k given in (7).

201 *Proof.* The proof follows similar steps as in [1, Lemma 1] and is omitted here for
 202 brevity. \square

To study the local asymptotic stability of the closed-loop system (8), the following delayed nonquadratic Lyapunov function candidate is considered:

$$V(x_k) = x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \quad (11)$$

203 where $\mathbb{P}_{G_0^P} \in \mathbb{R}^{n_x \times n_x}$ is the multiple fuzzy summation of matrices $P_{i_0^P} = P_{i_0^P}^\top \succ 0$,
 204 $i_0^P \in \mathbb{I}_{G_0^P}$.

205 **Remark 4.** The level set associated to the function (11) is defined by

$$\mathcal{L}_V = \left\{ x_k \in \mathbb{R}^{n_x} : x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \leq 1 \right\}.$$

206 Moreover, if $\Delta V = V(x_{k+1}) - V(x_k) < 0$ holds along trajectories of system (8) for
 207 all $x_k \in \mathcal{L}_V \setminus \{0\}$, then (11) is said to be a Lyapunov function and \mathcal{L}_V a contractively
 208 invariant set with respect to the closed-loop system (8). Note that \mathcal{L}_V is a subset of the
 209 DoA of the closed-loop system [32].

210 In the light of the previous discussions, this work is concerned with proposing
 211 sufficient conditions to solve the following control problem.

212 **Problem 1.** Consider the N-TS fuzzy model (6). Design a nonlinear fuzzy controller of
 213 the form (7) such that $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$, as large as possible, is a contractively invariant
 214 set with respect to the closed-loop system (8).

215 **4. Multiple-Parameterization Stabilization Conditions for Constrained N-TS Fuzzy**
 216 **Systems**

217 The following theorem provides sufficient conditions to guarantee that $\mathcal{L}_V \subset \mathcal{D}_x \cap$
 218 \mathcal{D}_u is a contractively invariant set with respect to system (8).

Theorem 1. Given $G_V = G_0^P \cup G_1^P \cup (G_0^A \oplus G_0^H) \cup (G_0^G \oplus G_0^L) \cup (G_0^B \oplus G_0^F) \cup (G_0^B \oplus G_0^K) \cup (G_0^B \oplus G_0^U)$. If there exist matrices $P_{i_j^P} = P_{i_j^P}^\top \succ 0$, $i_j^P = \text{pr}_{G_0^P}^i$, $j \in \{0, 1\}$, $H_{i_0^H}$, $i_0^H = \text{pr}_{G_0^H}^i$, $K_{i_0^K}$, $i_0^K = \text{pr}_{G_0^K}^i$, $F_{i_0^F}$, $i_0^F = \text{pr}_{G_0^F}^i$, $W_{i_0^W}$, $i_0^W = \text{pr}_{G_0^W}^i$, $J_{i_0^J}$, $i_0^J = \text{pr}_{G_0^J}^i$, and positive definite diagonal matrices $L_{i_0^L}$, $i_0^L = \text{pr}_{G_0^L}^i$, and $U_{i_0^U}$, $i_0^U = \text{pr}_{G_0^U}^i$, $i \in \mathbb{I}_{G_V}$, such that (12), (13) and (14) are feasible. Then, the control law (7) guarantees that $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$ is a contractively invariant set of the closed-loop system (8).

$$\begin{bmatrix} 1 & S_{(j)} \mathbb{P}_{G_0^P} \\ \star & \mathbb{P}_{G_0^P} \end{bmatrix} \succeq 0, \quad \forall j \in \mathcal{I}_{n_e} \quad (12)$$

$$\begin{bmatrix} \mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top - \mathbb{P}_{G_0^P} & -\mathbb{H}_{G_0^H}^\top E^\top & \left(\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W} \right)_{(l)}^\top \\ \star & 2\mathbb{L}_{G_0^L} & \left(\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J} \right)_{(l)}^\top \\ \star & \star & \bar{u}_{(l)}^2 \end{bmatrix} \succ 0, \quad \forall l \in \mathcal{I}_{n_u} \quad (13)$$

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \Phi_1 & \Phi_2 & -\mathbb{B}_{G_0^B} \mathbb{U}_{G_0^U} \\ \star & -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \mathbb{H}_{G_0^H}^\top E^\top & \mathbb{W}_{G_0^W}^\top \\ \star & \star & -2\mathbb{L}_{G_0^L} & \mathbb{J}_{G_0^J}^\top \\ \star & \star & \star & -2\mathbb{U}_{G_0^U} \end{bmatrix} \prec 0 \quad (14)$$

219 where $\Phi_1 = \mathbb{A}_{G_0^A} \mathbb{H}_{G_0^H} + \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F}$, and $\Phi_2 = \mathbb{G}_{G_0^G} \mathbb{L}_{G_0^L} + \mathbb{B}_{G_0^B} \mathbb{K}_{G_0^K}$.

Proof. Multiplying (12) with $\text{diag} \left(1, \mathbb{P}_{G_0^P}^{-1} \right)$ on the left and its transpose on the right, yields

$$\begin{bmatrix} 1 & S_{(j)} \\ \star & \mathbb{P}_{G_0^P}^{-1} \end{bmatrix} \succeq 0. \quad (15)$$

220 Then, pre- and post-multiplying (15) with $\begin{bmatrix} 1 & -x_k^\top \end{bmatrix}$ and its transpose, respectively,
 221 leads to

$$1 - x_k^\top S_{(j)} - S_{(j)} x_k + x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \geq 0.$$

222 Since for all $x_k \in \mathcal{L}_V$, one has $x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k \leq 1$, which implies that $S_{(j)} x_k \leq 1$. This
 223 proves the inclusion $\mathcal{L}_V \subseteq \mathcal{D}_x$.

