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A CLASSIFICATION OF TOTALLY GEODESIC AND TOTALLY UMBILICAL LEGENDRIAN SUBMANIFOLDS OF (κ,μ)-SPACES

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Abstract. We present classifications of totally geodesic and totally umbilical Legendrian submanifolds of (κ,μ)-spaces with Boeckx invariant $I \leq -1$. In particular, we prove that such submanifolds must be, up to local isometries, among the examples that we explicitly construct.

1. Introduction

Although under a different name, (κ,μ)-spaces were introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou in [2] (for technical details, we refer to the Preliminaries section). Actually, these manifolds have proven to be really useful, because they provide non-trivial examples for some important classes of contact metric manifolds (for instance, the unit tangent sphere bundle of any Riemannian manifold of constant sectional curvature carries such a structure). The theory of (κ,μ)-spaces was soon developed, with many interesting results. In particular, we can point out the outstanding paper [3], where E. Boeckx classified non-Sasakian (κ,μ)-spaces by using the invariant I (depending only on the values of κ and μ) introduced by himself. He also provided examples for all possible (κ,μ).

Nevertheless, the theory of submanifolds of (κ,μ)-spaces has not been developed in depth yet, even if we can find some very interesting papers about it. For example, in [4], B. Cappelletti Montano, L. Di Terlizzi and M. M. Tripathi proved that any invariant submanifold of a non-Sasakian contact (κ,μ)-space is always totally geodesic and, conversely, that every totally geodesic submanifold of a non-Sasakian contact (κ,μ)-space such that $\mu \neq 0$ and the characteristic vector field ξ is tangent to the submanifold is invariant. Motivated by these results, we consider the case of submanifolds which are normal to ξ. Moreover, we restrict our study to the case of Legendrian submanifolds, i.e., those with dimension n in a $(2n+1)$-dimensional ambient space.

From our point of view, a key step in continuing the analysis of submanifolds of (κ,μ)-spaces should be to understand the behavior of the so-called h operator of the ambient space with respect to the submanifold. Therefore, in this paper, we first establish in Section 3 a decomposition of that operator in its tangent and normal parts, and find its main properties. In Section 4 we present several examples of totally geodesic and totally umbilical Legendrian submanifolds of (κ,μ)-spaces with $I \leq -1$. Actually, we prove in Section 5 that these examples constitute the complete local classification of these kinds of submanifolds, given by our main results Theorems 5.1 and 5.2.

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2. Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold M. Then an *almost contact structure* is a triplet (φ,ξ,η), where φ is a $(1,1)$-tensor field, η a 1-form and ξ a vector field on M satisfying the following conditions

$$\varphi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1. \tag{2.1}$$

It follows from (2.1) that $\varphi\xi = 0$, $\eta\circ\varphi = 0$ and that $\operatorname{rank}(\varphi) = 2n$ ([1]).

Any almost contact manifold (M,φ,ξ,η) admits a *compatible metric*, i.e. a Riemannian metric g satisfying

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M. It follows that $\eta = g(\cdot,\xi)$ and $g(\cdot,\varphi\cdot) = -g(\varphi\cdot,\cdot)$. The manifold M is said to be an *almost contact metric manifold* with structure (φ,ξ,η,g).

We can define the *fundamental* 2*-form* Φ of an almost contact metric manifold by $\Phi(X,Y) = g(X,\varphi Y)$. If $\Phi = d\eta$, then η becomes a contact form, with ξ its Reeb/characteristic vector field and $\mathcal{D} = \ker(\eta)$ its corresponding contact distribution, and $M(\varphi,\xi,\eta,g)$ is called a *contact metric manifold*.

Every contact metric manifold satisfies

$$\nabla\xi = -\varphi - \varphi h, \tag{2.2}$$

where $2h$ is the Lie derivative of φ in the direction of ξ, i.e. $h = \frac{1}{2}L_\xi\varphi$. The tensor field h is symmetric with respect to g, satisfies $h\xi = 0$, anticommutes with φ and vanishes identically if and only if the Reeb vector field ξ is Killing. In this last case the contact metric manifold is said to be *K-contact*.

An almost contact metric manifold is said to be *normal* if $N_\varphi := [\varphi,\varphi] + 2d\eta\otimes\xi = 0$. A normal contact metric manifold is called a *Sasakian manifold*. Any Sasakian manifold is K-contact and the converse holds in dimension 3 but not in general.

A special class of contact metric manifold is that of (κ,μ)-spaces, first studied in [2] under the name of *contact metric manifolds with ξ belonging to the (κ,μ)-distribution*. A *contact metric (κ,μ)-space* is one satisfying the condition

$$R(X,Y)\xi = \kappa\left(\eta(Y)X - \eta(X)Y\right) + \mu\left(\eta(Y)hX - \eta(X)hY\right), \tag{2.3}$$

for some constants κ and μ. In this paper, all manifolds will be contact metric, so we will shorten "contact metric (κ,μ)-space" to "(κ,μ)-space".

Every (κ,μ)-space satisfies

$$h^2 = (\kappa-1)\varphi^2, \tag{2.4}$$

$$(\nabla_X\varphi)Y = g(X, Y+hY)\xi - \eta(Y)(X+hX), \tag{2.5}$$

$$\begin{aligned}(\nabla_X h)Y &= ((1-\kappa)g(X,\varphi Y) - g(X,\varphi hY))\xi \\ &\quad - \eta(Y)((1-\kappa)\varphi X + \varphi hX) - \mu\eta(X)\varphi hY.\end{aligned} \tag{2.6}$$

Moreover, we have the following result:

Theorem 2.1 ([2]). *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a (κ,μ)-space. Then $\kappa\le 1$. If $\kappa = 1$, then $h = 0$ and M^{2n+1} is a Sasakian manifold. If $\kappa < 1$, M^{2n+1} admits three mutually orthogonal and integrable distributions $E_M(0) = span(\xi)$, $E_M(\lambda)$ and $E_M(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

