



HAL
open science

Simultaneous Estimation of State and Unknown Input with Hinfinity Guarantee on Error-Bounds for Fuzzy Descriptor Systems

Tran Anh-Tu Nguyen, Thierry-Marie Guerra, Victor Campos

► **To cite this version:**

Tran Anh-Tu Nguyen, Thierry-Marie Guerra, Victor Campos. Simultaneous Estimation of State and Unknown Input with Hinfinity Guarantee on Error-Bounds for Fuzzy Descriptor Systems. IEEE Control Systems Letters, 2019, 3 (4), pp.1020-1025. 10.1109/LCSYS.2019.2920768 . hal-03480286

HAL Id: hal-03480286

<https://uphf.hal.science/hal-03480286>

Submitted on 25 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/333557692>

Simultaneous Estimation of State and Unknown Input with L^∞ Guarantee on Error-Bounds for Fuzzy Descriptor Systems

Article in *IEEE Control Systems Letters* · October 2019

DOI: 10.1109/LCSYS.2019.2920768

CITATIONS

18

READS

296

3 authors:



Anh-Tu Nguyen

Université Polytechnique Hauts-de-France

140 PUBLICATIONS 2,084 CITATIONS

SEE PROFILE



Thierry-Marie Guerra

Université Polytechnique Hauts-de-France

349 PUBLICATIONS 9,873 CITATIONS

SEE PROFILE



Victor Campos

Federal University of Minas Gerais

39 PUBLICATIONS 711 CITATIONS

SEE PROFILE

Simultaneous Estimation of State and Unknown Input with l_∞ Guarantee on Error-Bounds for Fuzzy Descriptor Systems

Anh-Tu Nguyen*, *Member, IEEE*, Thierry-Marie Guerra, and Víctor Campos

Abstract—This paper proposes a new l_∞ observer design for fuzzy descriptor systems with *unknown* inputs. The descriptor form is treated using a singular redundancy system representation. To keep the consistency of the resulting fuzzy observer structure, we make use of a virtual variable playing the role of the one-step ahead state estimate. As a result, the observer gain can be constructed with free-structure decision variables to reduce the design conservatism. Using a membership-function-dependent Lyapunov function, the observer design is reformulated as a convex optimization problem with a single line search parameter. In particular, the error bounds of both the state and the unknown input estimations can be minimized through the guaranteed l_∞ performance level. The effectiveness of our result is demonstrated with a challenging real-world application on robot manipulators.

Index Terms—Fuzzy models, fuzzy descriptor systems, fuzzy observers, unknown inputs, Lyapunov method.

I. INTRODUCTION

TAKAGI-Sugeno (T-S) fuzzy model-based approach has become an effective tool to deal with nonlinear systems [1]. Over the last decades, researchers have paid a great deal of attention to the analysis and synthesis issues of T-S fuzzy systems [2]–[5]. Since T-S fuzzy systems are constructed from local linear models blending with nonlinear membership functions (MFs), design conditions for fuzzy controllers/observers can be derived using Lyapunov method and linear matrix inequality (LMI) formulations, see [1], [6]–[8] and related references therein.

A large class of physical systems can be naturally represented using descriptor models [6], [9]. The descriptor forms appear in various engineering applications including robotics [10], healthcare devices [11], transportation and power systems [9] and so on. Fundamental results on analysis and design of descriptor systems have been extensively proposed for linear cases, see [9], [12]–[14] and related references. Various extensions to nonlinear/fuzzy descriptor systems have been also reported [7], [15]–[17]. Note that for fuzzy-model-based approaches, exploiting the descriptor forms allows reducing both the number of fuzzy rules and that of LMI-based design conditions [2]. As a result, the computational burden and the design conservatism can be significantly reduced [6], [17].

A.-T. Nguyen and T.-M. Guerra are with the laboratory LAMIH UMR CNRS 8201, Université Polytechnique Hauts-de-France, Valenciennes, France (e-mail: nguyen.tranahntu@gmail.com; thierry.guerra@uphf.fr).

V. Campos is with the Department of Electronics Engineering, Federal University of Minas Gerais, Brazil (e-mail: koztiah@gmail.com).

*Corresponding author: Anh-Tu Nguyen.

Up to now, existing results on T-S fuzzy systems are mostly concerned with known inputs [4], especially in cases of fuzzy descriptor forms [6], [7], [17]. However, in many practical situations, all input signals cannot be known [16]. Moreover, unmodeled dynamics, uncertain disturbances and faults in engineering systems can be all considered as unknown inputs (UIs) [18]. Therefore, designing UI observers is of crucial importance from both theoretical and practical viewpoints, especially in the areas of fault diagnosis [14], [16], [18], [19], and fault-tolerant control [19]–[21]. Unfortunately, due to the technical difficulties in dealing with the descriptor form, the design of UI observers for fuzzy descriptor systems still remains challenging [19]. For instance, some matrix-rank conditions are required for the UI decoupling in [16], [19], which can be restrictive for fuzzy descriptor observer design. The singular redundancy representation approach in [11] requires a special Luenberger-like observer structure, leading to a conservative design. This paper aims to avoid these technical difficulties. Our main contributions are summarized as follows.

- 1) A singular redundancy approach is exploited to deal with the descriptor form for observer design in presence of UIs. We introduce a *virtual* variable, playing the role of the one-step ahead state estimate, to guarantee the consistency of the resulting observers. Hence, together with specific convexification techniques, the proposed observer gains can be of free structures to reduce the conservatism. Moreover, our fuzzy observer design does not require any specific rank condition for UI decoupling.
- 2) Using Lyapunov-based arguments, the observer design procedure is recast as a convex optimization problem with a single line search parameter, which can be effectively solved with semidefinite programming [22]. In particular, the estimation errors of both the state and the UI can be minimized through a guaranteed l_∞ performance level.

