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# FOUR DIMENSIONAL BIHARMONIC HYPERSURFACES IN NONZERO SPACE FORM HAVE CONSTANT MEAN CURVATURE

ZHIDA GUAN, HAIZHONG LI, AND LUC VRANCKEN

ABSTRACT. In this paper, through making careful analysis of Gauss and Codazzi equations, we prove that four dimensional biharmonic hypersurfaces in nonzero space form have constant mean curvature. Our result gives the positive answer to the conjecture proposed by Balmus-Montaldo-Oniciuc in 2008 for four dimensional hypersurfaces.

## 1. INTRODUCTION

Biharmonic maps were introduced in 1964 by Eells and Sampson [ES] as a generalization of harmonic maps. In their paper, Eells and Sampson suggested considering the bi-energy of a map  $\phi : (M^n, g) \rightarrow (N^m, h)$  between two Riemannian manifolds defined by

$$(1.1) \quad E_2(\phi) = \frac{1}{2} \int_{M^n} |\tau(\phi)|^2 d\mu_g,$$

where  $\tau(\phi)$  is the tension field of  $\phi$  and  $d\mu_g$  is the volume element on  $(M^n, g)$ . Stationary points of the bi-energy functional are called biharmonic maps. Jiang (see [J1], [J2]) is the first mathematician who systematically studied the bi-energy functional, and he computed the first and second variations of  $E_2$ . The stationary points of the functional  $E_2$  satisfy the Euler-Lagrange equation

$$(1.2) \quad -\Delta\tau(\phi) = \sum_{i=1}^n R^{N^m}(d\phi(e_i), \tau(\phi))d\phi(e_i),$$

where  $\Delta$  is the Laplacian of  $(M^n, g)$ . Biharmonic submanifolds have attracted a lot of attentions from mathematicians and many important results on biharmonic submanifolds have been obtained since then (see [C1], [C2], [BMO1], [BMO2], [CMO1], [CMO2], [F1], [F2], [F3], [FH]).

The following conjecture was proposed by Balmus-Montaldo-Oniciuc in 2008 [BMO1] (also see Conjecture 7.2 of page 180 in [OC]).

**Conjecture.** *Any  $n$ -dimensional biharmonic submanifold in  $\mathbb{S}^{n+p}$  has constant mean curvature.*

When  $n = 2, p = 1$ , the conjecture was proved by Caddeo-Montaldo-Oniciuc in [CMO1]; when  $n = 3, p = 1$ , the conjecture was proved by Balmus-Montaldo-Oniciuc in [BMO2]. In this paper, we prove the conjecture for  $n = 4, p = 1$ . In fact, we prove the following theorem:

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**Theorem 1.1.** *Four dimensional biharmonic hypersurfaces in nonzero space form  $\mathbb{R}^5(c)$  ( $c \neq 0$ ) have constant mean curvature.*

In the study of biharmonic submanifolds, there are two other conjectures proposed by Chen in 1991 [C1], and by Caddeo-Montaldo-Oniciuc in 2001 [CMO1], respectively.

**Chen's Conjecture.** *Every  $n$ -dimensional biharmonic submanifold of Euclidean spaces  $\mathbb{R}^{n+p}$  is minimal.*

**Generalized Chen's Conjecture.** *Every  $n$ -dimensional biharmonic submanifold of a Riemannian manifold  $N^{n+p}$  with non-positive sectional curvature is minimal.*

When  $n = 2, p = 1$ , Chen's conjecture was proved by Chen and Jiang around 1987 independently; when  $n = 3, p = 1$ , Chen's conjecture was proved by Hasanis and Vlachos in 1995 [HV]. Recently, Fu-Hong-Zhan [FHZ] have made important progress about Chen's conjecture. In fact, they proved Chen's conjecture for  $n = 4, p = 1$ .

Ou and Tang [OT] constructed a family of counterexamples, where the generalized Chen's conjecture is false when the ambient space has nonconstant negative sectional curvature. However, the generalized Chen's conjecture remains open when the ambient spaces have constant sectional curvature. In particular, when  $p = 1, N^{n+1} = \mathbb{H}^{n+1}$ ,  $n = 2$  and  $n = 3$ , the generalized Chen's conjecture was proved by Caddeo-Montaldo-Oniciuc [CMO2] and Balmus-Montaldo-Oniciuc [BMO2], respectively. When  $c = -1$ , our Theorem 1.1 solves the generalized Chen's conjecture for  $p = 1, N^{n+1} = \mathbb{H}^{n+1}, n = 4$ .