As a consequence of this theorem, it was also proved in [2] that the sectional curvature of a plane section $\{X, Y\}$ normal to ξ is given by

$$K(X,Y) = \begin{cases} 2(1+\lambda)-\mu, \text{ for any } X, Y \in E_M(\lambda), \quad n>1, \\ 2(1-\lambda)-\mu, \text{ for any } X, Y \in E_M(-\lambda), \quad n>1, \\ -(\kappa+\mu)(g(X,\varphi Y))^2, \text{ for any unit vectors } X \in E_M(\lambda), Y \in E_M(-\lambda). \end{cases} \tag{2.7}$$

Given a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, a D_a-*homothetic deformation* is a change of structure tensors of the form

$$\tilde{\varphi} = \frac{1}{a}\varphi, \ \tilde{\xi} = \xi, \ \tilde{\eta} = a\eta, \ \tilde{g} = ag + a(a-1)\eta\otimes\eta, \tag{2.8}$$

where a is a positive constant. It is well known that $M^{2n+1}(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also a contact metric manifold.

It was also proved in [2] that the class of (κ,μ)-spaces remains invariant under D_a-homothetic deformations. Indeed, applying one of these deformations to a (κ,μ)-space yields a new $(\tilde{\kappa},\tilde{\mu})$-space, where

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \ \tilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

Many authors studied (κ,μ)-spaces later, as can be seen in [1]. We highlight here the work of Boeckx, who gave in [3] an explicit writing of the curvature tensor of these spaces:

$$\begin{aligned} R(X,Y)Z = {} & \left(1-\frac{\mu}{2}\right)(g(Y,Z)X - g(X,Z)Y) \\ & + g(Y,Z)hX - g(X,Z)hY - g(hX,Z)Y + g(hY,Z)X \\ & + \frac{1-\frac{\mu}{2}}{1-\kappa}(g(hY,Z)hX - g(hX,Z)hY) \\ & - \frac{\mu}{2}(g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y) + \mu g(\varphi X,Y)\varphi Z \\ & + \frac{\kappa-\frac{\mu}{2}}{1-\kappa}(g(\varphi hY,Z)\varphi hX - g(\varphi hX,Z)\varphi hY) \\ & - \eta(X)\eta(Z)\left(\left(\kappa-1+\frac{\mu}{2}\right)Y + (\mu-1)hY\right) \\ & + \eta(Y)\eta(Z)\left(\left(\kappa-1+\frac{\mu}{2}\right)X + (\mu-1)hX\right) \\ & + \eta(X)\left(\left(\kappa-1+\frac{\mu}{2}\right)g(Y,Z) + (\mu-1)g(hY,Z)\right)\xi \\ & - \eta(Y)\left(\left(\kappa-1+\frac{\mu}{2}\right)g(X,Z) + (\mu-1)g(hX,Z)\right)\xi. \end{aligned} \tag{2.9}$$

Boeckx [3] also classified the (κ,μ)-spaces in terms of an invariant that he introduced: $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$. Indeed, he proved that if M_1 and M_2 are two non-Sasakian (κ_i,μ_i)-spaces of the same dimension, then $I_{M_1} = I_{M_2}$ if and only if, up to a D_a-homothetic deformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a D_a-homothetic deformation.

It was also stated in paper [3] that "it follows that we know all non-Sasakian (κ,μ)-spaces locally as soon as we have, for every odd dimension $2n+1$ and for every possible value for the invariant I, one (κ,μ)-space M with $I_M = I$." For $I > -1$, we have the unit tangent sphere bundle $T_1M^n(c)$ of a space of constant curvature c ($c \neq 1$) for the appropriate c (see [2]). For $I \leq -1$, Boeckx presented in [3] the following examples for any possible odd dimension $2n+1$ and value of I.

Example 2.2 ([3]). Let $\mathfrak{g}$ be a $(2n+1)$-dimensional Lie algebra with basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and the Lie brackets given by

$$
\begin{array}{rclrcl}
[\xi, X_1] &=& -\dfrac{\alpha\beta}{2}X_2 - \dfrac{\alpha^2}{2}Y_1, & [Y_i, Y_j] &=& 0, \quad i,j \neq 2, \\
[\xi, X_2] &=& \dfrac{\alpha\beta}{2}X_1 - \dfrac{\alpha^2}{2}Y_2, & [X_1, Y_1] &=& -\beta X_2 + 2\xi, \\
[\xi, X_i] &=& -\dfrac{\alpha^2}{2}Y_i, \quad i \geq 3, & [X_1, Y_i] &=& 0, \quad i \geq 2, \\
[\xi, Y_1] &=& \dfrac{\beta^2}{2}X_1 - \dfrac{\alpha\beta}{2}Y_2, & [X_2, Y_1] &=& \beta X_1 - \alpha Y_2, \\
[\xi, Y_2] &=& \dfrac{\beta^2}{2}X_2 + \dfrac{\alpha\beta}{2}Y_1, & [X_2, Y_2] &=& \alpha Y_1 + 2\xi, \\
[\xi, Y_i] &=& \dfrac{\beta^2}{2}X_i, \quad i \geq 3, & [X_2, Y_i] &=& \beta X_i, \quad i \geq 3, \\
[X_1, X_i] &=& \alpha X_i, \quad i \neq 1, & [X_i, Y_1] &=& -\alpha Y_i, \quad i \geq 3, \\
[X_i, X_j] &=& 0, \quad i,j \neq 1, & [X_i, Y_2] &=& 0, \quad i \geq 3, \\
[Y_2, Y_i] &=& \beta Y_i, \quad i \neq 2, & [X_i, Y_j] &=& \delta_{ij}(-\beta X_2 + \alpha Y_1 + 2\xi), \quad i,j \geq 3,
\end{array}
\tag{2.10}
$$

for real numbers α and β. Next we define a left-invariant contact metric structure (φ, ξ, η, g) on the associated Lie group G as follows:

- the basis $\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ is orthonormal,
- the characteristic vector field is given by ξ,
- the one-form η is the metric dual of ξ,
- the $(1,1)$-tensor field φ is determined by $\varphi\xi = 0$, $\varphi X_i = Y_i$, $\varphi Y_i = -X_i$.