Notation: \mathbb{N} is the set of non-negative integers, and $\mathcal{I}_p = \{1, 2, \dots, p\} \subset \mathbb{N}$. For a matrix X , X^\top denotes its transpose, $X \succ 0$ means that X is positive definite, $\text{He}X = X + X^\top$, and $\|X\|$ denotes its maximum singular value. For a vector $x \in \mathbb{R}^n$, we denote its 2-norm as $\|x\| = \sqrt{x^\top x}$. For a sequence of vectors $\{x_k\}_{k \in \mathbb{N}}$ (respectively matrices $\{X_k\}_{k \in \mathbb{N}}$), we denote $\|x\|_\infty = \sup_{k \geq 0} \|x_k\|$ (respectively $\|X\|_\infty = \sup_{k \geq 0} \|X_k\|$). Then, $\{x_k\}_{k \in \mathbb{N}} \in l_\infty$ if $\|x\|_\infty < \infty$ and $\{X_k\}_{k \in \mathbb{N}} \in \mathcal{L}_\infty$ if $\|X\|_\infty < \infty$. $\text{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1, X_2 . I is the identity matrix of appropriate dimension. “ \star ” stands for the terms deduced by symmetry.

II. PROBLEM FORMULATION

This section formulates the l_∞ observer design of fuzzy descriptor systems using a singular redundancy approach.

A. Description of Fuzzy Descriptor Systems

T-S fuzzy systems are expressed by fuzzy IF-THEN rules, which locally represent linear input-output relations of nonlinear systems. A class of T-S fuzzy descriptor systems with r inference rules can be described as follows [2]:

$$\text{RULE } R_i: \text{ IF } \xi_{1k} \text{ is } \mathcal{M}_1^i \text{ and } \dots \text{ and } \xi_{pk} \text{ is } \mathcal{M}_p^i \\ \text{ THEN } \begin{cases} E_i x_{k+1} = A_i x_k + B_i u_k + D_i d_k \\ y_k = C_i x_k \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ the *known* input, $d_k \in \mathbb{R}^{n_d}$ the *unknown* input, and $y_k \in \mathbb{R}^{n_y}$ the measured output. $\xi_k = [\xi_{1k} \dots \xi_{pk}]^\top \in \mathbb{R}^p$ is the vector of measurable premise variables. $(A_i, B_i, C_i, D_i, E_i)$ are known constant matrices with appropriate dimensions, which represent the i th local linear model. R_i denotes the i th fuzzy inference rule. \mathcal{M}_j^i , with $i \in \mathcal{I}_r$ and $j \in \mathcal{I}_p$, is the fuzzy set. The fuzzy MFs are given by $h_i(\xi) = \frac{\prod_{j=1}^p \mu_j^i(\xi_j)}{\sum_{i=1}^r \prod_{j=1}^p \mu_j^i(\xi_j)}$, $i \in \mathcal{I}_r$, where $\mu_j^i(\xi_j)$ represents the membership grade of ξ_j in the respective fuzzy set \mathcal{M}_j^i . Note that the MFs satisfy the following property:

$$\sum_{i=1}^r h_i(\xi) = 1, \quad 0 \leq h_i(\xi) \leq 1, \quad \forall i \in \mathcal{I}_r. \quad (2)$$

Let Ω be the set of MFs satisfying (2). Using the center-of-gravity method for defuzzification, the fuzzy descriptor system (1) can be represented in the following compact form [2]:

$$\begin{aligned} E(h)x_{k+1} &= A(h)x_k + B(h)u_k + D(h)d_k \\ y_k &= C(h)x_k \end{aligned} \quad (3)$$

where

$$\Pi(h) = \sum_{i=1}^r h_i(\xi) \Pi_i, \quad \Pi \in \{A, B, C, D, E\}$$

and $h = (h_1(\xi), h_2(\xi), \dots, h_r(\xi)) \in \Omega$. We assume that $E(h)$ is a nonsingular matrix for $\forall h \in \Omega$. Note that various nonlinear systems in mechatronics and robotics engineering can be modeled as in (3), where $E(h)$ includes the inertia matrix [10], and is thus invertible [2], [11]. Assume also that $D(h)$ is of full-column rank for $\forall h \in \Omega$, and $\{d_k\}_{k \in \mathbb{N}} \in l_\infty$.

Remark 1. For a given nonlinear descriptor system, the sector nonlinearity approach [2] can be used to derive an *exact* fuzzy model (3) of this system on a compact set of the state space.

Let us define $\bar{x}_k = [x_k^\top \ x_{k+1}^\top]^\top$. To deal with the descriptor matrix $E(h)$ on the left-hand side, we rewrite system (3) in the following *equivalent* form:

$$\begin{aligned} \bar{E} \bar{x}_{k+1} &= \bar{A}(h) \bar{x}_k + \bar{B}(h) u_k + \bar{D}(h) d_k \\ y_k &= \bar{C}(h) \bar{x}_k \end{aligned} \quad (4)$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}(h) = \begin{bmatrix} 0 & I \\ A(h) & -E(h) \end{bmatrix}$, $\bar{B}(h) = \begin{bmatrix} 0 \\ B(h) \end{bmatrix}$, $\bar{D}(h) = \begin{bmatrix} 0 & D(h) \end{bmatrix}$, $\bar{C}(h) = [C(h) \ 0]$.

Remark 2. The singular redundancy form (4) has been exploited for continuous-time fuzzy descriptor systems [6], [11]. However, control and observer designs based on this form have not been yet reported for the discrete-time counterparts. Note that the discrete-time case requires a significantly different treatment compared to the continuous-time results [9].

