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# A CLASSIFICATION OF TOTALLY GEODESIC AND TOTALLY UMBILICAL LEGENDRIAN SUBMANIFOLDS OF $(\kappa, \mu)$ -SPACES

ALFONSO CARRIAZO, VERÓNICA MARTÍN-MOLINA, AND LUC VRANCKEN

ABSTRACT. We present classifications of totally geodesic and totally umbilical Legendrian submanifolds of  $(\kappa, \mu)$ -spaces with Boeckx invariant  $I \leq -1$ . In particular, we prove that such submanifolds must be, up to local isometries, among the examples that we explicitly construct.

## 1. INTRODUCTION

Although under a different name,  $(\kappa, \mu)$ -spaces were introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou in [2] (for technical details, we refer to the Preliminaries section). Actually, these manifolds have proven to be really useful, because they provide non-trivial examples for some important classes of contact metric manifolds (for instance, the unit tangent sphere bundle of any Riemannian manifold of constant sectional curvature carries such a structure). The theory of  $(\kappa, \mu)$ -spaces was soon developed, with many interesting results. In particular, we can point out the outstanding paper [3], where E. Boeckx classified non-Sasakian  $(\kappa, \mu)$ -spaces by using the invariant  $I$  (depending only on the values of  $\kappa$  and  $\mu$ ) introduced by himself. He also provided examples for all possible  $(\kappa, \mu)$ .

Nevertheless, the theory of submanifolds of  $(\kappa, \mu)$ -spaces has not been developed in depth yet, even if we can find some very interesting papers about it. For example, in [4], B. Cappelletti Montano, L. Di Terlizzi and M. M. Tripathi proved that any invariant submanifold of a non-Sasakian contact  $(\kappa, \mu)$ -space is always totally geodesic and, conversely, that every totally geodesic submanifold of a non-Sasakian contact  $(\kappa, \mu)$ -space such that  $\mu \neq 0$  and the characteristic vector field  $\xi$  is tangent to the submanifold is invariant. Motivated by these results, we consider the case of submanifolds which are normal to  $\xi$ . Moreover, we restrict our study to the case of Legendrian submanifolds, i.e., those with dimension  $n$  in a  $(2n + 1)$ -dimensional ambient space.

From our point of view, a key step in continuing the analysis of submanifolds of  $(\kappa, \mu)$ -spaces should be to understand the behavior of the so-called  $h$  operator of the ambient space with respect to the submanifold. Therefore, in this paper, we first establish in Section 3 a decomposition of that operator in its tangent and normal parts, and find its main properties. In Section 4 we present several examples of totally geodesic and totally umbilical Legendrian submanifolds of  $(\kappa, \mu)$ -spaces with  $I \leq -1$ . Actually, we prove in Section 5 that these examples constitute the complete local classification of these kinds of submanifolds, given by our main results Theorems 5.1 and 5.2.

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## 2. PRELIMINARIES

Let  $M$  be a  $(2n + 1)$ -dimensional smooth manifold  $M$ . Then an *almost contact structure* is a triplet  $(\varphi, \xi, \eta)$ , where  $\varphi$  is a  $(1, 1)$ -tensor field,  $\eta$  a 1-form and  $\xi$  a vector field on  $M$  satisfying the following conditions

$$(2.1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

It follows from (2.1) that  $\varphi\xi = 0$ ,  $\eta \circ \varphi = 0$  and that  $\text{rank}(\varphi) = 2n$  ([1]).

Any almost contact manifold  $(M, \varphi, \xi, \eta)$  admits a *compatible metric*, i.e. a Riemannian metric  $g$  satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields  $X, Y$  on  $M$ . It follows that  $\eta = g(\cdot, \xi)$  and  $g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)$ . The manifold  $M$  is said to be an *almost contact metric manifold* with structure  $(\varphi, \xi, \eta, g)$ .

We can define the *fundamental 2-form*  $\Phi$  of an almost contact metric manifold by  $\Phi(X, Y) = g(X, \varphi Y)$ . If  $\Phi = d\eta$ , then  $\eta$  becomes a contact form, with  $\xi$  its Reeb/characteristic vector field and  $\mathcal{D} = \ker(\eta)$  its corresponding contact distribution, and  $M(\varphi, \xi, \eta, g)$  is called a *contact metric manifold*.

Every contact metric manifold satisfies

$$(2.2) \quad \nabla\xi = -\varphi - \varphi h,$$

where  $2h$  is the Lie derivative of  $\varphi$  in the direction of  $\xi$ , i.e.  $h = \frac{1}{2}L_\xi\varphi$ . The tensor field  $h$  is symmetric with respect to  $g$ , satisfies  $h\xi = 0$ , anticommutes with  $\varphi$  and vanishes identically if and only if the Reeb vector field  $\xi$  is Killing. In this last case the contact metric manifold is said to be *K-contact*.

An almost contact metric manifold is said to be *normal* if  $N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi = 0$ . A normal contact metric manifold is called a *Sasakian manifold*. Any Sasakian manifold is K-contact and the converse holds in dimension 3 but not in general.

A special class of contact metric manifold is that of  $(\kappa, \mu)$ -spaces, first studied in [2] under the name of *contact metric manifolds with  $\xi$  belonging to the  $(\kappa, \mu)$ -distribution*. A *contact metric  $(\kappa, \mu)$ -space* is one satisfying the condition

$$(2.3) \quad R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for some constants  $\kappa$  and  $\mu$ . In this paper, all manifolds will be contact metric, so we will shorten “contact metric  $(\kappa, \mu)$ -space” to “ $(\kappa, \mu)$ -space”.