It can also be proved that G is a (κ, μ)-space with

$$\kappa = 1 - \frac{(\beta^2 - \alpha^2)^2}{16}, \ \mu = 2 + \frac{\alpha^2 + \beta^2}{2}.$$

Moreover, supposing $\beta^2 > \alpha^2$ gives us that $\lambda = \frac{\beta^2 - \alpha^2}{4} \neq 0$ and thus the (κ, μ)-space is not Sasakian. The orthonormal basis also satisfies that $hX_i = \lambda X_i$ and $hY_i = -\lambda Y_i$.

Finally, $I_G = -\frac{\beta^2 + \alpha^2}{\beta^2 - \alpha^2} \leq -1$, so for the appropriate choice of $\beta > \alpha \geq 0$, I_G attains any real value smaller than or equal to -1.

Lastly, we will recall some formulas from submanifolds theory in order to fix our notation. Let N be an n-dimensional submanifold isometrically immersed in an m-dimensional Riemannian manifold (M,g). Then, the Gauss and Weingarten formulas hold:

$$\nabla_X Y = \overline{\nabla}_X Y + \sigma(X,Y), \tag{2.11}$$

$$\nabla_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.12}$$

for any tangent vector fields X, Y and any normal vector field V. Here σ denotes the *second fundamental form*, A the *shape operator* and $\nabla^{\perp}$ the *normal connection*. It is well known that the second fundamental form and the shape operator are related the following way:

$$g(\sigma(X,Y),V) = g(A_V X, Y). \tag{2.13}$$

We denote by R and $\overline{R}$ the curvature tensors of M and N, respectively. They are related by Gauss and Codazzi's equations

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)), \tag{2.14}$$

$$(R(X,Y)Z)^{\perp} = (\nabla_X\sigma)(Y,Z) - (\nabla_Y\sigma)(X,Z), \tag{2.15}$$

respectively, where $R(X,Y)Z^{\perp}$ denotes the normal component of $R(X,Y)Z$ and

$$(\nabla_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z). \tag{2.16}$$

The submanifold N is said to be *totally geodesic* if the second fundamental form σ vanishes identically. It is said that it is *totally umbilical* if there exists a normal vector field V such that $\sigma(X,Y) = g(X,Y)V$, for any tangent vector fields X, Y. In fact, it can be proved that, in such a case, V has to be the *mean curvature* $\widetilde{H} = \frac{1}{n}\sum_{i=1}^{n}\sigma(e_i,e_i)$, where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame. It is clear that every totally geodesic submanifold is also totally umbilical but the converse is not true in general.

3. Decomposition of the h operator

Let N be a Legendrian submanifold of a $(2n+1)$-dimensional (κ,μ)-space M, that is, an n-dimensional submanifold such that ξ is normal to N. Therefore, $\eta(X) = 0$ for any tangent vector field X and so it follows from (2.1) that $\varphi^2 X = -X$. Moreover, it was proved in [6] that N is an anti-invariant submanifold, i.e., φX is normal for any tangent vector field X. Moreover, under our assumptions about the dimensions of M and N, it holds that every normal vector field V can be written as φX, for a certain tangent vector field X.

Therefore, we can decompose the h operator in the following way:

$$hX = h_1 X + \varphi h_2 X, \tag{3.1}$$

for any tangent vector field X, where $h_1 X$ (respectively $\varphi h_2 X$) denotes the tangent (resp. normal) component of hX.

We can prove the following properties:

Proposition 3.1. *Let N be a Legendrian submanifold of a (κ,μ)-space M. Then, h_1 and h_2 are symmetric operators that satisfy $h_1\xi = h_2\xi = 0$ and equations*

$$h_1^2 + h_2^2 = (1-\kappa)I, \tag{3.2}$$

$$h_1 h_2 = h_2 h_1. \tag{3.3}$$

Proof. The symmetry of h_1 and h_2 can be directly obtained from that of h and the compatibility of the metric g. Similarly, $h\xi = 0$ implies $h_1\xi = h_2\xi = 0$.

Furthermore, given a tangent vector field X, it follows from (2.1), (3.1) and the anticommutativity of h and φ that

$$h\varphi X = -\varphi hX = -\varphi h_1 X + h_2 X. \tag{3.4}$$

Using (2.4), we have that $h^2 X = (1-\kappa)X$. On the other hand, by virtue of (3.1) and (3.4), we obtain

$$h^2 X = h(h_1 X + \varphi h_2 X) = h_1^2 X + \varphi h_2 h_1 X - \varphi h_1 h_2 X + h_2^2 X.$$

Joining both expressions for h^2 and identifying the tangent and normal parts give us equations (3.2) and (3.3). □

Proposition 3.2. *Let N be a Legendrian submanifold of a (κ,μ)-space M. Then, h_1 and h_2 satisfy*

$$(\overline{\nabla}_X h_1)Y = -\varphi\sigma(X, h_2 Y) - h_2\varphi\sigma(X,Y), \tag{3.5}$$

$$(\overline{\nabla}_X h_2)Y = \varphi\sigma(X, h_1 Y) + h_1\varphi\sigma(X,Y), \tag{3.6}$$

for any tangent vector fields X, Y.