As in the continuous-time case, the representation (4) would naturally lead to the following fuzzy observer form [11]:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{k+1} \\ \hat{x}_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A(h) & -E(h) \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{x}_{k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ B(h) \end{bmatrix} u_k \\ + \begin{bmatrix} M_1(h) \\ M_2(h) \end{bmatrix} [C(h) \ 0] \begin{bmatrix} x_k - \hat{x}_k \\ x_{k+1} - \hat{x}_{k+1} \end{bmatrix} \quad (5)$$

where $M(h) = [M_1(h)^\top \ M_2(h)^\top]^\top$ is the membership-function-dependent observer gain, and \hat{x}_k is the estimate of the state x_k . However, it follows from the first row of (5) that

$$\hat{x}_{k+1} \equiv \hat{x}_{k+1} + M_1(h)C(h)(x_k - \hat{x}_k). \quad (6)$$

To guarantee the *consistency* of equation (6) for $\forall k \in \mathbb{N}$, and for any initial condition of the system state $x_0 \in \mathbb{R}^{n_x}$, one should impose $M_1(h) = 0$. This may lead to a conservative observer design [17]. To avoid any structure imposition on the observer gain, we consider the following fuzzy observer form:

$$\begin{aligned} \bar{E} \hat{x}_{k+1} &= \bar{A}(h) \hat{x}_k + \bar{B}(h) u_k + G(h)^{-1} L(h) (y_k - \hat{y}_k) \\ \hat{y}_k &= \bar{C}(h) \hat{x}_k \end{aligned} \quad (7)$$

with $\hat{x}_k = [\hat{x}_k^\top \ \psi_k^\top]^\top$. The membership-function-dependent observer gains to be determined

$$G(h) = \sum_{i=1}^r h_i(\xi_k) G_i, \quad L(h) = \sum_{i=1}^r h_i(\xi_k) L_i$$

are of appropriate dimensions. The existence of $G(h)^{-1}$ is further discussed in Theorem 1.

Remark 3. The key idea of the observer form (7) is to exploit the new *virtual* variable ψ_k as an estimate of x_{k+1} to guarantee the consistency of this form without requiring any special structure of the observer gains $G(h)$ and $L(h)$.

Let us define the state estimation errors as

$$e_k = x_k - \hat{x}_k, \quad \bar{e}_k = \bar{x}_k - \hat{\bar{x}}_k = \begin{bmatrix} e_k \\ x_{k+1} - \psi_k \end{bmatrix}.$$

From (4) and (7), the error dynamics is given by

$$\bar{E} \bar{e}_{k+1} = \mathcal{A}(h) \bar{e}_k + \bar{D}(h) d_k \quad (8)$$

where $\mathcal{A}(h) = \bar{A}(h) - G(h)^{-1} L(h) \bar{C}(h)$. We are now in the position to formulate the observer design problem.

Problem 1. Given a fuzzy descriptor system (3). Determine the observer gains $G(h)$ and $L(h)$ such that the error dynamics (8) verifies the following properties.

- (P1) If $d_k = 0$, $\forall k \in \mathbb{N}$, the error dynamics (8) is globally exponentially stable with respect to the origin.
- (P2) If $d_k \neq 0$, $\forall k \in \mathbb{N}$, the estimation error e_k is uniformly bounded for any initial condition e_0 and any input sequence $\{d_k\}_{k \in \mathbb{N}} \in l_\infty$. In addition, we have the property

$$\lim_{k \rightarrow \infty} \sup \|e_k\| < \gamma \|d\|_{l_\infty}, \quad \gamma > 0 \quad (9)$$

where the l_∞ -gain γ is specified in Theorem 1. Moreover, for $e_0 = 0$, it follows that $\|e_k\| < \gamma\|d\|_{l_\infty}, \forall k \in \mathbb{N}$.

The error dynamics (8) verifying the properties (P1)-(P2) is said to be globally uniformly l_∞ -stable with a performance level γ . A similar concept of l_∞ -stability can be found in [23].

III. LMI-BASED SYNTHESIS OF FUZZY DESCRIPTOR OBSERVERS WITH l_∞ PERFORMANCE GUARANTEE

This section provides a numerically tractable solution to design an l_∞ observer for the descriptor fuzzy system (3). We begin with the following theorem providing sufficient conditions to guarantee the global uniform l_∞ -stability of the error dynamics (8).

Theorem 1. Consider the error dynamics (8). Given a scalar $\alpha \in (0, 1)$, if there exist membership-function-dependent symmetric matrices $P(h) \in \mathbb{R}^{n_x \times n_x}$, $R(h) \in \mathbb{R}^{n_x \times n_x}$, matrices $Q(h) \in \mathbb{R}^{n_x \times n_x}$, $G_1(h) \in \mathbb{R}^{n_x \times n_x}$, $G_2(h) \in \mathbb{R}^{n_x \times n_x}$, $G_3(h) \in \mathbb{R}^{n_x \times n_x}$, $G_4(h) \in \mathbb{R}^{n_x \times n_x}$, $L_1(h) \in \mathbb{R}^{n_x \times n_y}$, $L_2(h) \in \mathbb{R}^{n_x \times n_y}$, and positive scalars τ, γ such that

$$\Sigma + \Phi + \Phi^\top < 0 \quad (10)$$

$$\begin{bmatrix} P(h) & \star \\ Q(h) & R(h) \end{bmatrix} \succ 0 \quad (11)$$

$$\begin{bmatrix} P(h) & \star \\ I & \gamma I \end{bmatrix} \succ 0 \quad (12)$$

for any $h, h_+ \in \Omega$, where

$$\Sigma = \begin{bmatrix} (\alpha - 1)P(h) & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & P(h_+) & \star & \star \\ 0 & 0 & Q(h_+) & R(h_+) & \star \\ 0 & 0 & 0 & 0 & -\alpha\gamma I \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \Gamma & \Upsilon & -G_1(h) & -G_2(h) & G_2(h)D(h) \\ \Xi & \Phi & -G_3(h) & -G_4(h) & G_4(h)D(h) \\ \tau\Gamma & \tau\Upsilon & -\tau G_1(h) & -\tau G_2(h) & \tau G_2(h)D(h) \\ \tau\Xi & \tau\Phi & -\tau G_3(h) & -\tau G_4(h) & \tau G_4(h)D(h) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma = G_2(h)A(h) - L_1(h)C(h), \quad \Upsilon = G_1(h) - G_2(h)E(h)$$