Inequality (13) implies clearly that $\mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top \succ \mathbb{P}_{G_0^P} \succ 0$. This, in its turn, implies that $\mathbb{H}_{G_0^H}$ is regular. Since $\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} \succeq \mathbb{H}_{G_0^H} + \mathbb{H}_{G_0^H}^\top - \mathbb{P}_{G_0^P}$, inequality

(13) implies that

$$\begin{bmatrix} \mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & -\mathbb{H}_{G_0^H}^\top E^\top & \left(\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W} \right)_{(l)}^\top \\ \star & 2\mathbb{L}_{G_0^L} & \left(\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J} \right)_{(l)}^\top \\ \star & \star & \bar{u}_{(l)}^2 \end{bmatrix} \succ 0. \quad (16)$$

Then, multiplying (16) with $\text{diag} \left(\mathbb{H}_{G_0^H}^{-\top}, \mathbb{L}_{G_0^L}^{-1}, 1 \right)$ on the left and its transpose on the right, followed by the well-known Schur complement lemma, we obtain

$$\begin{bmatrix} \mathbb{P}_{G_0^P}^{-1} & -E^\top \mathbb{L}_{G_0^L}^{-1} \\ \star & 2\mathbb{L}_{G_0^L}^{-1} \end{bmatrix} - \frac{1}{\bar{u}_{(l)}^2} \Sigma_{(l)}^\top \Sigma_{(l)} \succ 0, \quad \forall l \in \mathcal{I}_{n_u} \quad (17)$$

224 where $\Sigma = \left[\left(\mathbb{F}_{G_0^F} - \mathbb{W}_{G_0^W} \right) \mathbb{H}_{G_0^H}^{-1} \quad \left(\mathbb{K}_{G_0^K} - \mathbb{J}_{G_0^J} \right) \mathbb{L}_{G_0^L}^{-1} \right]$. Following the same steps as
 225 in [1], we can conclude from (17) that $\mathcal{L}_V \subset \mathcal{D}_u$.

Using again the property $-\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} \preceq \mathbb{P}_{G_0^P} - \mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top$, it follows from (14) that

$$\begin{bmatrix} -\mathbb{P}_{G_1^P} & \Phi_1 & \Phi_2 & -\mathbb{B}_{G_0^B} \mathbb{U}_{G_0^U} \\ \star & -\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & \mathbb{H}_{G_0^H}^\top E^\top & \mathbb{W}_{G_0^W}^\top \\ \star & \star & -2\mathbb{L}_{G_0^L} & \mathbb{J}_{G_0^J}^\top \\ \star & \star & \star & -2\mathbb{U}_{G_0^U} \end{bmatrix} \prec 0. \quad (18)$$

Then, multiplying (18) with $\text{diag} \left(\mathbb{P}_{G_1^P}^{-1}, \mathbb{H}_{G_0^H}^{-\top}, \mathbb{L}_{G_0^L}^{-1}, \mathbb{U}_{G_0^U}^{-1} \right)$ on the left and its transpose on the right, respectively, followed by the Schur complement lemma, the following inequality can be obtained:

$$\Psi^\top \mathbb{P}_{G_1^P}^{-1} \Psi + \begin{bmatrix} -\mathbb{P}_{G_0^P}^{-1} & E^\top \mathbb{L}_{G_0^L}^{-1} & \mathbb{H}_{G_0^H}^{-\top} \mathbb{W}_{G_0^W}^\top \mathbb{U}_{G_0^U}^{-1} \\ \star & -2\mathbb{L}_{G_0^L}^{-1} & \mathbb{L}_{G_0^L}^{-\top} \mathbb{J}_{G_0^J}^\top \mathbb{U}_{G_0^U}^{-1} \\ \star & \star & -2\mathbb{U}_{G_0^U}^{-1} \end{bmatrix} \prec 0 \quad (19)$$

226 where $\Psi = [\mathbb{A}_{cl} \quad \mathbb{G}_{cl} \quad -\mathbb{B}_{G_0^B}]$.

Pre- and postmultiplying (19) by $[x_k^\top \quad \varphi(x_k)^\top \quad \psi(u_k)^\top]$ and its transpose, it follows that

$$\begin{aligned} \Delta V + 2\varphi(x_k)^\top \mathbb{L}_{G_0^L}^{-1} (Ex_k - \varphi(x_k)) \\ - 2\psi(u_k)^\top \mathbb{U}_{G_0^U}^{-1} \left(\psi(u_k) - \mathbb{W}_{G_0^W} \mathbb{H}_{G_0^H}^{-1} x_k - \mathbb{J}_{G_0^J} \mathbb{L}_{G_0^L}^{-1} \varphi(x_k) \right) < 0. \end{aligned} \quad (20)$$

227 Since $\mathbb{L}_{G_0^L}^{-1} \succ 0$ and $\mathbb{U}_{G_0^U}^{-1} \succ 0$, by properties (2) and (10), it follows from (20) that
 228 $\Delta V < 0$, for $\forall x_k \in \mathcal{D}_x \cap \mathcal{D}_u$. This guarantees that the origin of the closed-loop
 229 system (8) is asymptotically stable and $\mathcal{L}_V \subset \mathcal{D}_x \cap \mathcal{D}_u$ is a contractively invariant set
 230 with respect to system (8). This completes the proof. \square

231 **Remark 5.** Without considering \mathcal{H}_∞ performance in the design condition of [15,
 232 Corollary 12], this condition can be seen as a special case of (14) in Theorem 1 for
 233 N-TS fuzzy systems without input saturation and state constraints. Indeed, the design
 234 conditions in [15] can be recovered from Theorem 1 by choosing the multisets of delays
 235 as $G_0^P = G_0^F = G_0^K = G_0^H = G_0^L = G_0^U = \{0\}$.