Proof. It follows from Gauss and Weingarten formulas (2.11) and (2.12) that

$$(\nabla_X\varphi)Y = \nabla_X\varphi Y - \varphi\nabla_X Y = -A_{\varphi Y}X + \nabla^{\perp}_X\varphi Y - \varphi\overline{\nabla}_X Y - \varphi\sigma(X,Y),$$

for any tangent vector fields X, Y. Therefore, by using (2.5) and identifying the tangent and normal components, we obtain:

$$A_{\varphi Y}X = -\varphi\sigma(X,Y), \tag{3.7}$$

$$\nabla^{\perp}_X\varphi Y = \varphi\overline{\nabla}_X Y + g(X, Y + h_1 Y)\xi. \tag{3.8}$$

On the other hand, using (2.6) and (3.1), we have

$$\nabla_X(h_1 Y + \varphi h_2 Y) - h(\nabla_X Y) = g(X, h_2 Y)\xi,$$

from where, by virtue of Gauss and Weingarten formulas (2.11) and (2.12), we deduce

$$\overline{\nabla}_X h_1 Y + \sigma(X, h_1 Y) - A_{\varphi h_2 Y}X + \nabla^{\perp}_X\varphi h_2 Y - h\overline{\nabla}_X Y - h\sigma(X,Y) = g(X, h_2 Y)\xi. \tag{3.9}$$

We can put $h\overline{\nabla}_X Y = h_1\overline{\nabla}_X Y + \varphi h_2\overline{\nabla}_X Y$ by (3.1). Now, by using (2.1), we can write $\sigma(X,Y) = -\varphi^2\sigma(X,Y) + \eta(\sigma(X,Y))\xi$, and hence $h\sigma(X,Y) = -h\varphi^2\sigma(X,Y) = \varphi h\varphi\sigma(X,Y)$. Again, equation (3.1) gives us $h\sigma(X,Y) = \varphi h_1\varphi\sigma(X,Y) - h_2\varphi\sigma(X,Y)$. Therefore, if we substitute these two expressions, together with (3.7) and (3.8), in (3.9), we obtain:

$$\begin{aligned}\overline{\nabla}_X h_1 Y + \sigma(X, h_1 Y) + \varphi\sigma(X, h_2 Y) + \varphi\overline{\nabla}_X h_2 Y + g(X, h_2 Y + h_1 h_2 Y)\xi \\ - h_1\overline{\nabla}_X Y - \varphi h_2\overline{\nabla}_X Y - \varphi h_1\varphi\sigma(X,Y) + h_2\varphi\sigma(X,Y) = g(X, h_2 Y)\xi.\end{aligned} \tag{3.10}$$

By identifying the tangent and normal parts of (3.10), equations (3.5) and (3.6) hold. □

It is clear that, if we multiply (3.10) by ξ, then we obtain

$$g(\sigma(X, h_1 Y), \xi) + g(X, h_1 h_2 Y) = 0,$$

for any tangent vector fields X, Y. In fact, we can prove a more general result, which will be very useful in the proof of our main theorems:

Lemma 3.3. *Let N be a Legendrian submanifold of a (κ,μ)-space M. Then,*

$$g(\sigma(X,Y),\xi)+g(X,h_2Y)=0, \tag{3.11}$$

for any tangent vector fields X,Y.

Proof. It follows from Weingarten equation (2.12) and from (2.13) that

$$g(X,\nabla_X\xi)+g(\sigma(X,Y),\xi)=0,$$

for any tangent vector fields X,Y. Then, it is enough to use (2.1), (2.2) and (3.1) to obtain (3.11). □

4. Examples

We will present in this section some examples of totally geodesic and totally umbilical Legendrian submanifolds of the (κ,μ)-spaces of Example 2.2. Let us begin with the totally geodesic ones.

Example 4.1. Let M be a (κ,μ)-space from Example 2.2 with invariant $I_M\le -1$. Then, the distribution $\mathcal{D}$ spanned by $\{X_1,\dots,X_n\}$ is involutive and any integral submanifold N of it is a totally geodesic submanifold of M. Indeed, the involutive condition can be easily checked from (2.10). In order to prove the totally geodesic one, it is enough to show that $\nabla_{X_i}X_j\in\mathcal{D}$, for any $i,j=1,\dots,n$, where ∇ denotes the Levi-Civita connection on M. In fact, in can be directly computed that:

$$\begin{aligned}&\nabla_{X_1}X_1=\nabla_{X_1}X_2=0,\quad \nabla_{X_2}X_1=-\alpha X_2,\quad \nabla_{X_2}X_2=\alpha X_1,\\ &\nabla_{X_1}X_i=\nabla_{X_2}X_i=0,\ \text{for any } i=3,\dots,n,\\ &\nabla_{X_i}X_1=-\alpha X_i,\quad \nabla_{X_i}X_2=0,\quad \nabla_{X_i}X_j=\delta_{ij}\alpha X_1,\ \text{for any } i,j=3,\dots,n.\end{aligned} \tag{4.1}$$

Moreover, since $hX_i=\lambda X_i$ for any $i=1,\dots,n$, then $TN=E_M(\lambda)$.

Example 4.2. Let M be a (κ,μ)-space from Example 2.2 with invariant $I_M\le -1$. Then, the distribution $\mathcal{D}$ spanned by $\{Y_1,\dots,Y_n\}$ is also involutive and any integral submanifold N of it is a totally geodesic submanifold of M. Indeed, both conditions can be checked the same way as in Example 4.1, by taking now into account that:

$$\begin{aligned}&\nabla_{Y_1}Y_1=\beta Y_2,\quad \nabla_{Y_1}Y_2=-\beta Y_1,\quad \nabla_{Y_2}Y_1=\nabla_{Y_2}Y_2=0,\\ &\nabla_{Y_1}Y_i=\nabla_{Y_2}Y_i=0,\ \text{for any } i=3,\dots,n,\\ &\nabla_{Y_i}Y_1=0,\quad \nabla_{Y_i}Y_2=-\beta Y_i,\quad \nabla_{Y_i}Y_j=\delta_{ij}\beta Y_2,\ \text{for any } i,j=3,\dots,n.\end{aligned} \tag{4.2}$$

In this case, since $hY_i=-\lambda Y_i$ for any $i=1,\dots,n$, then $TN=E_M(-\lambda)$.