Every  $(\kappa, \mu)$ -space satisfies

$$(2.4) \quad h^2 = (\kappa - 1)\varphi^2,$$

$$(2.5) \quad (\nabla_X\varphi)Y = g(X, Y + hY)\xi - \eta(Y)(X + hX),$$

$$(2.6) \quad \begin{aligned} (\nabla_X h)Y &= ((1 - \kappa)g(X, \varphi Y) - g(X, \varphi hY))\xi \\ &\quad - \eta(Y)((1 - \kappa)\varphi X + \varphi hX) - \mu\eta(X)\varphi hY. \end{aligned}$$

Moreover, we have the following result:

**Theorem 2.1** ([2]). *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a  $(\kappa, \mu)$ -space. Then  $\kappa \leq 1$ . If  $\kappa = 1$ , then  $h = 0$  and  $M^{2n+1}$  is a Sasakian manifold. If  $\kappa < 1$ ,  $M^{2n+1}$  admits three mutually orthogonal and integrable distributions  $E_M(0) = \text{span}(\xi)$ ,  $E_M(\lambda)$  and  $E_M(-\lambda)$  determined by the eigenspaces of  $h$ , where  $\lambda = \sqrt{1 - \kappa}$ .*

As a consequence of this theorem, it was also proved in [2] that the sectional curvature of a plane section  $\{X, Y\}$  normal to  $\xi$  is given by

$$(2.7) \quad K(X, Y) = \begin{cases} 2(1 + \lambda) - \mu, & \text{for any } X, Y \in E_M(\lambda), \quad n > 1, \\ 2(1 - \lambda) - \mu, & \text{for any } X, Y \in E_M(-\lambda), \quad n > 1, \\ -(\kappa + \mu)(g(X, \varphi Y))^2, & \text{for any unit vectors } X \in E_M(\lambda), Y \in E_M(-\lambda). \end{cases}$$

Given a contact metric manifold  $M^{2n+1}(\varphi, \xi, \eta, g)$ , a  $D_a$ -homothetic deformation is a change of structure tensors of the form

$$(2.8) \quad \tilde{\varphi} = \frac{1}{a}\varphi, \quad \tilde{\xi} = \xi, \quad \tilde{\eta} = a\eta, \quad \tilde{g} = ag + a(a-1)\eta \otimes \eta,$$

where  $a$  is a positive constant. It is well known that  $M^{2n+1}(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$  is also a contact metric manifold.

It was also proved in [2] that the class of  $(\kappa, \mu)$ -spaces remains invariant under  $D_a$ -homothetic deformations. Indeed, applying one of these deformations to a  $(\kappa, \mu)$ -space yields a new  $(\tilde{\kappa}, \tilde{\mu})$ -space, where

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \tilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

Many authors studied  $(\kappa, \mu)$ -spaces later, as can be seen in [1]. We highlight here the work of Boeckx, who gave in [3] an explicit writing of the curvature tensor of these spaces:

$$(2.9) \quad \begin{aligned} R(X, Y)Z &= \left(1 - \frac{\mu}{2}\right) (g(Y, Z)X - g(X, Z)Y) \\ &\quad + g(Y, Z)hX - g(X, Z)hY - g(hX, Z)Y + g(hY, Z)X \\ &\quad + \frac{1 - \frac{\mu}{2}}{1 - \kappa} (g(hY, Z)hX - g(hX, Z)hY) \\ &\quad - \frac{\mu}{2} (g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y) + \mu g(\varphi X, Y)\varphi Z \\ &\quad + \frac{\kappa - \frac{\mu}{2}}{1 - \kappa} (g(\varphi hY, Z)\varphi hX - g(\varphi hX, Z)\varphi hY) \\ &\quad - \eta(X)\eta(Z) \left( \left(\kappa - 1 + \frac{\mu}{2}\right) Y + (\mu - 1)hY \right) \\ &\quad + \eta(Y)\eta(Z) \left( \left(\kappa - 1 + \frac{\mu}{2}\right) X + (\mu - 1)hX \right) \\ &\quad + \eta(X) \left( \left(\kappa - 1 + \frac{\mu}{2}\right) g(Y, Z) + (\mu - 1)g(hY, Z) \right) \xi \\ &\quad - \eta(Y) \left( \left(\kappa - 1 + \frac{\mu}{2}\right) g(X, Z) + (\mu - 1)g(hX, Z) \right) \xi. \end{aligned}$$

Boeckx [3] also classified the  $(\kappa, \mu)$ -spaces in terms of an invariant that he introduced:  $I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}}$ . Indeed, he proved that if  $M_1$  and  $M_2$  are two non-Sasakian  $(\kappa_i, \mu_i)$ -spaces of the same dimension, then  $I_{M_1} = I_{M_2}$  if and only if, up to a  $D_a$ -homothetic deformation, the two spaces are locally isometric as contact metric spaces. In particular, if both spaces are simply connected and complete, they are globally isometric up to a  $D_a$ -homothetic deformation.

It was also stated in paper [3] that “it follows that we know all non-Sasakian  $(\kappa, \mu)$ -spaces locally as soon as we have, for every odd dimension  $2n + 1$  and for every possible value for the invariant  $I$ , one  $(\kappa, \mu)$ -space  $M$  with  $I_M = I$ .” For  $I > -1$ , we have the unit tangent sphere

bundle  $T_1M^n(c)$  of a space of constant curvature  $c$  ( $c \neq 1$ ) for the appropriate  $c$  (see [2]). For  $I \leq -1$ , Boeckx presented in [3] the following examples for any possible odd dimension  $2n + 1$  and value of  $I$ .

*Example 2.2* ([3]). Let  $\mathfrak{g}$  be a  $(2n + 1)$ -dimensional Lie algebra with basis  $\{\xi, X_1, \dots, X_n, Y_1, \dots, Y_n\}$  and the Lie brackets given by

$$\begin{aligned}
(2.10) \quad [\xi, X_1] &= -\frac{\alpha\beta}{2}X_2 - \frac{\alpha^2}{2}Y_1, & [Y_i, Y_j] &= 0, \quad i, j \neq 2, \\
[\xi, X_2] &= \frac{\alpha\beta}{2}X_1 - \frac{\alpha^2}{2}Y_2, & [X_1, Y_1] &= -\beta X_2 + 2\xi, \\
[\xi, X_i] &= -\frac{\alpha^2}{2}Y_i, \quad i \geq 3, & [X_1, Y_i] &= 0, \quad i \geq 2, \\
[\xi, Y_1] &= \frac{\beta^2}{2}X_1 - \frac{\alpha\beta}{2}Y_2, & [X_2, Y_1] &= \beta X_1 - \alpha Y_2, \\
[\xi, Y_2] &= \frac{\beta^2}{2}X_2 + \frac{\alpha\beta}{2}Y_1, & [X_2, Y_2] &= \alpha Y_1 + 2\xi, \\
[\xi, Y_i] &= \frac{\beta^2}{2}X_i, \quad i \geq 3, & [X_2, Y_i] &= \beta X_i, \quad i \geq 3, \\
[X_1, X_i] &= \alpha X_i, \quad i \neq 1, & [X_i, Y_1] &= -\alpha Y_i, \quad i \geq 3, \\
[X_i, X_j] &= 0, \quad i, j \neq 1, & [X_i, Y_2] &= 0, \quad i \geq 3, \\
[Y_2, Y_i] &= \beta Y_i, \quad i \neq 2, & [X_i, Y_j] &= \delta_{ij}(-\beta X_2 + \alpha Y_1 + 2\xi), \quad i, j \geq 3,
\end{aligned}$$

for real numbers  $\alpha$  and  $\beta$ . Next we define a left-invariant contact metric structure  $(\varphi, \xi, \eta, g)$  on the associated Lie group  $G$  as follows:

- the basis  $\{\xi, X_1, \dots, X_n, Y_1, \dots, Y_n\}$  is orthonormal,
- the characteristic vector field is given by  $\xi$ ,
- the one-form  $\eta$  is the metric dual of  $\xi$ ,
- the  $(1, 1)$ -tensor field  $\varphi$  is determined by  $\varphi\xi = 0$ ,  $\varphi X_i = Y_i$ ,  $\varphi Y_i = -X_i$ .