236 The proposed multiple-parameterization approach is based on the choice of mul-
 237 tisets of delays in Theorem 1, *i.e.*, by increasing the fuzzy summation dimensions in
 238 both delayed controller (7) and Lyapunov function (11). In particular, Theorem 1 al-
 239 lows to design both *non-delayed* and *delayed* controllers with appropriate choices of
 240 the multisets of delays.

For example, conditions to design a *non-delayed* controller can be obtained with
 $G_0^P = \{\langle \mathbf{1}_{G_0^F}(0), 0 \rangle\}$ and $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U =$
 $\{\langle \mathbf{1}_{G_0^F}(0), 0 \rangle\}$. If we consider $|G_0^P| = n_P$ and assume that $|G_0^F| = |G_0^P|$, the multiset
 of all delays in Theorem 1 is $G_V = \{0, \langle n_P, 0 \rangle, \langle n_P, 1 \rangle\}$. Hence, condition (14) cor-
 responds to a $(2n_P + 1)$ -dimensional fuzzy summation. For illustrations, taking $n_P = 1$,
 condition (14) becomes

$$\sum_{i_1=1}^r \sum_{i_2=1}^r \sum_{i_3=1}^r h_{i_1}(z_k) h_{i_2}(z_k) h_{i_3}(z_{k+1}) \Xi^* \prec 0 \quad (21)$$

241 where

$$\Xi^* = \begin{bmatrix} -P_{i_3} & A_{i_1} H_{i_2} + B_{i_1} F_{i_2} & G_{i_1} L_{i_2} + B_{i_1} K_{i_2} & -B_{i_1} U_{i_2} \\ \star & P_{i_2} - H_{i_2} - H_{i_2}^\top & H_{i_2} E^\top & W_{i_1}^\top \\ \star & \star & -2L_{i_2} & J_{i_2}^\top \\ \star & \star & \star & -2U_{i_2} \end{bmatrix}.$$

Moreover, the design conditions to obtain a *delayed* fuzzy controller can be obtained
 with $G_0^P = \{\langle \mathbf{1}_{G_0^F}(-1), -1 \rangle\}$ and $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W =$
 $G_0^U = \{\langle \mathbf{1}_{G_0^F}(0), 0 \rangle, \langle \mathbf{1}_{G_0^F}(-1), -1 \rangle\}$. In this case, with $|G_0^P| = n_P$ and $G_0^F =$
 $\{\langle n_P, 0 \rangle, \langle n_P, -1 \rangle\}$, the multiset G_V is $G_V = \{0, \langle n_P, 0 \rangle, \langle n_P, -1 \rangle\}$. For exam-
 ple, to obtain a 3-dimensional fuzzy summation based condition to design a *delayed*
 controller from (14), one can choose $G_0^P = \{-1\}$ and $G_0^F = \{0, -1\}$, which results in

$$\sum_{i_1=1}^r \sum_{i_2=1}^r \sum_{i_3=1}^r h_{i_1}(z_k) h_{i_2}(z_k) h_{i_3}(z_{k-1}) \Xi \prec 0 \quad (22)$$

242 where

$$\Xi = \begin{bmatrix} -P_{i_2} & A_{i_1} H_{(i_2, i_3)} + B_{i_1} F_{(i_2, i_3)} & G_{i_1} L_{(i_2, i_3)} + B_{i_1} K_{(i_2, i_3)} & -B_{i_1} U_{(i_2, i_3)} \\ \star & P_{i_3} - H_{(i_2, i_3)} - H_{(i_2, i_3)}^\top & H_{(i_2, i_3)} E^\top & W_{(i_2, i_3)}^\top \\ \star & \star & -2L_{(i_2, i_3)} & J_{(i_2, i_3)}^\top \\ \star & \star & \star & -2U_{(i_2, i_3)} \end{bmatrix}.$$

243 It can be clearly observed in (21) and (22) that for a fixed number of fuzzy summations,
 244 design conditions based on both *delayed* fuzzy controllers and Lyapunov functions

245 may lead to more degrees of freedom (thus, less conservativeness) than those for *non-*
 246 *delayed* ones. The extension to higher fuzzy summation dimensions is obvious from
 247 the above discussion.

248

249 **Remark 6.** *From practical aspects, there are two ways of reducing conservativeness*
 250 *with the proposed delayed multiple-parameterization approach. First, for the case of*
 251 *non-delayed fuzzy controllers, less conservative results can be obtained by increasing*
 252 *the multiple-parameterization dimension. Second, by introducing delayed membership*
 253 *functions in the fuzzy controller (7) and nonquadratic Lyapunov function (11), conser-*
 254 *vativeness can be further reduced since more degrees of freedom can be introduced into*
 255 *the design conditions without increasing the dimension of the overall fuzzy summation*
 256 *dimension related to the multiset of delays G_V in Theorem 1.*

257 5. LMI-Based Design Conditions of Constrained N-TS Fuzzy Systems

258 The stabilization conditions in Theorem 1 are expressed in terms of (possibly de-
 259 layed) membership functions. Hence, they cannot be solved directly by numerical
 260 solvers. For practical use, the following lemma provides a technical tool to derive a
 261 *finite* set of LMI-based design conditions from Theorem 1.