Example 4.3. Let M be a (κ,μ)-space from Example 2.2 with invariant $I_M\le -1$. Then, the distribution $\mathcal{D}$ spanned by $\{X_1,Y_2,Z_3,\dots,Z_n\}$, where Z_i is either X_i or Y_i, for any $i=3,\dots,n$, is also involutive and any integral submanifold N of it is a totally geodesic submanifold of M.

Indeed, both conditions can be checked the same way as in Examples 4.1 and 4.2, by using now (4.1), (4.2) and the following formulas:

$$\begin{aligned}
&\nabla_{X_1} Y_i = 0 \text{ for any } i = 2, \dots, n,\\
&\nabla_{Y_2} X_i = 0 \text{ for any } i = 1, 3, \dots, n,\\
&\nabla_{X_i} Y_2 = \nabla_{Y_i} X_1 = 0 \text{ for any } i = 3, \dots, n,\\
&\nabla_{X_i} Y_j = \nabla_{Y_i} X_j = 0 \text{ for any } i, j = 3, \dots, n, \text{ such that } i \neq j.
\end{aligned} \tag{4.3}$$

Finally, if we define $E(\pm\lambda) := E_M(\pm\lambda) \cap N$, we can write $TN = E(\lambda) \oplus E(-\lambda)$, with $\dim E(\lambda) = k$ (respectively $\dim E(-\lambda) = n - k$), where $k - 1$ (resp. $n - k - 1$) is the number of Z_i such that $Z_i = X_i$ (resp. $Z_i = Y_i$). Therefore, we can obtain an example for any value of k from 1 to $n - 1$.

We now present the family of totally umbilical examples:

Example 4.4. Let M be a (κ, μ)-space from Example 2.2 with invariant $I_M \leq -1$. Then, the distribution $\mathcal{D}$ spanned by $\{cX_1 + dY_1, \dots, cX_n + dY_n\}$, with c, d non-zero constants, is involutive and any integral submanifold N of it is a totally umbilical submanifold of M. Indeed, the involutive condition can be easily checked from (2.10). In order to prove the totally umbilical one, we will first show that $\sigma(cX_i + dY_i, cX_j + dY_j) = 2\delta_{ij} cd\lambda\xi$ by checking that the Levi-Civita connection on M satisfies $\nabla_{cX_i + dY_i}(cX_j + dY_j) = Z + 2\delta_{ij} cd\lambda\xi$, with $Z \in \mathcal{D}$, for any $i, j = 1, \dots, n$. In fact, it can be directly computed that:

$$\begin{aligned}
&\nabla_{cX_1+dY_1}(cX_1 + dY_1) = \beta d(cX_2 + dY_2) + 2cd\lambda\xi,\\
&\nabla_{cX_1+dY_1}(cX_2 + dY_2) = -\beta d(cX_1 + dY_1),\\
&\nabla_{cX_2+dY_2}(cX_1 + dY_1) = -\alpha c(cX_2 + dY_2),\\
&\nabla_{cX_2+dY_2}(cX_2 + dY_2) = \alpha c(cX_1 + dY_1) + 2cd\lambda\xi,\\
&\nabla_{cX_1+dY_1}(cX_j + dY_j) = \nabla_{cX_2+dY_2}(cX_j + dY_j) = 0, \text{ for any } j = 3, \dots, n,\\
&\nabla_{cX_i+dY_i}(cX_1 + dY_1) = -\alpha c(cX_i + dY_i),\\
&\nabla_{cX_i+dY_i}(cX_2 + dY_2) = -\beta d(cX_i + dY_i), \text{ for any } i = 3, \dots, n,\\
&\nabla_{cX_i+dY_i}(cX_j + dY_j) = \delta_{ij}(\alpha c(cX_1 + dY_1) + \beta d(cX_2 + dY_2) + 2cd\lambda\xi),\\
&\qquad \text{for any } i, j = 3, \dots, n.
\end{aligned}$$

Therefore, we can write $\sigma(cX_i + dY_i, cX_j + dY_j) = g(cX_i + dY_i, cX_j + dY_j)\frac{2cd\lambda}{c^2+d^2}\xi$ and, since $\frac{2cd\lambda}{c^2+d^2}\xi \neq 0$, the submanifold is totally umbilical but not totally geodesic.

Finally, we observe that $cX_i + dY_i$, $i = 1, \dots, n$, is not an eigenvector of h.

5. Main results

Theorem 5.1. *Let N be a Legendrian submanifold of a $(2n + 1)$-dimensional (κ, μ)-space M, with $\kappa < 1$ and $I_M \leq -1$. If N is totally geodesic, then, up to local isometries, it must be one of the submanifolds given in Examples 4.1, 4.2 or 4.3.*

Proof. Since the submanifold N is totally geodesic, if follows directly from (3.11) that $h_2 = 0$ and so $h|_N = h_1$ and $h_1^2 = (1 - \kappa)I$ (see (3.1) and (3.2)). The operator h_1 is differentiable and symmetric, so it is diagonalisable and it has two eigenvalues $\pm\lambda = \pm\sqrt{1 - \kappa}$, which are distinct and constant everywhere.

Let us denote by $E(\lambda)$ and $E(-\lambda)$ the eigenspaces of h_1 in TN and by k the dimension of $E(\lambda)$. This means that $\dim(E(-\lambda)) = n-k$ (because $\dim N = n$) and that $k \in \{0, \ldots, n\}$. The multiplicities of both eigenspaces must be the same at every point because the coefficients of the characteristic polynomial are differentiable. Indeed, the characteristic polynomial of h_1 is completely determined by k (thus, for different indices k, we get a different characteristic polynomial). Since k is an integer, it is impossible by continuity to go from one to the other one, thus the eigendistributions are differentiable. We can then write

$$TN = E(\lambda) \oplus E(-\lambda), \tag{5.1}$$

where $\dim(E(\lambda)) = k$ and $\dim(E(-\lambda)) = n-k$, for a certain $k \in \{0, \ldots, n\}$.