It can also be proved that  $G$  is a  $(\kappa, \mu)$ -space with

$$\kappa = 1 - \frac{(\beta^2 - \alpha^2)^2}{16}, \quad \mu = 2 + \frac{\alpha^2 + \beta^2}{2}.$$

Moreover, supposing  $\beta^2 > \alpha^2$  gives us that  $\lambda = \frac{\beta^2 - \alpha^2}{4} \neq 0$  and thus the  $(\kappa, \mu)$ -space is not Sasakian. The orthonormal basis also satisfies that  $hX_i = \lambda X_i$  and  $hY_i = -\lambda Y_i$ .

Finally,  $I_G = -\frac{\beta^2 + \alpha^2}{\beta^2 - \alpha^2} \leq -1$ , so for the appropriate choice of  $\beta > \alpha \geq 0$ ,  $I_G$  attains any real value smaller than or equal to  $-1$ .

Lastly, we will recall some formulas from submanifolds theory in order to fix our notation. Let  $N$  be an  $n$ -dimensional submanifold isometrically immersed in an  $m$ -dimensional Riemannian

manifold  $(M, g)$ . Then, the Gauss and Weingarten formulas hold:

$$(2.11) \quad \nabla_X Y = \bar{\nabla}_X Y + \sigma(X, Y),$$

$$(2.12) \quad \nabla_X V = -A_V X + \nabla_X^\perp V,$$

for any tangent vector fields  $X, Y$  and any normal vector field  $V$ . Here  $\sigma$  denotes the *second fundamental form*,  $A$  the *shape operator* and  $\nabla^\perp$  the *normal connection*. It is well known that the second fundamental form and the shape operator are related the following way:

$$(2.13) \quad g(\sigma(X, Y), V) = g(A_V X, Y).$$

We denote by  $R$  and  $\bar{R}$  the curvature tensors of  $M$  and  $N$ , respectively. They are related by Gauss and Codazzi's equations

$$(2.14) \quad R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)),$$

$$(2.15) \quad (R(X, Y)Z)^\perp = (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z),$$

respectively, where  $R(X, Y)Z^\perp$  denotes the normal component of  $R(X, Y)Z$  and

$$(2.16) \quad (\nabla_X \sigma)(Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\bar{\nabla}_X Y, Z) - \sigma(Y, \bar{\nabla}_X Z).$$

The submanifold  $N$  is said to be *totally geodesic* if the second fundamental form  $\sigma$  vanishes identically. It is said that it is *totally umbilical* if there exists a normal vector field  $V$  such that  $\sigma(X, Y) = g(X, Y)V$ , for any tangent vector fields  $X, Y$ . In fact, it can be proved that, in such a case,  $V$  has to be the *mean curvature*  $\tilde{H} = \frac{1}{n} \sum_{i=1}^n \sigma(e_i, e_i)$ , where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame. It is clear that every totally geodesic submanifold is also totally umbilical but the converse is not true in general.

### 3. DECOMPOSITION OF THE $h$ OPERATOR

Let  $N$  be a Legendrian submanifold of a  $(2n + 1)$ -dimensional  $(\kappa, \mu)$ -space  $M$ , that is, an  $n$ -dimensional submanifold such that  $\xi$  is normal to  $N$ . Therefore,  $\eta(X) = 0$  for any tangent vector field  $X$  and so it follows from (2.1) that  $\varphi^2 X = -X$ . Moreover, it was proved in [6] that  $N$  is an anti-invariant submanifold, i.e.,  $\varphi X$  is normal for any tangent vector field  $X$ . Moreover, under our assumptions about the dimensions of  $M$  and  $N$ , it holds that every normal vector field  $V$  can be written as  $\varphi X$ , for a certain tangent vector field  $X$ .

Therefore, we can decompose the  $h$  operator in the following way:

$$(3.1) \quad hX = h_1 X + \varphi h_2 X,$$

for any tangent vector field  $X$ , where  $h_1 X$  (respectively  $\varphi h_2 X$ ) denotes the tangent (resp. normal) component of  $hX$ .

We can prove the following properties:

**Proposition 3.1.** *Let  $N$  be a Legendrian submanifold of a  $(\kappa, \mu)$ -space  $M$ . Then,  $h_1$  and  $h_2$  are symmetric operators that satisfy  $h_1 \xi = h_2 \xi = 0$  and equations*

$$(3.2) \quad h_1^2 + h_2^2 = (1 - \kappa)I,$$

$$(3.3) \quad h_1 h_2 = h_2 h_1.$$

*Proof.* The symmetry of  $h_1$  and  $h_2$  can be directly obtained from that of  $h$  and the compatibility of the metric  $g$ . Similarly,  $h\xi = 0$  implies  $h_1\xi = h_2\xi = 0$ .

Furthermore, given a tangent vector field  $X$ , it follows from (2.1), (3.1) and the anticommutativity of  $h$  and  $\varphi$  that

$$(3.4) \quad h\varphi X = -\varphi hX = -\varphi h_1X + h_2X.$$

Using (2.4), we have that  $h^2X = (1 - \kappa)X$ . On the other hand, by virtue of (3.1) and (3.4), we obtain

$$h^2X = h(h_1X + \varphi h_2X) = h_1^2X + \varphi h_2h_1X - \varphi h_1h_2X + h_2^2X.$$

Joining both expressions for  $h^2$  and identifying the tangent and normal parts give us equations (3.2) and (3.3).  $\square$

**Proposition 3.2.** *Let  $N$  be a Legendrian submanifold of a  $(\kappa, \mu)$ -space  $M$ . Then,  $h_1$  and  $h_2$  satisfy*

$$(3.5) \quad (\overline{\nabla}_X h_1)Y = -\varphi\sigma(X, h_2Y) - h_2\varphi\sigma(X, Y),$$

$$(3.6) \quad (\overline{\nabla}_X h_2)Y = \varphi\sigma(X, h_1Y) + h_1\varphi\sigma(X, Y),$$

for any tangent vector fields  $X, Y$ .