Lemma 2. *Consider a multiple fuzzy summation with multiset of delays $G = D_1 \oplus$
 $\dots \oplus D_q$, where $D_k = \{\mathbf{1}_{D_k}(d_k), d_k\}$ for all $k \in \mathcal{I}_q$. The following inequality*

$$\sum_{\mathbf{i}_1 \in \mathbb{I}_{D_1}} \dots \sum_{\mathbf{i}_q \in \mathbb{I}_{D_q}} h_{\mathbf{i}_1}(z_{k+d_1}) \dots h_{\mathbf{i}_q}(z_{k+d_q}) \Upsilon_{(\mathbf{i}_1, \dots, \mathbf{i}_q)} \prec 0 \quad (23)$$

is verified if

$$\Theta = \sum_{\mathbf{j}_1 \in \mathcal{P}(\mathbf{j}_1)} \dots \sum_{\mathbf{j}_q \in \mathcal{P}(\mathbf{j}_q)} \Upsilon_{(\mathbf{i}_1, \dots, \mathbf{i}_q)} \prec 0, \quad \mathbf{j}_k \in \mathbb{I}_{D_k}^+, k \in \mathcal{I}_q. \quad (24)$$

Proof. Condition (23) can be *equivalently* rewritten in the following form:

$$\sum_{\mathbf{j}_1 \in \mathbb{I}_{D_1}^+} \dots \sum_{\mathbf{j}_q \in \mathbb{I}_{D_q}^+} \Theta \prec 0 \quad (25)$$

262 with Θ defined in (24). It is clear that inequality (24) is a sufficient condition to ensure
 263 (25). This concludes the proof. \square

264 **Remark 7.** *Given a multidimensional fuzzy summation with multiset of delays $G =$
 265 $D_1 \oplus \dots \oplus D_q$, with $D_k = \{\mathbf{1}_{D_k}(d_k), d_k\}$, $k \in \mathcal{I}_q$, the number of LMIs to ensure its
 266 negativity obtained with Lemma 2 is $\prod_{k=1}^q |\mathbb{I}_{D_k}^+|$, where $|\mathbb{I}_{D_k}^+| = \frac{(r+|D_k|-1)!}{|D_k|!(r-1)!}$.*

267 The following example illustrates the application of Lemma 2 to obtain a finite set
 268 of LMI-based conditions from a multiple fuzzy summation condition.

Example 3. Consider a 4-dimensional fuzzy summation in the form (23) with multiset of delays $G = \{\langle 2, 0 \rangle, \langle 2, 1 \rangle\}$. Then, $G = D_1 \oplus D_2$, with $D_1 = \{\langle 2, 0 \rangle\}$, $D_2 = \{\langle 2, 1 \rangle\}$ and $|D_1| = \mathbf{1}_G(0) = |D_2| = \mathbf{1}_G(1) = 2$. By Lemma 2, the negativity of this fuzzy summation is guaranteed if the following set of LMIs hold:

$$\begin{aligned} \Gamma_{(1,1,1,1)} \prec 0, \quad \Gamma_{(2,2,2,2)} \prec 0, \quad \Gamma_{(1,1,1,2)} + \Gamma_{(1,1,2,1)} \prec 0, \quad \Gamma_{(1,1,2,2)} \prec 0, \\ \Gamma_{(1,2,1,1)} + \Gamma_{(2,1,1,1)} \prec 0, \quad \Gamma_{(1,2,1,2)} + \Gamma_{(2,1,1,2)} + \Gamma_{(1,2,2,1)} + \Gamma_{(2,1,2,1)} \prec 0, \\ \Gamma_{(1,2,2,2)} + \Gamma_{(2,1,2,2)} \prec 0, \quad \Gamma_{(2,2,1,1)} \prec 0, \quad \Gamma_{(2,2,1,2)} + \Gamma_{(2,2,2,1)} \prec 0. \end{aligned}$$

269 **Remark 8.** Note that the explicit dependence of the set \mathcal{L}_V , obtained from Theorem
270 1, on the premise variables leads to difficulties in finding the largest contractively in-
271 variant set, especially in the case of delayed controllers [3]. A suitable alternative is
272 to consider the following shape-independent subset of \mathcal{L}_V :

$$\mathcal{E}_V = \bigcap_{\mathbf{i}_0^P \in \mathbb{I}_{G^P}} \mathcal{E}(P_{\mathbf{i}_0^P}^{-1}) \subset \mathcal{L}_V,$$

273 where $\mathcal{E}(P_{\mathbf{i}_0^P}^{-1}) = \{x_k \in \mathbb{R}^{n_x} : x_k^\top P_{\mathbf{i}_0^P}^{-1} x_k \leq 1\}$. The maximization of \mathcal{L}_V can be thus
274 performed by maximizing \mathcal{E}_V .

275 By Lemma 2, LMI-based sufficient conditions can be obtained to guarantee condi-
276 tions (12), (13) and (14) in Theorem 1. Hence, the maximization of \mathcal{E}_V can be
277 reformulated as a convex optimization problem in the the following corollary.

Corollary 1. Given G_V as in Theorem 1. If there exist matrices $\bar{P} = \bar{P}^\top \succ 0$, $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \text{pr}_{G^P}^{\mathbf{i}}$, $j \in \{0, 1\}$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \text{pr}_{G^H}^{\mathbf{i}}$, $K_{\mathbf{i}_0^K}$, $\mathbf{i}_0^K = \text{pr}_{G^K}^{\mathbf{i}}$, $F_{\mathbf{i}_j^F}$, $\mathbf{i}_j^F = \text{pr}_{G^F}^{\mathbf{i}}$, $W_{\mathbf{i}_0^W}$, $\mathbf{i}_0^W = \text{pr}_{G^W}^{\mathbf{i}}$, $J_{\mathbf{i}_j^J}$, $\mathbf{i}_j^J = \text{pr}_{G^J}^{\mathbf{i}}$, and diagonal matrices $L_{\mathbf{i}_0^L} \succ 0$, $\mathbf{i}_0^L = \text{pr}_{G^L}^{\mathbf{i}}$ and $U_{\mathbf{i}_0^U} \succ 0$, $\mathbf{i}_0^U = \text{pr}_{G^U}^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that the optimization problem

$$\begin{aligned} \max_{\bar{P}, P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, K_{\mathbf{i}_0^K}, F_{\mathbf{i}_j^F}, W_{\mathbf{i}_0^W}, J_{\mathbf{i}_j^J}, L_{\mathbf{i}_0^L}, U_{\mathbf{i}_0^U}} \det(\bar{P})^{1/n_x} \\ \text{subject to (12), (13), (14) and (27)} \end{aligned} \quad (26)$$

with

$$P_{\mathbf{i}_0^P} \succeq \bar{P}, \quad \mathbf{i}_0^P \in \mathbb{I}_{G^P}, \quad (27)$$