Moreover, we deduce from (3.5) that $\overline{\nabla} h_1 = 0$. Therefore, it is straightforward to check that, if $Y_\lambda \in E(\lambda)$, then $\overline{\nabla}_X Y_\lambda \in E(\lambda)$, for every tangent vector field X. Similarly, if $Y_{-\lambda} \in E(-\lambda)$, then $\overline{\nabla}_X Y_{-\lambda} \in E(-\lambda)$. Thus, $E(\lambda)$ and $E(-\lambda)$ are parallel and hence involutive. By virtue of Theorem 5.4 of [5], N can be locally decomposed as $M_1 \times M_2$, where M_1 and M_2 are leaves of the distributions $E(\lambda)$ and $E(-\lambda)$, respectively. Furthermore, it follows from (2.7) that, if $\dim M_1 \geq 2$ (resp. $\dim M_2 \geq 2$), then M_1 (resp. M_2) has constant curvature $2(1+\lambda) - \mu = 2\lambda(I_M + 1) \leq 0$ (resp. $2(1-\lambda) - \mu = 2\lambda(I_M - 1) < 0$).

Recall that we have examples of submanifolds with decomposition (5.1) for every value of k. Indeed, see Example 4.1 for $k = n$, Example 4.2 for $k = 0$ and Example 4.3 for any value of k from 1 to $n-1$. Now, we will prove that any example must be one of these, up to local isometries.

Let us denote by $F : N^n \to M^{2n+1}(\kappa, \mu)$ the immersion of N into M. Since $\kappa < 1$ and $I_M \leq -1$, we can suppose that, locally, $M^{2n+1}(\kappa, \mu)$ is one of the Lie groups from Example 2.2. Thus, it is homogeneous and we can fix a point $p_0 \in N$ such that $F(p_0) = e$, where e is the neutral element of the group.

We will give the explicit details when $2 \leq k \leq n-2$. The other cases can be done in a similar way. We have that $N = M_1(2\lambda(I_M + 1)) \times M_2(2\lambda(I_M - 1))$ and we also identify N with its image as the (totally geodesic) integral submanifold through e of the distribution spanned by $X_1, X_3, \ldots, X_{k+1}, Y_2, Y_{k+2}, \ldots Y_n$. We denote by G the latter immersion of N and we pick an orthonormal basis $\{e_1, \ldots, e_n\}$ at the point p_0 of N, with $G(p_0) = e$, such that $E_{p_0}(\lambda) = \langle e_1(p_0), \ldots, e_k(p_0) \rangle$, $E_{p_0}(-\lambda) = \langle e_{k+1}(p_0), \ldots, e_n(p_0) \rangle$ and

$$\begin{aligned}
&dG(e_1(p_0)) = X_1(e), \\
&dG(e_j(p_0)) = X_{j+1}(e), \quad j = 2, \ldots, k, \\
&dG(e_{k+1}(p_0)) = Y_2(e), \\
&dG(e_j(p_0)) = Y_j(e), \quad j = k+2, \ldots, n,
\end{aligned}$$

Note that by construction both

$$X_1(e), X_3(e), \ldots, X_{k+1}(e), \varphi Y_2(e), \varphi Y_{k+2}(e), \ldots, \varphi Y_n(e)$$

and

$$dF(e_1(p_0)), \ldots, dF(e_k(p_0)), \varphi dF(e_{k+1}(p_0)), \ldots, \varphi dF(e_n(p_0))$$

are basis of $E_e(\lambda)$. So, in view of Theorem 3 of [3], there exists an isometry H of $M^{2n+1}(\kappa, \mu)$ preserving the structure such that $H(e) = e$ and H maps one basis of $E_e(\lambda)$ into the other one. As a consequence, we have that $H \circ F(e) = G(e)$ and $d(H \circ F)(e_i) = dG(e_i)$.

We now take a geodesic γ in N through the point p_0. Since N is totally geodesic, both with respect to the immersions $H \circ F$ and G, the curves $H \circ F(\gamma)$ and $G(\gamma)$ are both geodesics in $M^{2n+1}(\kappa, \mu)$ through e. Since $d(H \circ F)(e_i) = dG(e_i)$, they are also determined by the same initial conditions. Therefore, both curves need to coincide, so $H \circ F(\gamma(s)) = G(\gamma(s))$ for all s and thus F and G are congruent. □

Theorem 5.2. *Let N be a Legendrian submanifold of a $(2n+1)$-dimensional (κ, μ)-space M, with $n \geq 3$, $\kappa < 1$ and $I_M \leq -1$. If N is totally umbilical (but not totally geodesic), then, up to local isometries, it must be one of the submanifolds given in Example 4.4.*

Proof. Since N is totally umbilical (but not totally geodesic), then there exists a normal vector field $V \neq 0$ such that $\sigma(X, Y) = g(X, Y)V$. It follows from (3.11) that $g(X, Y)\eta(V) + g(X, h_2Y) = 0$, for any tangent vector fields X, Y, and thus

$$h_2Y = aY, \tag{5.2}$$

with $a = -\eta(V)$.

We will now prove that $a \neq 0$. Indeed, if we suppose that $a = 0$, then $h_2 = 0$ and, as in the proof of Theorem 5.1, we have that $h = h_1$, $h_1^2 = (1-\kappa)I$ and $\overline{\nabla} h_1 = 0$. Moreover, since $h_2 = 0$, it is clear that $\overline{\nabla} h_2 = 0$ and we obtain from (3.6) that $\varphi\sigma(X, h_1Y) + h_1\varphi\sigma(X, Y) = 0$, which, by using that N is totally umbilical, becomes

$$g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V = 0, \tag{5.3}$$

for any tangent vector fields X, Y. Let us now choose unit vector fields $X_\lambda \in E(\lambda)$ and $X_{-\lambda} \in E(-\lambda)$. Then, taking $X = Y = X_\lambda$ in (5.3) implies $h_1\varphi V = -\lambda\varphi V$ and taking $X = Y = X_{-\lambda}$ in (5.3) implies $h_1\varphi V = \lambda\varphi V$. Since $V \neq 0$, this yields a contradiction.