*Proof.* It follows from Gauss and Weingarten formulas (2.11) and (2.12) that

$$(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y = -A_{\varphi Y} X + \nabla_X^\perp \varphi Y - \varphi \overline{\nabla}_X Y - \varphi \sigma(X, Y),$$

for any tangent vector fields  $X, Y$ . Therefore, by using (2.5) and identifying the tangent and normal components, we obtain:

$$(3.7) \quad A_{\varphi Y} X = -\varphi\sigma(X, Y),$$

$$(3.8) \quad \nabla_X^\perp \varphi Y = \varphi \overline{\nabla}_X Y + g(X, Y + h_1Y)\xi.$$

On the other hand, using (2.6) and (3.1), we have

$$\nabla_X(h_1Y + \varphi h_2Y) - h(\nabla_X Y) = g(X, h_2Y)\xi,$$

from where, by virtue of Gauss and Weingarten formulas (2.11) and (2.12), we deduce

$$(3.9) \quad \overline{\nabla}_X h_1Y + \sigma(X, h_1Y) - A_{\varphi h_2Y} X + \nabla_X^\perp \varphi h_2Y - h\overline{\nabla}_X Y - h\sigma(X, Y) = g(X, h_2Y)\xi.$$

We can put  $h\overline{\nabla}_X Y = h_1\overline{\nabla}_X Y + \varphi h_2\overline{\nabla}_X Y$  by (3.1). Now, by using (2.1), we can write  $\sigma(X, Y) = -\varphi^2\sigma(X, Y) + \eta(\sigma(X, Y))\xi$ , and hence  $h\sigma(X, Y) = -h\varphi^2\sigma(X, Y) = \varphi h\varphi\sigma(X, Y)$ . Again, equation (3.1) gives us  $h\sigma(X, Y) = \varphi h_1\varphi\sigma(X, Y) - h_2\varphi\sigma(X, Y)$ . Therefore, if we substitute these two expressions, together with (3.7) and (3.8), in (3.9), we obtain:

$$(3.10) \quad \begin{aligned} & \overline{\nabla}_X h_1Y + \sigma(X, h_1Y) + \varphi\sigma(X, h_2Y) + \varphi \overline{\nabla}_X h_2Y + g(X, h_2Y + h_1h_2Y)\xi \\ & - h_1\overline{\nabla}_X Y - \varphi h_2\overline{\nabla}_X Y - \varphi h_1\varphi\sigma(X, Y) + h_2\varphi\sigma(X, Y) = g(X, h_2Y)\xi. \end{aligned}$$

By identifying the tangent and normal parts of (3.10), equations (3.5) and (3.6) hold.  $\square$

It is clear that, if we multiply (3.10) by  $\xi$ , then we obtain

$$g(\sigma(X, h_1Y), \xi) + g(X, h_1h_2Y) = 0,$$

for any tangent vector fields  $X, Y$ . In fact, we can prove a more general result, which will be very useful in the proof of our main theorems:

**Lemma 3.3.** *Let  $N$  be a Legendrian submanifold of a  $(\kappa, \mu)$ -space  $M$ . Then,*

$$(3.11) \quad g(\sigma(X, Y), \xi) + g(X, h_2 Y) = 0,$$

for any tangent vector fields  $X, Y$ .

*Proof.* It follows from Weingarten equation (2.12) and from (2.13) that

$$g(X, \nabla_X \xi) + g(\sigma(X, Y), \xi) = 0,$$

for any tangent vector fields  $X, Y$ . Then, it is enough to use (2.1), (2.2) and (3.1) to obtain (3.11).  $\square$

#### 4. EXAMPLES

We will present in this section some examples of totally geodesic and totally umbilical Legendrian submanifolds of the  $(\kappa, \mu)$ -spaces of Example 2.2. Let us begin with the totally geodesic ones.

*Example 4.1.* Let  $M$  be a  $(\kappa, \mu)$ -space from Example 2.2 with invariant  $I_M \leq -1$ . Then, the distribution  $\mathcal{D}$  spanned by  $\{X_1, \dots, X_n\}$  is involutive and any integral submanifold  $N$  of it is a totally geodesic submanifold of  $M$ . Indeed, the involutive condition can be easily checked from (2.10). In order to prove the totally geodesic one, it is enough to show that  $\nabla_{X_i} X_j \in \mathcal{D}$ , for any  $i, j = 1, \dots, n$ , where  $\nabla$  denotes the Levi-Civita connection on  $M$ . In fact, it can be directly computed that:

$$(4.1) \quad \begin{aligned} \nabla_{X_1} X_1 &= \nabla_{X_1} X_2 = 0, & \nabla_{X_2} X_1 &= -\alpha X_2, & \nabla_{X_2} X_2 &= \alpha X_1, \\ \nabla_{X_1} X_i &= \nabla_{X_2} X_i = 0, & \text{for any } i &= 3, \dots, n, \\ \nabla_{X_i} X_1 &= -\alpha X_i, & \nabla_{X_i} X_2 &= 0, & \nabla_{X_i} X_j &= \delta_{ij} \alpha X_1, \text{ for any } i, j = 3, \dots, n. \end{aligned}$$

Moreover, since  $hX_i = \lambda X_i$  for any  $i = 1, \dots, n$ , then  $TN = E_M(\lambda)$ .

*Example 4.2.* Let  $M$  be a  $(\kappa, \mu)$ -space from Example 2.2 with invariant  $I_M \leq -1$ . Then, the distribution  $\mathcal{D}$  spanned by  $\{Y_1, \dots, Y_n\}$  is also involutive and any integral submanifold  $N$  of it is a totally geodesic submanifold of  $M$ . Indeed, both conditions can be checked the same way as in Example 4.1, by taking now into account that:

$$(4.2) \quad \begin{aligned} \nabla_{Y_1} Y_1 &= \beta Y_2, & \nabla_{Y_1} Y_2 &= -\beta Y_1, & \nabla_{Y_2} Y_1 &= \nabla_{Y_2} Y_2 = 0, \\ \nabla_{Y_1} Y_i &= \nabla_{Y_2} Y_i = 0, & \text{for any } i &= 3, \dots, n, \\ \nabla_{Y_i} Y_1 &= 0, & \nabla_{Y_i} Y_2 &= -\beta Y_i, & \nabla_{Y_i} Y_j &= \delta_{ij} \beta Y_2, \text{ for any } i, j = 3, \dots, n. \end{aligned}$$

In this case, since  $hY_i = -\lambda Y_i$  for any  $i = 1, \dots, n$ , then  $TN = E_M(-\lambda)$ .