278 is feasible. Then, Problem 1 is solved with the guaranteed contractively invariant set
279 $\mathcal{E}_V \subset \mathcal{L}_V$.

280 *Proof.* Note that inequality (27) directly implies the inclusion $\mathcal{E}(\bar{P}^{-1}) \subseteq \mathcal{E}_V$. The rest
281 of the proof is a direct consequence of Theorem 1 and Remark 8. \square

282 It should be highlighted that the index $\det(\bar{P})^{1/n_x}$ is proportional to the volume of
283 the ellipsoid $\mathcal{E}(\bar{P}^{-1})$.

284 **Remark 9.** The computational complexity of an LMI solver based on interior point
 285 methods can be estimated as being proportional to $\log_{10}(N_d^3 N_l)$, where N_d is the
 286 number of decision variables and N_l the number of LMI rows [1, 29]. For the opti-
 287 mization problem in Corollary 1 with LMI conditions derived using Lemma 2, these
 288 quantities can be computed as follows:

$$\begin{aligned} N_d &= n_x \left(\frac{n_x + 1}{2} \right) \left(1 + r^{|G_0^P|} \right) + r^{|G_0^L|} n_\varphi + r^{|G_0^U|} n_u + \\ &\quad r^{|G_0^H|} n_x^2 + n_x n_u \left(r^{|G_0^F|} + r^{|G_0^W|} \right) + n_u n_\varphi \left(r^{|G_0^K|} + r^{|G_0^J|} \right), \\ N_l &= N_{l_1} + N_{l_2} + N_{l_3} + N_{l_4}, \end{aligned}$$

289 where $N_{l_1} = (1 + n_x) n_e \prod_{k=1}^q |\mathbb{I}_{D_k}^+|$ is the number of LMI rows in (12). In this case,
 290 the multisets D_k , $k \in \mathcal{I}_q$, are defined as in Lemma 2 for $G = G_0^P$; $N_{l_2} = (n_x + n_\varphi +$
 291 $1) n_u \prod_{k=1}^q |\mathbb{I}_{\bar{D}_k}^+|$, with \bar{D}_k related to the multiset $G = G_0^P \cup G_0^H \cup G_0^L \cup G_0^F \cup G_0^K$, is
 292 the number of LMI rows in (13); $N_{l_3} = (2n_x + n_\varphi + n_u) \prod_{k=1}^q |\mathbb{I}_{\bar{D}_k}^+|$, with \bar{D}_k related
 293 to G_V in Theorem 1, is the number of LMI rows in (14); $N_{l_4} = n_x r^{|G_0^P|}$ is the number
 294 of LMI rows in (27). Note that the LMI rows computation is based on the procedure
 295 described in Remark 7.

296 6. Illustrative Examples

297 Three numerical examples are given in this section to demonstrate effectiveness of
 298 the proposed multiple-parameterization approach in reducing control design conserva-
 299 tiveness and enlarging the estimation of DoA. The first two are academic examples,
 300 while the third example is physically motivated.

301 6.1. Example 1

Consider the following nonlinear input-saturated system:

$$\begin{aligned} \begin{bmatrix} x_{k+1(1)} \\ x_{k+1(2)} \end{bmatrix} &= \begin{bmatrix} x_{k(1)} - x_{k(1)} x_{k(2)} \\ -x_{k(1)} - 0.5 x_{k(2)} \end{bmatrix} \\ &+ \begin{bmatrix} 0.5 x_{k(1)}^3 \\ 0.2 x_{k(2)} (1.2 + \sin x_{k(2)}) \end{bmatrix} + \begin{bmatrix} 5 + x_{k(1)} \\ 2 x_{k(1)} \end{bmatrix} \text{sat}(u_k), \quad (28) \end{aligned}$$

302 where $x_{k(1)} \in [-b, b]$, $b > 0$, $x_{k(2)} \in [-2, 2]$ and $\bar{u} = 0.7$. System (28) has two
 303 sector nonlinearities, namely $\varphi_{k(1)} = x_{k(1)}^3 \in \text{co}\{0, b^2 x_{k(1)}\}$ and $\bar{\varphi}_{k(2)} = x_{k(2)} (1.2 +$
 304 $\sin x_{k(2)}) \in \text{co}\{0.2 x_{k(2)}, 2.2 x_{k(2)}\}$. Applying a simple loop transformation [32] to
 305 $\bar{\varphi}_{k(2)}$, the following nonlinearity can be derived $\varphi_{k(2)} = x_{k(2)} (1.2 + \sin x_{k(2)}) -$
 306 $0.2 x_{k(2)} \in \text{co}\{0, 2 x_{k(2)}\}$. Considering $z_{k(1)} = x_{k(1)}$ as a premise variable, the non-
 307 linear system (28) can be *exactly* represented by a 2-rule N-TS fuzzy model (1) with
 308 the membership functions $h_1(x_{k(1)}) = \frac{x_{k(1)} + b}{2b}$, $h_2(x_{k(1)}) = 1 - h_1(x_{k(1)})$, and local
 309 state-space matrices

$$A_1 = \begin{bmatrix} 1 & -b \\ -1 & -0.46 \end{bmatrix}, B_1 = \begin{bmatrix} 5 + b \\ 2b \end{bmatrix}, A_2 = \begin{bmatrix} 1 & b \\ -1 & -0.46 \end{bmatrix}, B_2 = \begin{bmatrix} 5 - b \\ -2b \end{bmatrix}$$