Therefore, we can suppose from now on that (5.2) holds for $a \neq 0$. We deduce from equation (3.6) that

$$X(a)Y = \varphi\sigma(X, h_1Y) + h_1\varphi\sigma(X, Y) = g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V,$$

for every X, Y tangent vector fields.

Since $\dim N \geq 3$, we can take Y linearly independent from φV and $h_1\varphi V$. Then we deduce from the previous equation that $X(a) = 0$, for every X, thus a is a constant. Moreover, $g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V = 0$, for every X, Y tangent vector fields. Taking unit $X = Y$, we obtain that $h_1\varphi V = -g(X, h_1X)\varphi V$, which is only possible if $h_1 = 0$ or $\varphi V = 0$. If $h_1 = 0$, then substituting (5.2) in (3.6) gives that $2ag(X, Y)\varphi V = 0$, so again $\varphi V = 0$.

In both cases, we have obtained that $\varphi V = 0$, so V is parallel to ξ and it follows from $a = -\eta(V)$ that $V = -a\xi$ and $\sigma(X, Y) = -ag(X, Y)\xi$ holds, for every X, Y tangent, where $a \neq 0$ is a constant.

Let us now recall Codazzi's equation (2.15):

$$(R(X, Y)Z)^\perp = (\nabla_X\sigma)(Y, Z) - (\nabla_Y\sigma)(X, Z).$$

The first term is the normal component of $R(X,Y)Z$, so by equation (2.9) and the fact that $h_2X = h_1X + a\varphi X$, we can write

$$\begin{aligned}(R(X,Y)Z)^{\perp} &= a(g(Y,Z)\varphi X - g(X,Z)\varphi Y)\\ &+ a\frac{1-\frac{\mu}{2}}{1-\kappa}(g(h_1Y,Z)\varphi X - g(h_1X,Z)\varphi Y)\\ &- a\frac{\kappa-\frac{\mu}{2}}{1-\kappa}(g(Y,Z)\varphi h_1X - g(X,Z)\varphi h_1Y).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\nabla_X\sigma)(Y,Z) &= \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\overline{\nabla}_XY,Z) - \sigma(Y,\overline{\nabla}_XZ) =\\ &= \nabla^{\perp}_X(-ag(Y,Z)\xi) + ag(\overline{\nabla}_XY,Z)\xi + ag(\overline{\nabla}_XZ,X)\xi =\\ &= -ag(Y,Z)\overline{\nabla}^{\perp}_X\xi = ag(Y,Z)(\varphi X + \varphi h_1X).\end{aligned}$$

Therefore, the second term of Codazzi's equation is

$$\begin{aligned}(\nabla_X\sigma)(Y,Z) - (\nabla_Y\sigma)(X,Z) &= ag(Y,Z)(\varphi X + \varphi h_1X) - ag(X,Z)(\varphi Y + \varphi h_1Y)\\ &= a(g(Y,Z)\varphi X - g(X,Z)\varphi Y) + a(g(Y,Z)\varphi h_1X - g(X,Z)\varphi h_1Y).\end{aligned}$$

Joining both terms, and bearing in mind that $a \neq 0$, we obtain

$$\begin{aligned}&\frac{1-\frac{\mu}{2}}{1-\kappa}(g(h_1Y,Z)\varphi X - g(h_1X,Z)\varphi Y) =\\ &= \frac{1-\frac{\mu}{2}}{1-\kappa}(g(Y,Z)\varphi h_1X - g(X,Z)\varphi h_1Y).\end{aligned}$$

Since we are supposing that $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}} \leq -1$, then $\frac{1-\frac{\mu}{2}}{1-\kappa} \neq 0$ and applying φ to both terms of the previous equation gives us that

$$g(h_1Y,Z)X - g(h_1X,Z)Y = g(Y,Z)h_1X - g(X,Z)h_1Y,$$

for every X, Y, Z tangent vector fields.

Since $\dim(N) \geq 3$, we can choose $Y = Z$ unit and orthogonal to X, h_1X, and we obtain that

$$h_1X = g(h_1Y,Y)X, \tag{5.4}$$

and thus $h_1X = bX$ for some function b.

From (3.2), we have that $a^2 + b^2 = 1 - \kappa = \lambda^2 \neq 0$, and in particular that b must be constant. We can also write that $a = \lambda\cos(\theta)$ and $b = \lambda\sin(\theta)$ for some constant $\theta \in [-\pi,\pi]$. Since $a \neq 0$, then $\theta \neq \pm\frac{\pi}{2}$.

By Gauss equation (2.14) and the fact that $h_2X = aX$, then

$$\begin{aligned}R(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)) =\\ &= \overline{R}(X,Y,Z,W) - a^2(g(X,W)g(Y,Z) + g(X,Z)g(Y,W)),\end{aligned}$$

for every X, Y, Z, W tangent vector fields.