$$\Xi = G_4(h)A(h) - L_2(h)C(h), \quad \Phi = G_3(h) - G_4(h)E(h)$$

$$h_+ = (h_1(\xi_{k+1}), h_2(\xi_{k+1}), \dots, h_r(\xi_{k+1})).$$

Then, the error dynamics (8) is globally uniformly l_∞ -stable with performance level γ . Moreover, the fuzzy observer structure corresponding to the initial system (3) is given by

$$\begin{aligned} E(h)\hat{x}_{k+1} &= A(h)\hat{x}_k + B(h)u_k + [E(h) \quad I] \mathcal{K}(h) (y_k - \hat{y}_k) \\ \hat{y}_k &= C(h)\hat{x}_k \end{aligned} \quad (13)$$

with $\mathcal{K}(h) = G(h)^{-1}L(h)$ and

$$G(h) = \begin{bmatrix} G_1(h) & G_2(h) \\ G_3(h) & G_4(h) \end{bmatrix}, \quad L(h) = \begin{bmatrix} L_1(h) \\ L_2(h) \end{bmatrix}. \quad (14)$$

Proof. To begin with, let us define

$$\bar{P}(h) = \begin{bmatrix} P(h) & \star \\ Q(h) & R(h) \end{bmatrix}, \quad \mathcal{K}(h) = \begin{bmatrix} \mathcal{K}_1(h) \\ \mathcal{K}_2(h) \end{bmatrix}. \quad (15)$$

It is important to note that inequality (10) implies

$$\bar{P}(h_+) - \tau(G(h) + G(h)^\top) < 0. \quad (16)$$

Since $\bar{P}(h_+) \succ 0$ due to (11), inequality (16) guarantees the nonsingularity of $G(h)$, thus the existence of $G(h)^{-1}$.

For the observer design, let us consider the following membership-function-dependent Lyapunov function:

$$V_k = \bar{e}_k^\top \bar{E}^\top \bar{P}(h) \bar{E} \bar{e}_k. \quad (17)$$

Due to the special structure of \bar{E} , it follows that

$$\bar{E}^\top \bar{P}(h) \bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P(h) & \star \\ Q(h) & R(h) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P(h) & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, V_k can be rewritten as $V_k = e_k^\top P(h) e_k$, which is positive definite for $\forall k \in \mathbb{N}$ since $P(h) \succ 0$. From the matrix partitions in (14)-(15), inequality (10) can be represented as

$$\begin{bmatrix} (\alpha - 1)\bar{E}^\top \bar{P}(h) \bar{E} & \star & \star \\ 0 & \bar{P}(h_+) & \star \\ 0 & 0 & -\alpha\gamma I \end{bmatrix} + \Psi + \Psi^\top < 0 \quad (18)$$

where $\Psi = [G(h)^\top \quad \tau G(h)^\top \quad 0]^\top [A(h) \quad -I \quad \bar{D}(h)]$. From the dynamics equation (8), we note that

$$2\Lambda_k (\mathcal{A}(h)\bar{e}_k - \bar{E}\bar{e}_{k+1} + \bar{D}(h)d_k) = 0, \quad \forall k \in \mathbb{N} \quad (19)$$

with $\Lambda_k = \bar{e}_k^\top G(h) + \tau \bar{e}_{k+1}^\top \bar{E}^\top G(h)$. Pre- and postmultiplying (18) with $[\bar{e}_k^\top \quad \bar{e}_{k+1}^\top \bar{E}^\top \quad d_k^\top]$ and its transpose while taking into account (19), we obtain

$$\begin{aligned} \bar{e}_{k+1}^\top \bar{E}^\top \bar{P}(h_+) \bar{E} \bar{e}_{k+1} + (\alpha - 1)\bar{e}_k^\top \bar{E}^\top \bar{P}(h) \bar{E} \bar{e}_k \\ - \alpha\gamma d_k^\top d_k < 0, \quad \forall k \in \mathbb{N}. \end{aligned} \quad (20)$$

Condition (20) can be represented in the compact form

$$\Delta V_k + \alpha(V_k - \gamma d_k^\top d_k) < 0, \quad \forall k \in \mathbb{N} \quad (21)$$

where V_k is defined in (17), and $\Delta V_k = V_{k+1} - V_k$ is its difference along the trajectories of the error dynamics (8). It follows from (21) that

$$V_k < (1 - \alpha)V_{k-1} + \alpha\gamma\|d_{k-1}\|^2, \quad \forall k \geq 1. \quad (22)$$

By recursivity and since $\alpha \in (0, 1)$, we derive from (22) that

$$\begin{aligned} V_k &< (1 - \alpha)^k V_0 + \alpha\gamma \sum_{i=0}^{k-1} (1 - \alpha)^i \|d_{k-1-i}\|^2 \\ &< (1 - \alpha)^k V_0 + \alpha\gamma \|d\|_{l_\infty}^2 \sum_{i=0}^{k-1} (1 - \alpha)^i \\ &< (1 - \alpha)^k V_0 + \gamma \|d\|_{l_\infty}^2, \quad \forall k \geq 1. \end{aligned} \quad (23)$$

It is clear that inequality (23) guarantees the property (P1) defined in Problem 1 when $d_k = 0$, for $\forall k \in \mathbb{N}$. By Schur complement lemma [22], it follows from (12) and (23) that

$$\|e_k\|^2 < \gamma V_k < \gamma(1 - \alpha)^k V_0 + \gamma^2 \|d\|_{l_\infty}^2, \quad \forall k \geq 1. \quad (24)$$

Remark that inequality (24) implies (9). Moreover, we have $V_0 = 0$, for $e_0 = 0$. Then, it follows from (24) that

$$\|e_k\|^2 < \gamma^2 \|d\|_{l_\infty}^2, \quad \forall k \geq 1.$$

As a consequence, the property (P2) is verified.