310 and $G_1 = G_2 = \text{diag}(0.5, 0.2)$. The validity domain \mathcal{D}_x of the nonlinear system
311 (28) explicitly depends on the parameter b . Denote b^* as the maximal value of b for
312 which the control design is feasible. This value is used to illustrate the design con-
313 servativeness of the proposed control approach according to the procedure discussed
314 in Remark 6. Applying Corollary 1 with different choices of multisets of delays, the
315 obtained values for b^* are given in Table 1. We assume that $G_0^F = G_0^H = G_0^K =$
316 $G_0^L = G_0^J = G_0^W = G_0^U$ and $|G_0^P|$ is up to 3.

317 Note that taking into account both input and state constraints, the proposed 3-
318 dimensional fuzzy summation based conditions to design non-delayed controllers are
319 an extension of those given in [15]. These conditions can be extended to 6-dimensional
320 fuzzy summation based conditions by setting $G_0^P = \{\langle 2, 0 \rangle\}$ and $G_0^F = \{\langle 3, 0 \rangle\}$,
321 which corresponds to $G_V = \{\langle 0, 4 \rangle, \langle 2, 1 \rangle\}$ in Corollary 1. For this multiset of delays,
322 (14) can be rewritten as a 6-dimensional summation similar to (21).

323 Moreover, a 3-dimensional delayed based condition can be obtained with $G_0^P =$
324 $\{-1\}$ and $G_0^F = \{0, -1\}$, which results in $G_V = \{\langle 2, 0 \rangle, -1\}$. To obtain 6-dimensional
325 fuzzy summation based conditions, we consider $G_0^P = \{-1\}$ and $G_0^F = \{\langle 0, 4 \rangle, -1\}$,
326 which leads to $G_V = \{\langle 0, 5 \rangle, -1\}$. Similarly, for $G_0^P = \{\langle 2, -1 \rangle\}$ and $G_0^F =$
327 $\{\langle 3, 0 \rangle, \langle 2, -1 \rangle\}$, one has $G_V = \{\langle 4, 0 \rangle, \langle 2, -1 \rangle\}$. Then, note that delayed condi-
328 tions offer more flexibility to design since different multisets of delays can be chosen
329 without increasing $|G_V|$.

Table 1: Parameter b^* obtained with Corollary 1 for different multisets of delays.

G_0^P	G_0^F	$ G_V $	b^*
$\{0\}$ (extension of [15])	$\{0\}$	3	1.367
$\{-1\}$	$\{0, -1\}$	3	1.369
$\{\langle 2, 0 \rangle\}$	$\{\langle 3, 0 \rangle\}$	6	1.432
$\{-1\}$	$\{\langle 4, 0 \rangle, -1\}$	6	1.437
$\{\langle 2, -1 \rangle\}$	$\{\langle 3, 0 \rangle, \langle 2, -1 \rangle\}$	6	1.442
$\{\langle 3, 0 \rangle\}$	$\{\langle 3, 0 \rangle\}$	7	1.436
$\{-1\}$	$\{\langle 5, 0 \rangle, -1\}$	7	1.438
$\{\langle 2, -1 \rangle\}$	$\{\langle 4, 0 \rangle, \langle 2, -1 \rangle\}$	7	1.443
$\{\langle 3, -1 \rangle\}$	$\{\langle 3, 0 \rangle, \langle 3, -1 \rangle\}$	7	1.445

330 From the results in Table 1, it is clear that increasing the fuzzy summation dimen-
331 sion leads to less design conservativeness. However, observe also that there is a kind of
332 limit bound for which the conservativeness can be reduced since in comparison to the
333 6-dimensional based conditions, the results with $|G_V| = 7$ are just slightly improved.

334 To further illustrate the advantages of the proposed control approach, let us consider
335 $b = 1.367$, the value for which all design conditions in Table 1 are feasible. As depicted
336 in Figure 1 (a), the contractively invariant set \mathcal{E}_V obtained with the 6-dimensional non-
337 delayed design conditions encompasses the one provided by the 3-dimensional non-
338 delayed conditions (*i.e.*, extension of [15]). Consider now $b = 1.389$, a value for
339 which the 3-dimensional conditions are infeasible. Figure 1 (b) depicts the set $\mathcal{E}(P_{10}^{-1})$
340 together with several closed-loop trajectories