On the other hand, we know from equation (2.9) and the fact that $hX = bX + a\varphi X$, that

$$\begin{aligned}R(X,Y,Z,W) = &\left(1 - \frac{\mu}{2} + 2b + b^2\frac{1-\frac{\mu}{2}}{1-\kappa} + a^2\frac{\kappa-\frac{\mu}{2}}{1-\kappa}\right)\\ &(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)).\end{aligned}$$

Joining the last two equations, we obtain

$$\begin{aligned}
\overline{R}(X,Y,Z,W) = &\left(1-\frac{\mu}{2}+2b+b^2\frac{1-\frac{\mu}{2}}{1-\kappa}+a^2\left(\frac{\kappa-\frac{\mu}{2}}{1-\kappa}+1\right)\right)\\
&(g(X,W)g(Y,Z)-g(X,Z)g(Y,W))\\
=&\left(1-\frac{\mu}{2}+2b+(a^2+b^2)\frac{1-\frac{\mu}{2}}{1-\kappa}\right)\\
&(g(X,W)g(Y,Z)-g(X,Z)g(Y,W))\\
=&2(1-\frac{\mu}{2}+b)(g(X,W)g(Y,Z)-g(X,Z)g(Y,W)).
\end{aligned}$$

This means that the submanifold is a space form with constant curvature $2(1-\frac{\mu}{2}+b)$. Moreover, since $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}} \leq -1$ and $b = \lambda\sin(\theta) \neq \lambda$, then $1-\frac{\mu}{2}+b < 1-\frac{\mu}{2}+\lambda \leq 0$ and the submanifold is a hyperbolic space $N = \mathbb{H}(2(1-\frac{\mu}{2}+\lambda\sin(\theta)))$.

Summing up, there exists $\theta \in [-\pi,\pi]$, $\theta \neq \pm\frac{\pi}{2}$, such that

$$\begin{aligned}
N &= \mathbb{H}(2(1-\frac{\mu}{2}+\lambda\sin(\theta))),\\
h_1X &= \lambda\sin(\theta)X,\\
h_2X &= \lambda\cos(\theta)X,\\
\sigma(X,Y) &= -\lambda\cos(\theta)g(X,Y)\xi.
\end{aligned} \tag{5.5}$$

We have examples of submanifolds with these properties for every value of θ. Indeed, Examples 4.4 with $c=\cos(\pi/4-\theta/2)$, $d=-\sin(\pi/4-\theta/2)$ satisfy

$$\begin{aligned}
\sigma(cX_i+dY_i,cX_j+dY_j) = 2\delta_{ij}cd\lambda\xi &= -2\delta_{ij}\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\lambda\xi =\\
&= -\delta_{ij}\sin\left(\frac{\pi}{2}-\theta\right)\lambda\xi = -\delta_{ij}\lambda\cos(\theta)\xi =\\
&= -\lambda\cos(\theta)g(cX_i+dY_i,cX_j+dY_j)\xi,
\end{aligned}$$

and the rest of conditions also hold.

Now, we will prove that any totally umbilical submanifold N must be one of these, up to local isometries. Let us denote by $F: N^n \to M^{2n+1}(\kappa,\mu)$ the immersion of N into $M(\kappa,\mu)$. Since $\kappa < 1$ and $I_M \leq -1$, we can suppose that, locally, $M(\kappa,\mu)$ is one of the Lie groups from Example 2.2. Thus, it is homogeneous and we can fix a point $p_0 \in N$ such that $F(p_0) = e$, where e is the neutral element of the group.

We have that $N = \mathbb{H}(2(1-\frac{\mu}{2}+\lambda\sin(\theta)))$ and we can identify N with its image as the (totally umbilical) integral submanifold through e of the distribution spanned by $\{cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)X_i(e) - \sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)Y_i(e), i=1,\dots,n\}$. We denote by G this immersion of N and we take an orthonormal basis $\{e_1,\dots,e_n\}$ at the point p_0 of N such that

$$dG(e_i) = cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)X_i(e) - \sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)Y_i(e),\ i=1,\dots,n.$$

On the other hand, we have that

$$\begin{aligned} h(dF(e_i)) &= dF(\lambda \sin(\theta) e_i) + \varphi dF(\lambda \cos(\theta) e_i) \\ &= \lambda \sin(\theta) dF(e_i) + \lambda \cos(\theta) \varphi dF(e_i), \end{aligned} \tag{5.6}$$

$$h\varphi(dF(e_i)) = -\varphi h(dF(e_i)) = \lambda \cos(\theta) dF(e_i) - \lambda \sin(\theta) \varphi dF(e_i). \tag{5.7}$$

Therefore, using (5.6) and (5.7), we can construct eigenvectors of h associated with the eigenvalue λ the following way:

$$\begin{aligned} & h\left(cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) dF(e_i) + \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \varphi(dF(e_i)) \right) = \\ & = \lambda \left(\left(cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \sin(\theta) + \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos(\theta) \right) dF(e_i) \right. \\ & \quad \left. + \left(cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos(\theta) - \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \sin(\theta) \right) \varphi(dF(e_i)) \right) = \\ & = \lambda \left(\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) dF(e_i) + \cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \varphi(dF(e_i)) \right) = \\ & = \lambda \left(cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) dF(e_i) + \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \varphi(dF(e_i)) \right), \end{aligned}$$

for any $i = 1, \ldots, n$.

Note that, by construction, both

$$\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) dF(e_i) + \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \varphi(dF(e_i)),\ i = 1, \ldots, n$$

and

$$X_1(e), \ldots, X_n(e)$$

are basis of $E_e(\lambda)$. So, in view of Theorem 3 of [3], there exists an isometry H of $M^{2n+1}(\kappa, \mu)$ preserving the structure such that $H(e) = e$ and H maps one basis of $E_e(\lambda)$ into the other one. As a consequence, we have that $H \circ F(e) = G(e)$ and $d(H \circ F)(e_i) = dG(e_i)$.

We now take a geodesic γ in N through the point p_0. Since N is totally umbilical with respect to both $H \circ F$ and G, then $\gamma_1 = H \circ F(\gamma)$ and $\gamma_2 = G(\gamma)$ are curves in $M(\kappa, \mu)$ passing through e that satisfy $\nabla_{\gamma_1'} \gamma_1' = \nabla_{\gamma_2'} \gamma_2' = -\lambda \sin(\theta) \xi$. Since $d(H \circ F)(e_i) = dG(e_i)$, they are also determined by the same initial conditions. Therefore, both curves need to coincide, so $H \circ F(\gamma(s)) = G(\gamma(s))$ for all s and thus F and G are congruent. □

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