The paper is organized as follows. In Section 2, we recall some fundamental concepts and formulas for  $n$ -dimensional biharmonic hypersurfaces in space forms  $\mathbb{R}^{n+1}(c)$ . In Section 3, for 4-dimensional biharmonic hypersurfaces, we derive some equations and lemmas. In Section 4, we give the proof of Theorem 1.1.

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## 2. PRELIMINARIES

Let  $M^n$  be an  $n$ -dimensional hypersurface in  $(n + 1)$ -dimensional space form  $\mathbb{R}^{n+1}(c)$  with constant sectional curvature  $c$ . When  $c = 0$ ,  $\mathbb{R}^{n+1}(c)$  is  $(n + 1)$ -dimensional Euclidean space; when  $c = 1$ ,  $\mathbb{R}^{n+1}(c)$  is  $(n + 1)$ -dimensional unit sphere; when  $c = -1$ ,  $\mathbb{R}^{n+1}(c)$  is  $(n + 1)$ -dimensional hyperbolic space. Let  $\nabla$  and  $\tilde{\nabla}$  be the Levi-Civita connections of  $M^n$  and  $\mathbb{R}^{n+1}(c)$ . Denote  $X$  and  $Y$  tangent vector fields of  $M^n$  and  $\xi$  the unit normal vector field. Then the Gauss formula and Weingarten formula (for example, see [C3]) are

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -AX,$$

where  $h$  is the second fundamental form, and  $A$  is the Weingarten operator. The mean curvature function  $H$  is defined by

$$(2.3) \quad H = \frac{1}{n} \text{trace } h.$$

Moreover, the Gauss and Codazzi equations are given by

$$(2.4) \quad R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$(2.5) \quad (\nabla_X A)Y = (\nabla_Y A)X,$$

where the Riemannian curvature  $R(X, Y)Z$  is defined by

$$(2.6) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

We have the following characterization result for  $M^n$  to be a biharmonic hypersurface in  $\mathbb{R}^{n+1}(c)$  (see [J1], [J2], [BMO2], [CMO2], [F3], [FH]).

**Proposition 2.1.** A hypersurface  $M^n$  in a space form  $\mathbb{R}^{n+1}(c)$  is biharmonic if and only if  $H$  and  $A$  satisfy

$$(2.7) \quad \Delta H + H \text{ trace } A^2 = ncH,$$

$$(2.8) \quad 2A \text{grad } H + nH \text{grad } H = 0,$$

where the Laplacian operator  $\Delta$  acting on a smooth function  $f$  on  $M^n$  is defined by

$$(2.9) \quad \Delta f = -\text{div}(\nabla f).$$

Let  $M^n$  be an  $n$ -dimensional biharmonic hypersurface in  $(n+1)$ -dimensional space form  $\mathbb{R}^{n+1}(c)$ . Suppose the mean curvature function  $H$  is not constant. From (2.8), we have that  $\text{grad } H$  is an eigenvector of the Weingarten operator  $A$  with the corresponding principal curvature  $-nH/2$ . Without loss of generality, we can choose  $e_1$  such that  $e_1$  is parallel to  $\text{grad } H$ , and we can choose suitable orthonormal frame  $\{e_1, e_2, \dots, e_n\}$  such that

$$(2.10) \quad Ae_i = \lambda_i e_i,$$

where  $\lambda_1 = -nH/2$ .

Denote the connection coefficients  $\omega_{ij}^k$  by

$$(2.11) \quad \nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k, \quad \omega_{ij}^k + \omega_{ik}^j = 0, \quad i, j = 1, \dots, n.$$

**Lemma 2.2.** ([FH]) Let  $M^n$  be an biharmonic hypersurface in a space form  $\mathbb{R}^{n+1}(c)$  and assume the mean curvature  $H$  is non-constant. Then the multiplicity of the principal curvature  $\lambda_1 (= -nH/2)$  is one, that is,  $\lambda_j \neq \lambda_1$  for  $2 \leq j \leq n$ .