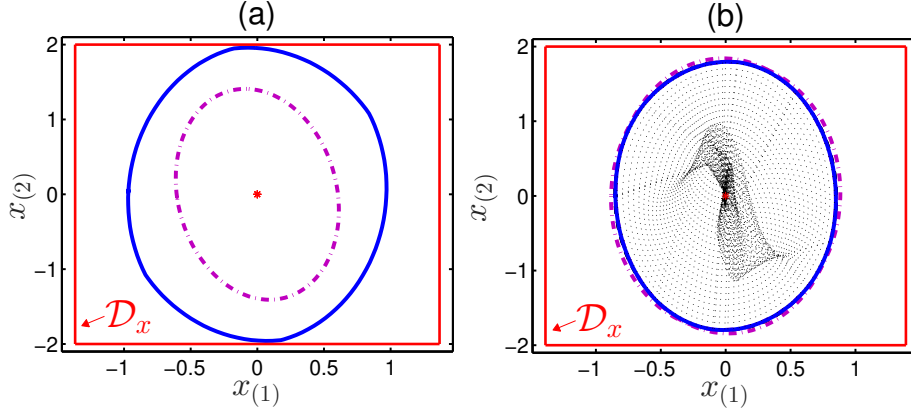


Figure 1: Illustrations on the contractively invariant sets. (a) \mathcal{E}_V obtained from Corollary 1 using the non-delayed conditions with 3 (dash-dotted line) and 6 (solid line) fuzzy summations for $b = 1.367$. (b) $\mathcal{E}(P_{i_0}^{-1})$ (dash-dotted line), \mathcal{E}_V (solid line) and closed-loop trajectories obtained with 6 fuzzy summations and $G_0^P = \{(2, -1)\}$ for $b = 1.389$.

341 6.2. Example 2

342 Consider the discrete-time N-TS fuzzy model (1) described in [14, Example 1] with
 343 the following local matrices:

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 1+T & T & \frac{2}{\pi}\sin\frac{\pi}{2}\bar{T} & -0.1T \\ \bar{T} & 1-2\bar{T} & 0 & 0 \\ \bar{T} & a^2\bar{T} & 1-0.3\bar{T} & 0 \\ 0 & 0 & \frac{2}{\pi}\sin\frac{\pi}{2}T & 1-T \end{bmatrix}, & B_1 &= \begin{bmatrix} (1+a^2)T \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 1+T & \bar{T} & \frac{2}{\pi}\sin\frac{\pi}{2}T & -0.1T \\ T & 1-2\bar{T} & 0 & 0 \\ T & 0 & 1-0.3\bar{T} & 0 \\ 0 & 0 & \frac{2}{\pi}\sin\frac{\pi}{2}T & 1-T \end{bmatrix}, & B_2 &= \begin{bmatrix} \bar{T} \\ 0 \\ 0 \\ 0 \end{bmatrix}, & G_1 = G_2 &= \begin{bmatrix} T \\ 0 \\ 0 \\ T \end{bmatrix}
 \end{aligned}$$

344 where $x(1) \in [-a, a]$ with $a > 0$, $x(3) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the sampling time is $T = 0.5$ s, the
 345 control input bound is $\bar{u} = 2$. The sector nonlinearity is chosen as $\varphi(x) = \sin x(3) -$
 346 $(\frac{2}{\pi}\sin\frac{\pi}{2})x(3) \in \text{co}\{0, (1 - (\frac{2}{\pi}\sin\frac{\pi}{2})x(3))\}$. In addition, the normalized membership
 347 functions are defined by $h_1(x_1) = x_1^2/a^2$ and $h_2(x_1) = 1 - h_1(x_1)$. Similar to Exam-
 348 ple 6.1, it is evaluated the maximal a , denoted by a^* , for which a given control design
 349 condition is feasible. The results obtained with $G_0^F = G_0^H = G_0^K = G_0^L = G_0^J = G_0^W = G_0^U$
 350 are summarized in Table 2. It can be confirmed that fuzzy summation based design
 351 conditions with higher dimensions can provide less conservative results.

352 Consider the case with $a = 7$, for which the design conditions based on 3-dimensional
 353 fuzzy summations are *infeasible*. The conservativeness reduction is evaluated in terms
 354 of the index $\det(\bar{P})^{1/n_x}$ obtained from Corollary 1. The values of this index for differ-
 355 ent multisets of delays are summarized in Table 3. Observe that the volume of the ellip-
 356 soid $\mathcal{E}(\bar{P}^{-1})$ can be increased with higher dimensions in the multiple-parameterization
 357 conditions.

Table 2: Parameter a^* obtained with Corollary 1 for different multisets of delays.

G_0^P	G_0^F	$ G_V $	a^*
$\{0\}$ (extension of [15])	$\{0\}$	3	6.592
$\{-1\}$	$\{0, -1\}$	3	6.592
$\{2, 0\}$	$\{2, 0\}$	5	7.931
$\{-1\}$	$\{3, 0, -1\}$	5	8.224
$\{2, 0\}$	$\{3, 0\}$	6	8.952
$\{-1\}$	$\{4, 0, -1\}$	6	9.854

Table 3: Values of index $\det(\bar{P})^{1/n_x}$ obtained with different multisets of delays in Corollary 1 for $a = 7$.

G_0^P	G_0^F	$ G_V $	$\det(\bar{P})^{1/n_x}$
$\{2, 0\}$	$\{2, 0\}$	5	0.3015
$\{-1\}$	$\{3, 0, -1\}$	5	0.3864
$\{2, 0\}$	$\{3, 0\}$	6	0.3935
$\{-1\}$	$\{4, 0, -1\}$	6	0.4285

358 6.3. Example 3 (Stabilization of Connected Inverted Pendulums)

359 Let us consider the stabilization problem of a constrained system composed of two
 360 inverted pendulums interconnected via a linear spring, *i.e.*, the nonlinear system (4)
 361 with $N = 2$. The system parameters are $T = 0.2s$, $g = 9.8m/s^2$, $l_1 = 0.5m$, $l_2 =$
 362 $0.7m$, $m_1 = 0.25kg$, $m_2 = 0.35kg$, $k_1 = 60N/m$ and $a = 0.4m$.