To recover the observer structure (13), let us put (7) in its explicit form. Then, we can directly obtain

$$\psi_k = \hat{x}_{k+1} - \mathcal{K}_1(h)(y_k - \hat{y}_k) \quad (25)$$

$$E(h)\psi_k = A(h)\hat{x}_k + B(h)u_k + \mathcal{K}_2(h)(y_k - \hat{y}_k). \quad (26)$$

Eliminating the virtual variable ψ_k in (25) and (26), the observer structure (13) is straightforwardly recovered. The proof can be now concluded. \square

Remark 4. Note from (25) that the virtual variable ψ_k plays a similar role as \hat{x}_{k+1} in (5). If $d_k = 0$, for $\forall k \in \mathbb{N}$, and the estimation error \bar{e}_k in (8) converges to the origin, then $\hat{x}_k \rightarrow x_k$ and $\psi_k \rightarrow \hat{x}_{k+1}$ when $k \rightarrow \infty$.

Remark 5. The proposed observer design offers several extra degrees of freedom to enable less conservative results. First, the use of the singular redundancy form (4) of system (3) together with the virtual estimation variable ψ_k in (8) allows introducing slack variables with free structures into the design problem. Second, the observer gain $\mathcal{K}(h)$ is constructed from $G(h)$ and $L(h)$ which are *independent* to $\bar{P}(h)$. Moreover, introducing the extra variable $G(h)$, the Lyapunov matrix $\bar{P}(h)$ is decoupled from any product with the state-space matrices $\bar{A}(h)$, $\bar{C}(h)$, $\bar{D}(h)$, $\bar{E}(h)$. Third, all decision variables can be membership-function-dependent for the observer design.

Remark 6. Consider the error dynamics (8). By minimizing the l_∞ -gain level γ , it follows directly from (9) that the estimation error e_k can be minimized.

Remark 7. The observer design in Theorem 1 is directly applied to the standard T-S fuzzy systems by setting $E(h) = I$. Hence, the proposed method can be considered as a generalization of the l_∞ observer design for fuzzy descriptor systems.

Conditions (10)-(12) are expressed in terms of membership-function-dependent matrix inequalities. Hence, these cannot be directly used to design l_∞ fuzzy observers. Theorem 2 aims to convert these inequalities into a finite set of numerically tractable design conditions.

Theorem 2. Consider the error dynamics (8). For a given scalar $\alpha \in (0, 1)$, if there exist symmetric matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, $R_i \in \mathbb{R}^{n_x \times n_x}$, matrices $Q_i \in \mathbb{R}^{n_x \times n_x}$, $G_{1i} \in \mathbb{R}^{n_x \times n_x}$, $G_{2i} \in \mathbb{R}^{n_x \times n_x}$, $G_{3i} \in \mathbb{R}^{n_x \times n_x}$, $G_{4i} \in \mathbb{R}^{n_x \times n_x}$, $L_{1i} \in \mathbb{R}^{n_x \times n_y}$, $L_{2i} \in \mathbb{R}^{n_x \times n_y}$, $i \in \mathcal{I}_r$, and positive scalars τ , γ such that the following optimization problem is achievable:

$$\underset{\zeta_i, i \in \mathcal{I}_r}{\text{minimize}} \quad \gamma \quad (27)$$

subject to

$$\Sigma_{Pli} + \text{He}(\Phi_{\phi ii}) \prec 0, \quad i, l \in \mathcal{I}_r \quad (28)$$

$$\Sigma_{Pli} + \text{He}(\Phi_{\phi ij} + \Phi_{\phi ji}) \prec 0, \quad i, j, l \in \mathcal{I}_r, \quad i < j \quad (29)$$

$$\begin{bmatrix} P_i & \star \\ Q_i & R_i \end{bmatrix} \succ 0, \quad i \in \mathcal{I}_r \quad (30)$$

$$\begin{bmatrix} P_i & \star \\ I & \gamma I \end{bmatrix} \succ 0, \quad i \in \mathcal{I}_r \quad (31)$$

where $\zeta_i = (\tau, \gamma, P_i, R_i, Q_i, L_{1i}, L_{2i}, G_{1i}, G_{2i}, G_{3i}, G_{4i})$ and

$$\Sigma_{Pli} = \begin{bmatrix} (\alpha - 1)P_i & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & P_l & \star & \star \\ 0 & 0 & Q_l & R_l & \star \\ 0 & 0 & 0 & 0 & -\alpha\gamma I \end{bmatrix}$$

$$\Phi_{\phi ij} = \begin{bmatrix} \Gamma_{ij} & \Upsilon_{ij} & -G_{1i} & -G_{2i} & G_{2i}D_j \\ \Xi_{ij} & \Phi_{ij} & -G_{3i} & -G_{4i} & G_{4i}D_j \\ \tau\Gamma_{ij} & \tau\Upsilon_{ij} & -\tau G_{1i} & -\tau G_{2i} & \tau G_{2i}D_j \\ \tau\Xi_{ij} & \tau\Phi_{ij} & -\tau G_{3i} & -\tau G_{4i} & \tau G_{4i}D_j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_{ij} = G_{2i}A_j - L_{1i}C_j, \quad \Xi_{ij} = G_{4i}A_j - L_{2i}C_j$$

$$\Upsilon_{ij} = G_{1i} - G_{2i}E_j, \quad \Phi_{ij} = G_{3i} - G_{4i}E_j.$$

Then, the error dynamics (8) is globally uniformly l_∞ -stable with performance level γ . Moreover, the fuzzy observer structure corresponding to the initial system (3) is given in (13).