**Lemma 2.3.** ([FH]) *Let  $M^n$  be an biharmonic hypersurface in a space form  $\mathbb{R}^{n+1}(c)$  and assume the mean curvature  $H$  is non-constant. Then the principal curvatures  $\lambda_i$  and the connection coefficients  $\omega_{ii}^1$  satisfy*

$$(2.12) \quad e_1 e_1(\lambda_1) = e_1(\lambda_1) \left( \sum_{j=2}^n \omega_{jj}^1 \right) + \lambda_1 (n(n-2)c - R + 4\lambda_1^2),$$

$$(2.13) \quad e_1(\lambda_i) = (\lambda_i - \lambda_1) \omega_{ii}^1, \quad 2 \leq i \leq n,$$

$$(2.14) \quad e_1(\omega_{ii}^1) = (\omega_{ii}^1)^2 + \lambda_1 \lambda_i + c, \quad 2 \leq i \leq n,$$

where  $R$  is the scalar curvature.

In 2015, Fu [F2] proved the following theorem.

**Theorem 2.4.** ([F2]) *Let  $M^n$  be a biharmonic hypersurface with at most three distinct principal curvatures in  $\mathbb{R}^{n+1}(c)$ . Then  $M^n$  has constant mean curvature.*

### 3. FOUR DIMENSIONAL BIHARMONIC HYPERSURFACES IN $\mathbb{R}^5(c)$

From now on, we study the biharmonicity of a hypersurface  $M^4$  in a space form  $\mathbb{R}^5(c)$ . By Theorem 2.4, we only need to work on the case that  $M^4$  has four distinct principal curvatures, and we assume that the mean curvature  $H$  is non-constant. Then there exists a neighborhood of  $p$  such that  $\text{grad } H \neq 0$ . The squared length of the second fundamental form of  $M$  is

$$(3.1) \quad S = \sum_{i=1}^4 \lambda_i^2 = 4H^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2.$$

By using  $\lambda_1 = -2H$ , Gauss equation is

$$(3.2) \quad R = 12c + 16H^2 - S = 12c + 12H^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2.$$

Since  $e_1$  is parallel to  $\text{grad } H$ , it follows that

$$(3.3) \quad e_1(H) \neq 0, \quad e_2(H) = e_3(H) = e_4(H) = 0.$$

The following result can be found in [F3].

**Lemma 3.1.** ([F3]) *Let  $M^4$  be a biharmonic hypersurface with four distinct principal curvatures in space forms  $\mathbb{R}^5(c)$ , then we have*

$$(3.4) \quad \nabla_{e_1} e_i = 0, \quad i = 1, 2, 3, 4,$$

$$(3.5) \quad \nabla_{e_i} e_1 = -\omega_{ii}^1 e_i, \quad i = 2, 3, 4,$$

$$(3.6) \quad \nabla_{e_i} e_i = \sum_{k=1, k \neq i}^4 \omega_{ii}^k e_k, \quad i = 2, 3, 4,$$

$$(3.7) \quad \nabla_{e_i} e_j = -\omega_{ii}^j e_i + \omega_{ij}^k e_k \quad \text{for distinct } i, j, k = 2, 3, 4,$$

where

$$(3.8) \quad \omega_{ii}^j = -\frac{e_j(\lambda_i)}{\lambda_j - \lambda_i}.$$

Note that we have

$$(3.9) \quad \lambda_2 + \lambda_3 + \lambda_4 = -3\lambda_1,$$

$$(3.10) \quad S = \sum_{i=1}^4 \lambda_i^2 = \lambda_1^2 + \sum_{i=2}^4 \lambda_i^2,$$

$$(3.11) \quad e_1(\lambda_1) \neq 0, \quad e_i(\lambda_1) = 0, \quad 2 \leq i \leq 4.$$

Following [FH] and [FHZ], the functions  $f_k$  are defined by

$$(3.12) \quad f_k = (\omega_{22}^1)^k + (\omega_{33}^1)^k + (\omega_{44}^1)^k, \quad k = 1, \dots, 5.$$

The following lemma was proved by Fu-Hong-Zhan in [FHZ].