363 A 4-rule N-TS fuzzy model can be obtained applying the sector nonlinearity ap-
 364 proach with $z_{(\kappa)} = \sin x_{(j)} \in [\frac{2}{\pi}, 1]$ and $\varphi_{\kappa}(x) = \sin x_{(j)} \cos x_{(j)} \in \text{co}\{0, x_{(j)}\}$,
 365 $j = 2\kappa - 1$, $\kappa \in \mathcal{I}_2$. Note however that 16 fuzzy rules are required to derive an
 366 exact representation of this nonlinear system in the classical TS fuzzy form. The
 367 membership functions are $h_1(z_k) = w_0^1(z_{(1)})w_0^2(z_{(2)})$, $h_2(z) = w_0^1(z_{(1)})w_1^2(z_{(2)})$,
 368 $h_3(z) = w_1^1(z_{(1)})w_0^2(z_{(2)})$, $h_4(z) = w_1^1(z_{(1)})w_1^2(z_{(2)})$, where

$$w_0^{\kappa}(z_{(\kappa)}) = \begin{cases} 1, & z_{(\kappa)} = 0, \\ \frac{z_{(\kappa)} - \frac{2}{\pi} \text{asin} z_{(\kappa)}}{(1 - \frac{2}{\pi}) \text{asin} z_{(\kappa)}}, & z_{(\kappa)} \neq 0, \end{cases}$$

369 and $w_1^{\kappa}(z_{(\kappa)}) = 1 - w_0^{\kappa}(z_{(\kappa)})$, $\kappa \in \mathcal{I}_2$. The local state-space matrices are given as

370 follows:

$$A_i = \begin{bmatrix} 1 & \bar{T} & 0 & 0 \\ \frac{gT}{l_1} b_j & 1 & 0 & 0 \\ 0 & 0 & 1 & \bar{T} \\ 0 & 0 & \frac{gT}{l_2} d_i & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & 0 \\ \frac{T}{m_1 l_1^2} & 0 \\ 0 & 0 \\ 0 & \frac{T}{m_2 l_2^2} \end{bmatrix},$$

$$G_i = \begin{bmatrix} 0 & -\frac{k_1 a^2 \bar{T}}{m_1 l_1^2} & 0 & \frac{k_1 a^2 T}{m_2 l_2^2} \\ 0 & \frac{k_1 a^2 \bar{T}}{m_1 l_1^2} & 0 & -\frac{k_1 a^2 T}{m_2 l_2^2} \end{bmatrix}^\top, \quad i = l + 2(j - 1),$$

371 for $j, l \in \mathcal{I}_2$, $b_1 = d_1 = 1$ and $b_2 = d_2 = \frac{2}{\pi}$. Moreover, the input and state con-
 372 straints are respectively given as $\bar{u} = [8 \ 6]^\top$ and $|x_{(1)}| \leq \frac{\pi}{3}$, $|x_{(3)}| \leq \frac{\pi}{3}$. For the
 373 control design, the 3-dimensional delayed conditions with multisets of delays used in
 374 Examples 6.1 and 6.2 are considered. The number of decision variables involved in the
 375 solution is $N_d = 754$ and the number of LMI rows is $N_l = 800$. Hence, the numerical
 376 complexity is estimated as $\log_{10}(N_d^3 N_l) = 11.535$.

377 The closed-loop behavior of the nonlinear system corresponding to the initial condi-
 378 tion $x_0 = [\frac{\pi}{6} \ 0 \ -\frac{\pi}{4} \ 0]^\top$ is depicted in Figure 2. Note that although the input
 379 signal u_k exceeds its amplitude bound \bar{u} , the control signal actually applied to the
 380 system is $\text{sat}(u_k)$, which is always bounded by $\bar{u}_{(l)}$. Observe also that despite the
 381 saturation of both actuators at the beginning of the simulation, the proposed nonlin-
 382 ear controller can guarantee an asymptotic stability of the closed-loop system. This
 383 illustrates the effectiveness of the proposed control approach in dealing with complex
 384 constrained nonlinear systems in real-world applications.

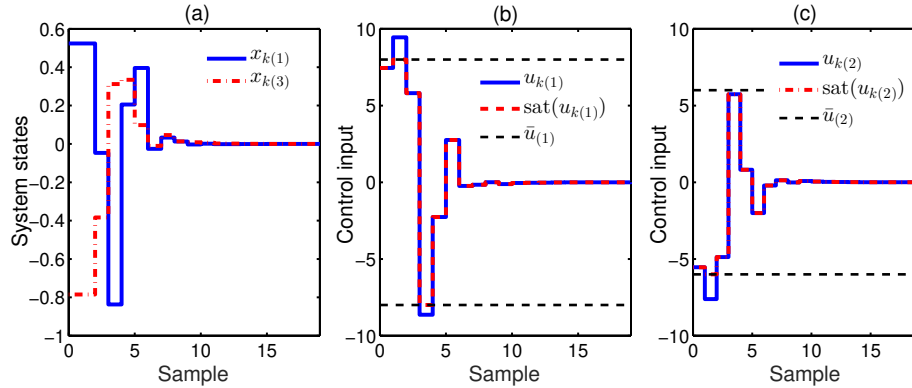


Figure 2: Closed-loop behaviors of the 2 connected inverted pendulums system. (a) Position trajectories. (b) and (c) Control input.

385 7. Concluding Remarks

386 This work has proposed a multiple-parameterization approach for stabilization of
387 N-TS fuzzy systems with input saturation subject to state constraints. This control ap-
388 proach offers the possibility to design both non-delayed and delayed nonlinear fuzzy
389 controllers. In addition, a procedure to obtain convex design conditions in terms of
390 LMIs was proposed. Several numerical tests were performed to illustrate the advan-
391 tages of the proposed control approach compared to existing fuzzy summation based
392 results. Future research directions are related to the development of fuzzy output
393 feedback control designs based on the proposed delayed multiple-parameterization ap-
394 proach and switching mechanisms [30] to further reduce the design conservativeness
395 as well as the online computational burden.

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