Proof. Based on the convex combination form of the state-space matrices $A(h)$, $C(h)$, $D(h)$, $E(h)$, and the decision matrices $P(h)$, $R(h)$, $Q(h)$, $L_1(h)$, $L_2(h)$, $G_1(h)$, $G_2(h)$, $G_3(h)$, $G_4(h)$, inequality (10) can be exactly represented as

$$\sum_{l=1}^r \sum_{i=1}^r h_l(\xi_{k+1})h_i^2(\xi_k)(\Sigma_{Pli} + \Phi_{\phi ii} + \Phi_{\phi ii}^\top)$$

$$+ \sum_{l=1}^r \sum_{i=1}^N \sum_{i < j}^N h_l(\xi_{k+1})h_i(\xi_k)h_j(\xi_k)\Theta_{lij} \prec 0 \quad (32)$$

where $\Theta_{lij} = \Sigma_{Pli} + \text{He}(\Phi_{\phi ij} + \Phi_{\phi ji})$. Since $h_i(\xi) \geq 0$, $\forall i \in \mathcal{I}_r$, it is clear that (28) and (29) guarantee condition (32). Moreover, condition (11) can be also rewritten as

$$\sum_{i=1}^r h_i(\xi) \begin{bmatrix} P_i & \star \\ Q_i & R_i \end{bmatrix} \succ 0. \quad (33)$$

We can easily see that (30) ensures (33). Similarly, condition (12) is also verified due to (31). The rest of the proof follows the results of Theorem 1. \square

Remark 8. Note that the positive scalar τ is introduced in the observer design through the equality (19) to reduce the design conservatism. A similar idea has been exploited in [3] for the control design of non-descriptor fuzzy systems. The design conditions in Theorem 2 are expressed in terms of linear matrix inequalities with a line search over the scalar τ . A gridding method can be performed to search for τ . Then, the optimization problem (27) under LMI constraints (28)-(31) can be effectively solved with available numerical solvers [22].

IV. UNKNOWN INPUT ESTIMATION WITH GUARANTEE ON ULTIMATE NORM-BOUNDED ERROR

The reconstruction of the UI of fuzzy descriptor systems is discussed in this section. Moreover, we also provide a characterization on the ultimate norm upper-bound of the UI estimation error as in [23].

To begin with, we derive from (3) that

$$D(h)d_k = E(h)x_{k+1} - A(h)x_k - B(h)u_k. \quad (34)$$

Since $D(h)$ is of full-column rank for $\forall h \in \Omega$, its pseudo-inverse $D(h)^\dagger$ exists and $D(h)^\dagger D(h) = I$. Premultiplying (34) with $D(h)^\dagger$ yields

$$d_k = D(h)^\dagger (E(h)x_{k+1} - A(h)x_k - B(h)u_k).$$

The estimate \hat{d}_k of d_k is proposed in the form

$$\hat{d}_k = D(h)^\dagger (E(h)\hat{x}_{k+1} - A(h)\hat{x}_k - B(h)u_k). \quad (35)$$

Hence, the UI estimation error $e_{dk} = d_k - \hat{d}_k$ is given by

$$e_{dk} = D(h)^\dagger (E(h)e_{k+1} - A(h)e_k).$$

It follows from (35) that

$$\begin{aligned} \|e_{dk}\| &\leq \|D(h)^\dagger E(h)\| \|e_{k+1}\| + \|D(h)^\dagger A(h)\| \|e_k\| \\ &\leq \|D(h)^\dagger\| (\|E(h)\| \|e_{k+1}\| + \|A(h)\| \|e_k\|). \end{aligned} \quad (36)$$

If the optimization problem (27) is feasible. Then, we have

$$\limsup_{k \rightarrow \infty} \|e_k\| < \gamma \|d\|_{l_\infty}, \quad \limsup_{k \rightarrow \infty} \|e_{k+1}\| < \gamma \|d\|_{l_\infty}. \quad (37)$$

Combining (36) and (37), the ultimate norm upper-bound of the estimation error $d_k - \hat{d}_k$ is characterized by

$$\limsup_{k \rightarrow \infty} \|e_{dk}\| \leq \gamma \eta \|d\|_{l_\infty} \quad (38)$$

where $\eta = \|D(h)^\dagger\|_\infty (\|E(h)\|_\infty + \|A(h)\|_\infty)$. The following result summarizes the above discussions.

Corollary 1. Consider the error dynamics (8). If the optimization problem (27) in Theorem 2 is feasible. Then, there exists a fuzzy descriptor observer (13) such that

- (i) The error dynamics (8) is globally uniformly l_∞ -stable with a guaranteed performance level γ ,
- (ii) The UI estimation error is ultimately bounded as shown in (38), where \hat{d}_k is defined in (35).

Remark that if the l_∞ -gain level γ is arbitrarily small and

$$\{\hat{\Phi}(h(\xi_k))\}_{k \in \mathbb{N}} \in \mathcal{L}_\infty, \quad \text{with } \hat{\Phi} \in \{A, D^\dagger, E\}.$$

Then, the state and the UI can be estimated with an *arbitrary* degree of accuracy as respectively shown in (37) and (38).

V. ILLUSTRATIVE EXAMPLE

For illustration purposes, let us consider a nonlinear two-link robot manipulator [10] whose dynamics is given by

$$M(\theta(t))\ddot{\theta}(t) - N(\theta(t)) = u(t) + d(t) \quad (39)$$

where

$$\begin{aligned} M(\theta) &= \begin{bmatrix} m_1 l^2 + m_2 L_1^2 + J_1 & m_2 l L_2 \cos(\theta_1 - \theta_2) \\ * & m_2 L_2^2 + J_2 \end{bmatrix} \\ N(\theta) &= \begin{bmatrix} (m_1 L_1 + m_2 l) g \sin(\theta_1) \\ m_2 L_2 g \sin(\theta_2) \end{bmatrix} \end{aligned}$$

and $\theta = [\theta_1 \ \theta_2]^\top$ is the position vector, $u = [u_1 \ u_2]^\top$ is the known torque input, $d = [d_1 \ d_2]^\top$ is the unknown input at the joints. The robot nomenclature is given in Table I.

A discrete-time representation of the robot model (39) can be obtained by the Euler discretization method as

$$E(\xi_k)x_{k+1} = A(\xi_k)x_k + B(u_k + d_k), \quad y_k = Cx_k \quad (40)$$

TABLE I
ROBOT NOMENCLATURE.