**Lemma 3.2.** ([FHZ]) *With the notations  $f_k$ , the following two relations hold*

$$(3.13) \quad f_1^4 - 6f_1^2f_2 + 3f_2^2 + 8f_1f_3 - 6f_4 = 0,$$

$$(3.14) \quad f_1^5 - 5f_1^3f_2 + 5f_1^2f_3 + 5f_2f_3 - 6f_5 = 0.$$

For simplicity, given a function  $g$  on  $M^4$ , we write  $g' = e_1(g)$ ,  $g'' = e_1e_1(g)$ ,  $g''' = e_1e_1e_1(g)$  and  $g'''' = e_1e_1e_1e_1(g)$ . Also, we write  $\lambda = \lambda_1$  and  $f_1 = T$ . Note that for  $i = 2, 3, 4$ , using Lemma 3.1, we have

$$(3.15) \quad \begin{aligned} e_i(\lambda') &= e_i e_1(\lambda) \\ &= e_1 e_i(\lambda) + [e_i, e_1](\lambda) \\ &= (\nabla_{e_i} e_1 - \nabla_{e_1} e_i)(\lambda) = 0, \end{aligned}$$

similarly we have

$$(3.16) \quad e_i(\lambda'') = e_i(\lambda''') = e_i(\lambda''''') = e_i(\lambda''''') = 0.$$

Following the argument of Fu-Hong-Zhan in [FHZ], we can prove

**Lemma 3.3.**  *$f_k$  can be written as*

$$(3.17) \quad \left\{ \begin{array}{l} f_1 = T, \\ f_2 = T' + 3\lambda^2 - 3c, \\ f_3 = \frac{1}{2}T'' - (\lambda^2 + c)T + 6\lambda\lambda', \\ f_4 = \frac{1}{6}T''' - \frac{4}{3}(\lambda^2 + c)T' - \frac{5}{3}\lambda\lambda'T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4 - c\lambda^2 + 3c^2, \\ f_5 = \frac{1}{24}T'''' - \frac{5}{6}(\lambda^2 + c)T'' + 2\lambda\lambda''' + \frac{5}{3}\lambda'\lambda'' - \frac{26}{3}\lambda^3\lambda' - \frac{47}{6}c\lambda\lambda' \\ \quad - \frac{25}{12}\lambda\lambda'T' - \frac{1}{12}(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4 - 24c\lambda^2 - 12c^2)T. \end{array} \right.$$

*Proof.* Since  $e_1(\lambda) \neq 0$ , from (2.12), we have

$$(3.18) \quad f_1 = \frac{e_1 e_1(\lambda) - \lambda(S - 4c)}{e_1(\lambda)} = \frac{\lambda''}{\lambda'} - \frac{\lambda}{\lambda'}(S - 4c) =: T.$$

Taking the sum of  $i$  from 2 to 4 in (2.14) and (2.13) and using (3.9), we have

$$(3.19) \quad f_2 = 3\lambda^2 + e_1(f_1) - 3c = T' + 3\lambda^2 - 3c,$$

$$(3.20) \quad \begin{aligned} g_1 &:= \sum_{i=2}^4 \lambda_i \omega_{ii}^1 \\ &= \lambda T - 3e_1(\lambda) = \lambda T - 3\lambda'. \end{aligned}$$

Multiplying  $\omega_{ii}^1$  on both sides of (2.14), we have

$$\frac{1}{2}e_1\left((\omega_{ii}^1)^2\right) = (\omega_{ii}^1)^3 + \lambda\lambda_i\omega_{ii}^1 + c\omega_{ii}^1.$$

Taking the sum of  $i$  from 2 to 4, we have

$$(3.21) \quad \begin{aligned} f_3 &= \frac{1}{2}e_1(f_2) - \lambda g_1 - cT \\ &= \frac{1}{2}T'' - (\lambda^2 + c)T + 6\lambda\lambda'. \end{aligned}$$