*Example 4.3.* Let  $M$  be a  $(\kappa, \mu)$ -space from Example 2.2 with invariant  $I_M \leq -1$ . Then, the distribution  $\mathcal{D}$  spanned by  $\{X_1, Y_2, Z_3, \dots, Z_n\}$ , where  $Z_i$  is either  $X_i$  or  $Y_i$ , for any  $i = 3, \dots, n$ , is also involutive and any integral submanifold  $N$  of it is a totally geodesic submanifold of  $M$ .



Indeed, both conditions can be checked the same way as in Examples 4.1 and 4.2, by using now (4.1), (4.2) and the following formulas:

$$(4.3) \quad \begin{aligned} \nabla_{X_1} Y_i &= 0 \text{ for any } i = 2, \dots, n, \\ \nabla_{Y_2} X_i &= 0 \text{ for any } i = 1, 3, \dots, n, \\ \nabla_{X_i} Y_2 &= \nabla_{Y_i} X_1 = 0 \text{ for any } i = 3, \dots, n, \\ \nabla_{X_i} Y_j &= \nabla_{Y_i} X_j = 0 \text{ for any } i, j = 3, \dots, n, \text{ such that } i \neq j. \end{aligned}$$

Finally, if we define  $E(\pm\lambda) := E_M(\pm\lambda) \cap N$ , we can write  $TN = E(\lambda) \oplus E(-\lambda)$ , with  $\dim E(\lambda) = k$  (respectively  $\dim E(-\lambda) = n - k$ ), where  $k - 1$  (resp.  $n - k - 1$ ) is the number of  $Z_i$  such that  $Z_i = X_i$  (resp.  $Z_i = Y_i$ ). Therefore, we can obtain an example for any value of  $k$  from 1 to  $n - 1$ .

We now present the family of totally umbilical examples:

*Example 4.4.* Let  $M$  be a  $(\kappa, \mu)$ -space from Example 2.2 with invariant  $I_M \leq -1$ . Then, the distribution  $\mathcal{D}$  spanned by  $\{cX_1 + dY_1, \dots, cX_n + dY_n\}$ , with  $c, d$  non-zero constants, is involutive and any integral submanifold  $N$  of it is a totally umbilical submanifold of  $M$ . Indeed, the involutive condition can be easily checked from (2.10). In order to prove the totally umbilical one, we will first show that  $\sigma(cX_i + dY_i, cX_j + dY_j) = 2\delta_{ij}cd\lambda\xi$  by checking that the Levi-Civita connection on  $M$  satisfies  $\nabla_{cX_i + dY_i}(cX_j + dY_j) = Z + 2\delta_{ij}cd\lambda\xi$ , with  $Z \in \mathcal{D}$ , for any  $i, j = 1, \dots, n$ . In fact, it can be directly computed that:

$$\begin{aligned} \nabla_{cX_1 + dY_1}(cX_1 + dY_1) &= \beta d(cX_2 + dY_2) + 2cd\lambda\xi, \\ \nabla_{cX_1 + dY_1}(cX_2 + dY_2) &= -\beta d(cX_1 + dY_1), \\ \nabla_{cX_2 + dY_2}(cX_1 + dY_1) &= -\alpha c(cX_2 + dY_2), \\ \nabla_{cX_2 + dY_2}(cX_2 + dY_2) &= \alpha c(cX_1 + dY_1) + 2cd\lambda\xi, \\ \nabla_{cX_1 + dY_1}(cX_j + dY_j) &= \nabla_{cX_2 + dY_2}(cX_j + dY_j) = 0, \text{ for any } j = 3, \dots, n, \\ \nabla_{cX_i + dY_i}(cX_1 + dY_1) &= -\alpha c(cX_i + dY_i), \\ \nabla_{cX_i + dY_i}(cX_2 + dY_2) &= -\beta d(cX_i + dY_i), \text{ for any } i = 3, \dots, n, \\ \nabla_{cX_i + dY_i}(cX_j + dY_j) &= \delta_{ij}(\alpha c(cX_1 + dY_1) + \beta d(cX_2 + dY_2) + 2cd\lambda\xi), \\ &\text{for any } i, j = 3, \dots, n. \end{aligned}$$

Therefore, we can write  $\sigma(cX_i + dY_i, cX_j + dY_j) = g(cX_i + dY_i, cX_j + dY_j) \frac{2cd\lambda}{c^2 + d^2} \xi$  and, since  $\frac{2cd\lambda}{c^2 + d^2} \xi \neq 0$ , the submanifold is totally umbilical but not totally geodesic.

Finally, we observe that  $cX_i + dY_i$ ,  $i = 1, \dots, n$ , is not an eigenvector of  $h$ .

## 5. MAIN RESULTS

**Theorem 5.1.** *Let  $N$  be a Legendrian submanifold of a  $(2n + 1)$ -dimensional  $(\kappa, \mu)$ -space  $M$ , with  $\kappa < 1$  and  $I_M \leq -1$ . If  $N$  is totally geodesic, then, up to local isometries, it must be one of the submanifolds given in Examples 4.1, 4.2 or 4.3.*

*Proof.* Since the submanifold  $N$  is totally geodesic, it follows directly from (3.11) that  $h_2 = 0$  and so  $h|_N = h_1$  and  $h_1^2 = (1 - \kappa)I$  (see (3.1) and (3.2)). The operator  $h_1$  is differentiable and symmetric, so it is diagonalisable and it has two eigenvalues  $\pm\lambda = \pm\sqrt{1 - \kappa}$ , which are distinct and constant everywhere.

Let us denote by  $E(\lambda)$  and  $E(-\lambda)$  the eigenspaces of  $h_1$  in  $TN$  and by  $k$  the dimension of  $E(\lambda)$ . This means that  $\dim(E(-\lambda)) = n - k$  (because  $\dim N = n$ ) and that  $k \in \{0, \dots, n\}$ . The multiplicities of both eigenspaces must be the same at every point because the coefficients of the characteristic polynomial are differentiable. Indeed, the characteristic polynomial of  $h_1$  is completely determined by  $k$  (thus, for different indices  $k$ , we get a different characteristic polynomial). Since  $k$  is an integer, it is impossible by continuity to go from one to the other one, thus the eigendistributions are differentiable. We can then write

$$(5.1) \quad TN = E(\lambda) \oplus E(-\lambda),$$

where  $\dim(E(\lambda)) = k$  and  $\dim(E(-\lambda)) = n - k$ , for a certain  $k \in \{0, \dots, n\}$ .