Symbol	Description	Value
L_1, L_2	Distances of joints from arm mass centers (m)	0.5, 0.5
l	Distance between joint-1 and joint-2 (m)	0.3
m_1, m_2	Masses of robot arms (kg)	10, 3
J_1, J_2	Inertia of robot arms (kgm^2)	0.83, 0.41
g	Gravity acceleration (m/s^2)	9.81
T_s	Sampling time (ms)	5

where $x_k = [\theta_{1k} \ \theta_{2k} \ \dot{\theta}_{1k} \ \dot{\theta}_{2k}]^\top$ and

$$E(\xi_k) = \begin{bmatrix} I & 0 \\ 0 & M(\xi_k) \end{bmatrix}, \quad A(\xi_k) = \begin{bmatrix} I & T_s I \\ T_s R(\xi_k) & M(\xi_k) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ T_s I \end{bmatrix}, \quad C = [I \ 0]$$

$$M(\xi_k) = \begin{bmatrix} m_1 l^2 + m_2 L_1^2 + J_1 & m_2 l L_2 \xi_{1k} \\ * & m_2 L_2^2 + J_2 \end{bmatrix}$$

$$R(\xi_k) = \begin{bmatrix} (m_1 L_1 + m_2 l) g \xi_{2k} & 0 \\ 0 & m_2 L_2 g \xi_{3k} \end{bmatrix}$$

$$\xi_{1k} = \cos(\theta_{1k} - \theta_{2k}), \quad \xi_{2k} = \frac{\sin(\theta_{1k})}{\theta_{1k}}, \quad \xi_{3k} = \frac{\sin(\theta_{2k})}{\theta_{2k}}.$$

For system (40), the vector of premise variables is defined as $\xi_k = [\xi_{1k} \ \xi_{2k} \ \xi_{3k}]^\top$, with $\xi_{i \min} \leq \xi_{ik} \leq \xi_{i \max}$, $\forall i \in \mathcal{I}_3$,

$$\xi_{1 \min} = -1, \quad \xi_{1 \max} = 1$$

$$\xi_{2 \min} = \xi_{3 \min} = -0.2172, \quad \xi_{2 \max} = \xi_{3 \max} = 1.$$

Applying sector nonlinearity approach [2, Chapter 2], an equivalent fuzzy descriptor representation (3) with eight fuzzy rules ($r = 8$) can be obtained for the nonlinear system (40). The state-space matrices and the nonlinear MFs corresponding to this fuzzy representation are omitted for brevity.

Theorem 2 is employed with $\alpha = 0.05$ and a logarithmically scaled grid over $\tau \in [10^{-5}, 10^5]$ with 100 points. The suboptimal values of γ found for each τ are given in Fig. 1. To illustrate the interest of using the *virtual* variable ψ_k for the observer design, the case in which $L_{1i} = 0$, $\forall i \in \mathcal{I}_8$, *i.e.*, design conditions corresponding to the observer structure (5), is also presented. We assume that γ is infinite when the optimization problem (27) is unfeasible. Note that the classical observer structure (5) always leads to more conservative results compared to the new observer (7). The design conditions in Theorem 2 are feasible for $\tau \in [1.1233, 10^5]$, with a minimum at $\tau = 3853.5$ for $\gamma = 0.071$. The corresponding observer solution is used for estimation purposes. For illustrations, we select $x_0 = [\pi \ \pi \ 1 \ -1]^\top$, $\hat{x}_0 = [\pi \ \pi \ 0 \ 0]^\top$, $u_k = [0 \ 0]^\top$ as the known input, and the UI vector d_k is as

$$d_k = \begin{cases} [0 \ 0]^\top, & k \leq 20 \\ [5 \cos(0.05k) \ 5 \cos(0.05k + \frac{\pi}{3})]^\top, & k > 20. \end{cases}$$

The above choice of d_k allows us to compare the estimation performance for two cases: *with* (Case 1) and *without* (Case 2) the presence of unknown inputs. The estimation performance is summarized in Fig. 2. Observe in Fig. 2(a) that for Case 1, the state estimation error e_k asymptotically converges toward

the origin from the initial condition. We remark in Case 2 that the system state can be estimated with a satisfactory accuracy and the l_∞ -gain performance is verified. Moreover, as represented in Figs. 2(b) and (c), both unknown inputs are effectively reconstructed with the estimate given in (35).

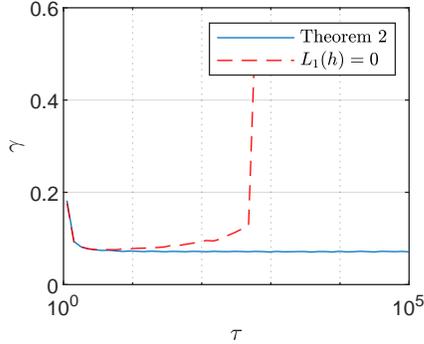


Fig. 1. Suboptimal values of γ as a function of the scalar parameter τ in Theorem 2 and Theorem 2 with $L_{1i} = 0, \forall i \in \mathcal{I}_8$, for the nonlinear two-link robot manipulator with $\alpha = 0.05$.

VI. CONCLUSIONS

A new l_∞ observer design has been proposed for fuzzy descriptor systems. Based on Lyapunov stability arguments, the design conditions are derived in terms of LMI constraints over a single line search parameter. In particular, exploiting a singular redundancy representation and a virtual estimation variable, all involved decision variables for the observer design can be of free structure to reduce the design conservatism. The estimation performance of both the state and the UI is guaranteed with a suboptimal l_∞ -gain level. We show the application of our result to the observer problem of a nonlinear robot manipulator. This work focuses on the case where the premise variables are online measurable. Designing

UI observers for fuzzy descriptor systems with unmeasurable premise variables is not trivial, which is currently investigated.