Moreover, we deduce from (3.5) that  $\overline{\nabla} h_1 = 0$ . Therefore, it is straightforward to check that, if  $Y_\lambda \in E(\lambda)$ , then  $\overline{\nabla}_X Y_\lambda \in E(\lambda)$ , for every tangent vector field  $X$ . Similarly, if  $Y_{-\lambda} \in E(-\lambda)$ , then  $\overline{\nabla}_X Y_{-\lambda} \in E(-\lambda)$ . Thus,  $E(\lambda)$  and  $E(-\lambda)$  are parallel and hence involutive. By virtue of Theorem 5.4 of [5],  $N$  can be locally decomposed as  $M_1 \times M_2$ , where  $M_1$  and  $M_2$  are leaves of the distributions  $E(\lambda)$  and  $E(-\lambda)$ , respectively. Furthermore, it follows from (2.7) that, if  $\dim M_1 \geq 2$  (resp.  $\dim M_2 \geq 2$ ), then  $M_1$  (resp.  $M_2$ ) has constant curvature  $2(1 + \lambda) - \mu = 2\lambda(I_M + 1) \leq 0$  (resp.  $2(1 - \lambda) - \mu = 2\lambda(I_M - 1) < 0$ ).

Recall that we have examples of submanifolds with decomposition (5.1) for every value of  $k$ . Indeed, see Example 4.1 for  $k = n$ , Example 4.2 for  $k = 0$  and Example 4.3 for any value of  $k$  from 1 to  $n - 1$ . Now, we will prove that any example must be one of these, up to local isometries.

Let us denote by  $F : N^n \rightarrow M^{2n+1}(\kappa, \mu)$  the immersion of  $N$  into  $M$ . Since  $\kappa < 1$  and  $I_M \leq -1$ , we can suppose that, locally,  $M^{2n+1}(\kappa, \mu)$  is one of the Lie groups from Example 2.2. Thus, it is homogeneous and we can fix a point  $p_0 \in N$  such that  $F(p_0) = e$ , where  $e$  is the neutral element of the group.

We will give the explicit details when  $2 \leq k \leq n - 2$ . The other cases can be done in a similar way. We have that  $N = M_1(2\lambda(I_M + 1)) \times M_2(2\lambda(I_M - 1))$  and we also identify  $N$  with its image as the (totally geodesic) integral submanifold through  $e$  of the distribution spanned by  $X_1, X_3, \dots, X_{k+1}, Y_2, Y_{k+2}, \dots, Y_n$ . We denote by  $G$  the latter immersion of  $N$  and we pick an orthonormal basis  $\{e_1, \dots, e_n\}$  at the point  $p_0$  of  $N$ , with  $G(p_0) = e$ , such that  $E_{p_0}(\lambda) = \langle e_1(p_0), \dots, e_k(p_0) \rangle$ ,  $E_{p_0}(-\lambda) = \langle e_{k+1}(p_0), \dots, e_n(p_0) \rangle$  and

$$\begin{aligned} dG(e_1(p_0)) &= X_1(e), \\ dG(e_j(p_0)) &= X_{j+1}(e), \quad j = 2, \dots, k, \\ dG(e_{k+1}(p_0)) &= Y_2(e), \\ dG(e_j(p_0)) &= Y_j(e), \quad j = k + 2, \dots, n, \end{aligned}$$

Note that by construction both

$$X_1(e), X_3(e), \dots, X_{k+1}(e), \varphi Y_2(e), \varphi Y_{k+2}(e), \dots, \varphi Y_n(e)$$

and

$$dF(e_1(p_0)), \dots, dF(e_k(p_0)), \varphi dF(e_{k+1}(p_0)), \dots, \varphi dF(e_n(p_0))$$

are basis of  $E_e(\lambda)$ . So, in view of Theorem 3 of [3], there exists an isometry  $H$  of  $M^{2n+1}(\kappa, \mu)$  preserving the structure such that  $H(e) = e$  and  $H$  maps one basis of  $E_e(\lambda)$  into the other one. As a consequence, we have that  $H \circ F(e) = G(e)$  and  $d(H \circ F)(e_i) = dG(e_i)$ .

We now take a geodesic  $\gamma$  in  $N$  through the point  $p_0$ . Since  $N$  is totally geodesic, both with respect to the immersions  $H \circ F$  and  $G$ , the curves  $H \circ F(\gamma)$  and  $G(\gamma)$  are both geodesics in  $M^{2n+1}(\kappa, \mu)$  through  $e$ . Since  $d(H \circ F)(e_i) = dG(e_i)$ , they are also determined by the same initial conditions. Therefore, both curves need to coincide, so  $H \circ F(\gamma(s)) = G(\gamma(s))$  for all  $s$  and thus  $F$  and  $G$  are congruent.  $\square$

**Theorem 5.2.** *Let  $N$  be a Legendrian submanifold of a  $(2n + 1)$ -dimensional  $(\kappa, \mu)$ -space  $M$ , with  $n \geq 3$ ,  $\kappa < 1$  and  $I_M \leq -1$ . If  $N$  is totally umbilical (but not totally geodesic), then, up to local isometries, it must be one of the submanifolds given in Example 4.4.*

*Proof.* Since  $N$  is totally umbilical (but not totally geodesic), then there exists a normal vector field  $V \neq 0$  such that  $\sigma(X, Y) = g(X, Y)V$ . It follows from (3.11) that  $g(X, Y)\eta(V) + g(X, h_2Y) = 0$ , for any tangent vector fields  $X, Y$ , and thus

$$(5.2) \quad h_2Y = aY,$$

with  $a = -\eta(V)$ .

We will now prove that  $a \neq 0$ . Indeed, if we suppose that  $a = 0$ , then  $h_2 = 0$  and, as in the proof of Theorem 5.1, we have that  $h = h_1$ ,  $h_1^2 = (1 - \kappa)I$  and  $\bar{\nabla}h_1 = 0$ . Moreover, since  $h_2 = 0$ , it is clear that  $\bar{\nabla}h_2 = 0$  and we obtain from (3.6) that  $\varphi\sigma(X, h_1Y) + h_1\varphi\sigma(X, Y) = 0$ , which, by using that  $N$  is totally umbilical, becomes

$$(5.3) \quad g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V = 0,$$

for any tangent vector fields  $X, Y$ . Let us now choose unit vector fields  $X_\lambda \in E(\lambda)$  and  $X_{-\lambda} \in E(-\lambda)$ . Then, taking  $X = Y = X_\lambda$  in (5.3) implies  $h_1\varphi V = -\lambda\varphi V$  and taking  $X = Y = X_{-\lambda}$  in (5.3) implies  $h_1\varphi V = \lambda\varphi V$ . Since  $V \neq 0$ , this yields a contradiction.

Therefore, we can suppose from now on that (5.2) holds for  $a \neq 0$ . We deduce from equation (3.6) that

$$X(a)Y = \varphi\sigma(X, h_1Y) + h_1\varphi\sigma(X, Y) = g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V,$$

for every  $X, Y$  tangent vector fields.

Since  $\dim N \geq 3$ , we can take  $Y$  linearly independent from  $\varphi V$  and  $h_1\varphi V$ . Then we deduce from the previous equation that  $X(a) = 0$ , for every  $X$ , thus  $a$  is a constant. Moreover,  $g(X, h_1Y)\varphi V + g(X, Y)h_1\varphi V = 0$ , for every  $X, Y$  tangent vector fields. Taking unit  $X = Y$ , we obtain that  $h_1\varphi V = -g(X, h_1X)\varphi V$ , which is only possible if  $h_1 = 0$  or  $\varphi V = 0$ . If  $h_1 = 0$ , then substituting (5.2) in (3.6) gives that  $2ag(X, Y)\varphi V = 0$ , so again  $\varphi V = 0$ .

In both cases, we have obtained that  $\varphi V = 0$ , so  $V$  is parallel to  $\xi$  and it follows from  $a = -\eta(V)$  that  $V = -a\xi$  and  $\sigma(X, Y) = -ag(X, Y)\xi$  holds, for every  $X, Y$  tangent, where  $a \neq 0$  is a constant.

Let us now recall Codazzi's equation (2.15):

$$(R(X, Y)Z)^\perp = (\nabla_X\sigma)(Y, Z) - (\nabla_Y\sigma)(X, Z).$$

The first term is the normal component of  $R(X, Y)Z$ , so by equation (2.9) and the fact that  $h_2X = h_1X + a\varphi X$ , we can write

$$\begin{aligned} (R(X, Y)Z)^\perp &= a(g(Y, Z)\varphi X - g(X, Z)\varphi Y) \\ &\quad + a\frac{1-\frac{\mu}{2}}{1-\kappa}(g(h_1Y, Z)\varphi X - g(h_1X, Z)\varphi Y) \\ &\quad - a\frac{\kappa-\frac{\mu}{2}}{1-\kappa}(g(Y, Z)\varphi h_1X - g(X, Z)\varphi h_1Y). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\nabla_X\sigma)(Y, Z) &= \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z) = \\ &= \nabla_X^\perp(-ag(Y, Z)\xi) + ag(\overline{\nabla}_X Y, Z)\xi + ag(\overline{\nabla}_X Z, X)\xi = \\ &= -ag(Y, Z)\overline{\nabla}_X^\perp\xi = ag(Y, Z)(\varphi X + \varphi h_1X). \end{aligned}$$

Therefore, the second term of Codazzi's equation is

$$\begin{aligned} (\nabla_X\sigma)(Y, Z) - (\nabla_Y\sigma)(X, Z) &= ag(Y, Z)(\varphi X + \varphi h_1X) - ag(X, Z)(\varphi Y + \varphi h_1Y) \\ &= a(g(Y, Z)\varphi X - g(X, Z)\varphi Y) + a(g(Y, Z)\varphi h_1X - g(X, Z)\varphi h_1Y). \end{aligned}$$

Joining both terms, and bearing in mind that  $a \neq 0$ , we obtain

$$\begin{aligned} \frac{1-\frac{\mu}{2}}{1-\kappa}(g(h_1Y, Z)\varphi X - g(h_1X, Z)\varphi Y) &= \\ = \frac{1-\frac{\mu}{2}}{1-\kappa}(g(Y, Z)\varphi h_1X - g(X, Z)\varphi h_1Y). \end{aligned}$$

Since we are supposing that  $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}} \leq -1$ , then  $\frac{1-\frac{\mu}{2}}{1-\kappa} \neq 0$  and applying  $\varphi$  to both terms of the previous equation gives us that

$$g(h_1Y, Z)X - g(h_1X, Z)Y = g(Y, Z)h_1X - g(X, Z)h_1Y,$$

for every  $X, Y, Z$  tangent vector fields.

Since  $\dim(N) \geq 3$ , we can choose  $Y = Z$  unit and orthogonal to  $X, h_1X$ , and we obtain that

$$(5.4) \quad h_1X = g(h_1Y, Y)X,$$

and thus  $h_1X = bX$  for some function  $b$ .

From (3.2), we have that  $a^2 + b^2 = 1 - \kappa = \lambda^2 \neq 0$ , and in particular that  $b$  must be constant. We can also write that  $a = \lambda \cos(\theta)$  and  $b = \lambda \sin(\theta)$  for some constant  $\theta \in [-\pi, \pi]$ . Since  $a \neq 0$ , then  $\theta \neq \pm\frac{\pi}{2}$ .

By Gauss equation (2.14) and the fact that  $h_2X = aX$ , then

$$\begin{aligned} R(X, Y, Z, W) &= \overline{R}(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) = \\ &= \overline{R}(X, Y, Z, W) - a^2(g(X, W)g(Y, Z) + g(X, Z)g(Y, W)), \end{aligned}$$

for every  $X, Y, Z, W$  tangent vector fields.