REFERENCES

- [1] A.-T. Nguyen, T. Taniguchi, L. Eciolaza, V. Campos, R. Palhares, and M. Sugeno, "Fuzzy control systems: Past, present and future," *IEEE Comput. Intell. Mag.*, vol. 14, no. 1, pp. 56–68, 2019.
- [2] K. Tanaka and H. Wang, *Fuzzy Control Systems Design and Analysis: a Linear Matrix Inequality Approach*. John Wiley & Sons, 2004.
- [3] L. Mozelli, R. Palhares, and G. Avellar, "A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems," *Inf. Sci.*, vol. 179, no. 8, pp. 1149–1162, 2009.
- [4] M. Chadli and H. Karimi, "Robust observer design for unknown inputs Takagi-Sugeno models," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 1, pp. 158–164, 2013.
- [5] A.-T. Nguyen, K. Tanaka, A. Dequidt, and M. Dambrine, "Static output feedback design for a class of constrained Takagi-Sugeno fuzzy systems," *J. Franklin Inst.*, vol. 354, no. 7, pp. 2856–2870, 2017.
- [6] K. Tanaka, H. Ohtake, and H. O. Wang, "A descriptor system approach to fuzzy control system design via fuzzy Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 333–341, 2007.
- [7] F. Li, P. Shi, L. Wu, and X. Zhang, "Fuzzy-model-based \mathcal{D} -stability and nonfragile control for discrete-time descriptor systems with multiple delays," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 1019–1025, 2014.
- [8] X. Su, L. Wu, P. Shi, and Y.-D. Song, "A novel approach to output feedback control of fuzzy stochastic systems," *Automatica*, vol. 50, no. 12, pp. 3268–3275, 2014.
- [9] S. Xu and J. Lam, *Robust Control and Filtering of Singular Systems*. Berlin: Springer, 2006, vol. 332.
- [10] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot Modeling and Control*. John Wiley & Sons, 2006.
- [11] K. Guelton, S. Delprat, and T.-M. Guerra, "An alternative to inverse dynamics joint torques estimation in human stance based on a Takagi-Sugeno unknown-inputs observer in the descriptor form," *Control Eng. Pract.*, vol. 16, no. 12, pp. 1414–1426, 2008.
- [12] M. Chadli and M. Darouach, "Novel bounded real lemma for discrete-time descriptor systems: Application to \mathcal{H}_∞ control design," *Automatica*, vol. 48, no. 2, pp. 449–453, 2012.
- [13] Z. Wang, Y. Shen, X. Zhang, and Q. Wang, "Observer design for discrete-time descriptor systems: an LMI approach," *Syst. Control Lett.*, vol. 61, no. 6, pp. 683–687, 2012.
- [14] Z. Wang, P. Shi, and C.-C. Lim, " $\mathcal{H}_\infty/\mathcal{H}_\infty$ fault detection observer in finite frequency domain for linear parameter-varying descriptor systems," *Automatica*, vol. 86, pp. 38–45, 2017.
- [15] L. Hassan, A. Zemouche, and M. Boutayeb, " \mathcal{H}_∞ observers design for a class of nonlinear time-delay systems in descriptor form," *Int. J. Control*, vol. 84, no. 10, pp. 1653–1663, 2011.
- [16] B. Marx, D. Koenig, and J. Ragot, "Design of observers for T-S descriptor systems with unknown inputs and application to fault diagnosis," *IET Control Theory Appl.*, vol. 1, no. 5, pp. 1487–1495, 2007.
- [17] T. M. Guerra, V. Estrada-Manzo, and Z. Lendek, "Observer design for Takagi-Sugeno descriptor models: An LMI approach," *Automatica*, vol. 52, pp. 154–159, 2015.
- [18] J. Zhang, X. Zhao, F. Zhu, and H. Karimi, "Reduced-order observer design for switched descriptor systems with unknown inputs," *IEEE Trans. Autom. Control*, pp. 1–1, 2019.
- [19] Q. Jia, W. Chen, Y. Zhang, and H. Li, "Fault reconstruction and fault-tolerant control via learning observers in Takagi-Sugeno fuzzy descriptor systems with time delays," *IEEE Trans. Ind. Electron.*, vol. 62, no. 6, pp. 3885–3895, 2015.
- [20] Z. Gao, C. Cecati, and S. X. Ding, "A survey of fault diagnosis and fault-tolerant techniques—Part I: Fault diagnosis with model-based and signal-based approaches," *IEEE Trans. Ind. Electron.*, vol. 62, no. 6, pp. 3757–3767, 2015.
- [21] S. Yin, H. Yang, and O. Kaynak, "Sliding mode observer-based FTC for Markovian jump systems with actuator and sensor faults," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3551–3558, 2017.
- [22] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994, vol. 15.
- [23] A. Chakrabarty, M. J. Corless, G. T. Buzzard, S. H. Żak, and A. E. Rundell, "State and unknown input observers for nonlinear systems with bounded exogenous inputs," *IEEE Trans. Autom. Control*, vol. 62, no. 11, pp. 5497–5510, 2017.

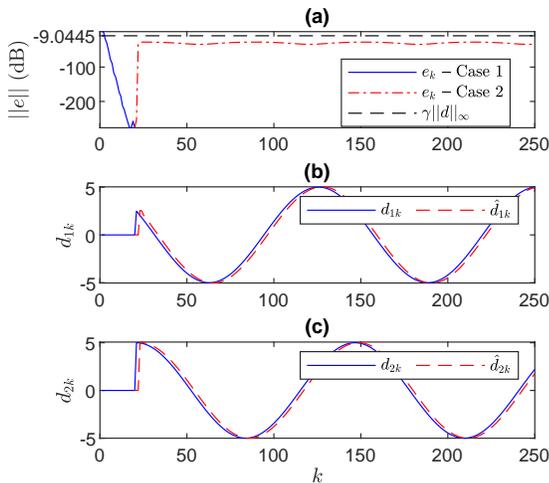


Fig. 2. Estimation performance. (a) State estimation with (Case 1) and without (Case 2) the presence of unknown inputs. (b) and (c) Estimation of the unknown inputs in Case 2.