On the other hand, we know from equation (2.9) and the fact that  $hX = bX + a\varphi X$ , that

$$\begin{aligned} R(X, Y, Z, W) &= \left(1 - \frac{\mu}{2} + 2b + b^2\frac{1-\frac{\mu}{2}}{1-\kappa} + a^2\frac{\kappa-\frac{\mu}{2}}{1-\kappa}\right) \\ &\quad (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)). \end{aligned}$$

Joining the last two equations, we obtain

$$\begin{aligned}
\overline{R}(X, Y, Z, W) &= \left( 1 - \frac{\mu}{2} + 2b + b^2 \frac{1 - \frac{\mu}{2}}{1 - \kappa} + a^2 \left( \frac{\kappa - \frac{\mu}{2}}{1 - \kappa} + 1 \right) \right) \\
&\quad (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \\
&= \left( 1 - \frac{\mu}{2} + 2b + (a^2 + b^2) \frac{1 - \frac{\mu}{2}}{1 - \kappa} \right) \\
&\quad (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \\
&= 2 \left( 1 - \frac{\mu}{2} + b \right) (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)).
\end{aligned}$$

This means that the submanifold is a space form with constant curvature  $2(1 - \frac{\mu}{2} + b)$ . Moreover, since  $I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}} \leq -1$  and  $b = \lambda \sin(\theta) \neq \lambda$ , then  $1 - \frac{\mu}{2} + b < 1 - \frac{\mu}{2} + \lambda \leq 0$  and the submanifold is a hyperbolic space  $N = \mathbb{H}(2(1 - \frac{\mu}{2} + \lambda \sin(\theta)))$ .

Summing up, there exists  $\theta \in [-\pi, \pi]$ ,  $\theta \neq \pm \frac{\pi}{2}$ , such that

$$\begin{aligned}
(5.5) \quad N &= \mathbb{H}(2(1 - \frac{\mu}{2} + \lambda \sin(\theta))), \\
h_1 X &= \lambda \sin(\theta) X, \\
h_2 X &= \lambda \cos(\theta) X, \\
\sigma(X, Y) &= -\lambda \cos(\theta) g(X, Y) \xi.
\end{aligned}$$

We have examples of submanifolds with these properties for every value of  $\theta$ . Indeed, Examples 4.4 with  $c = \cos(\pi/4 - \theta/2)$ ,  $d = -\sin(\pi/4 - \theta/2)$  satisfy

$$\begin{aligned}
\sigma(cX_i + dY_i, cX_j + dY_j) &= 2\delta_{ij} cd \lambda \xi = -2\delta_{ij} \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \lambda \xi = \\
&= -\delta_{ij} \sin\left(\frac{\pi}{2} - \theta\right) \lambda \xi = -\delta_{ij} \lambda \cos(\theta) \xi = \\
&= -\lambda \cos(\theta) g(cX_i + dY_i, cX_j + dY_j) \xi,
\end{aligned}$$

and the rest of conditions also hold.

Now, we will prove that any totally umbilical submanifold  $N$  must be one of these, up to local isometries. Let us denote by  $F : N^n \rightarrow M^{2n+1}(\kappa, \mu)$  the immersion of  $N$  into  $M(\kappa, \mu)$ . Since  $\kappa < 1$  and  $I_M \leq -1$ , we can suppose that, locally,  $M(\kappa, \mu)$  is one of the Lie groups from Example 2.2. Thus, it is homogeneous and we can fix a point  $p_0 \in N$  such that  $F(p_0) = e$ , where  $e$  is the neutral element of the group.

We have that  $N = \mathbb{H}(2(1 - \frac{\mu}{2} + \lambda \sin(\theta)))$  and we can identify  $N$  with its image as the (totally umbilical) integral submanifold through  $e$  of the distribution spanned by  $\{\cos(\frac{\pi}{4} - \frac{\theta}{2}) X_i(e) - \sin(\frac{\pi}{4} - \frac{\theta}{2}) Y_i(e), i = 1, \dots, n\}$ . We denote by  $G$  this immersion of  $N$  and we take an orthonormal basis  $\{e_1, \dots, e_n\}$  at the point  $p_0$  of  $N$  such that

$$dG(e_i) = \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) X_i(e) - \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) Y_i(e), \quad i = 1, \dots, n.$$

On the other hand, we have that

$$(5.6) \quad \begin{aligned} h(dF(e_i)) &= dF(\lambda \sin(\theta)e_i) + \varphi dF(\lambda \cos(\theta)e_i) \\ &= \lambda \sin(\theta)dF(e_i) + \lambda \cos(\theta)\varphi dF(e_i), \end{aligned}$$

$$(5.7) \quad h\varphi(dF(e_i)) = -\varphi h(dF(e_i)) = \lambda \cos(\theta)dF(e_i) - \lambda \sin(\theta)\varphi dF(e_i).$$

Therefore, using (5.6) and (5.7), we can construct eigenvectors of  $h$  associated with the eigenvalue  $\lambda$  the following way:

$$\begin{aligned} &h \left( \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) dF(e_i) + \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \varphi(dF(e_i)) \right) = \\ &= \lambda \left( \left( \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \sin(\theta) + \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \cos(\theta) \right) dF(e_i) \right. \\ &\quad \left. + \left( \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \cos(\theta) - \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \sin(\theta) \right) \varphi(dF(e_i)) \right) = \\ &= \lambda \left( \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right) dF(e_i) + \cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \varphi(dF(e_i)) \right) = \\ &= \lambda \left( \cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) dF(e_i) + \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \varphi(dF(e_i)) \right), \end{aligned}$$

for any  $i = 1, \dots, n$ .

Note that, by construction, both

$$\cos \left( \frac{\pi}{4} - \frac{\theta}{2} \right) dF(e_i) + \sin \left( \frac{\pi}{4} - \frac{\theta}{2} \right) \varphi(dF(e_i)), \quad i = 1, \dots, n$$

and

$$X_1(e), \dots, X_n(e)$$

are basis of  $E_e(\lambda)$ . So, in view of Theorem 3 of [3], there exists an isometry  $H$  of  $M^{2n+1}(\kappa, \mu)$  preserving the structure such that  $H(e) = e$  and  $H$  maps one basis of  $E_e(\lambda)$  into the other one. As a consequence, we have that  $H \circ F(e) = G(e)$  and  $d(H \circ F)(e_i) = dG(e_i)$ .

We now take a geodesic  $\gamma$  in  $N$  through the point  $p_0$ . Since  $N$  is totally umbilical with respect to both  $H \circ F$  and  $G$ , then  $\gamma_1 = H \circ F(\gamma)$  and  $\gamma_2 = G(\gamma)$  are curves in  $M(\kappa, \mu)$  passing through  $e$  that satisfy  $\nabla_{\gamma'_1} \gamma'_1 = \nabla_{\gamma'_2} \gamma'_2 = -\lambda \sin(\theta)\xi$ . Since  $d(H \circ F)(e_i) = dG(e_i)$ , they are also determined by the same initial conditions. Therefore, both curves need to coincide, so  $H \circ F(\gamma(s)) = G(\gamma(s))$  for all  $s$  and thus  $F$  and  $G$  are congruent.  $\square$

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