Differentiating (3.20) with respect to  $e_1$  and using (2.13) and (2.14), we have

$$(3.22) \quad e_1(g_1) = 2 \sum_{i=2}^4 \lambda_i (\omega_{ii}^1)^2 + \lambda \sum_{i=2}^4 \lambda_i^2 - \lambda \sum_{i=2}^4 (\omega_{ii}^1)^2 - 3c\lambda.$$

From (3.9), (3.10), (3.18), we have

$$\begin{aligned} g_2 &:= \sum_{i=2}^4 \lambda_i (\omega_{ii}^1)^2 = \frac{1}{2} \{e_1(g_1) - \lambda(S - \lambda^2) + \lambda f_2 + 3c\lambda\} \\ &= \frac{1}{2} \{e_1(g_1) - \lambda'' + \lambda'T + \lambda^3 + \lambda f_2 - c\lambda\}. \end{aligned}$$

Using (3.19) and (3.20), we have

$$(3.23) \quad g_2 = \lambda T' + \lambda'T - 2\lambda'' + 2\lambda^3 - 2c\lambda.$$

Multiplying  $(\omega_{ii}^1)^2$  on both sides of (2.14), we have

$$\frac{1}{3}e_1\left((\omega_{ii}^1)^3\right) = (\omega_{ii}^1)^4 + \lambda\lambda_i(\omega_{ii}^1)^2 + c(\omega_{ii}^1)^2.$$

Taking the sum of  $i$  from 2 to 4, we have

$$(3.24) \quad \begin{aligned} f_4 &= \frac{1}{3}e_1(f_3) - \lambda g_2 - cf_2 \\ &= \frac{1}{6}T''' - \frac{4}{3}(\lambda^2 + c)T' - \frac{5}{3}\lambda\lambda'T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4 - c\lambda^2 + 3c^2. \end{aligned}$$

Multiplying  $\lambda_i$  on both sides of (2.13), we have

$$\lambda_i^2 \omega_{ii}^1 = \frac{1}{2}e_1(\lambda_i^2) + \lambda\lambda_i \omega_{ii}^1.$$

Using (3.10), we have

$$\begin{aligned}
(3.25) \quad g_3 &:= \sum_{i=2}^4 \lambda_i^2 \omega_{ii}^1 = \frac{1}{2} e_1 (S - \lambda^2) + \lambda g_1 \\
&= \frac{1}{2} \left( \frac{\lambda'' - \lambda' T}{\lambda} - \lambda^2 \right)' + \lambda g_1 \\
&= -\frac{\lambda'}{2\lambda} T' + \left( \lambda^2 - \frac{\lambda'' \lambda - \lambda'^2}{2\lambda^2} \right) T - 4\lambda \lambda' + \frac{\lambda''' \lambda - \lambda'' \lambda'}{2\lambda^2}.
\end{aligned}$$

Differentiating (3.23) with respect to  $e_1$  and using (2.13) and (2.14), we have

$$e_1(g_2) = 3 \sum_{i=2}^4 \lambda_i (\omega_{ii}^1)^3 - \lambda \sum_{i=2}^4 (\omega_{ii}^1)^3 + 2\lambda \sum_{i=2}^4 \lambda_i^2 \omega_{ii}^1 + 2c \sum_{i=2}^4 \lambda_i \omega_{ii}^1,$$

that is

$$\begin{aligned}
(3.26) \quad g_4 &:= \sum_{i=2}^4 \lambda_i (\omega_{ii}^1)^3 \\
&= \frac{1}{3} (e_1(g_2) + \lambda f_3 - 2\lambda g_3 - 2c g_1) \\
&= \frac{1}{2} \lambda T'' + \lambda' T' + \frac{1}{3} \left( 2\lambda'' - 3\lambda^3 - \frac{\lambda'^2}{\lambda} - 3c\lambda \right) T \\
&\quad - \lambda''' + \frac{20}{3} \lambda^2 \lambda' + \frac{\lambda'' \lambda'}{3\lambda} + \frac{4}{3} c \lambda'.
\end{aligned}$$

Multiplying  $(\omega_{ii}^1)^3$  on both sides of (2.14), we have

$$\frac{1}{4} e_1 \left( (\omega_{ii}^1)^4 \right) = (\omega_{ii}^1)^5 + \lambda \lambda_i (\omega_{ii}^1)^3 + c (\omega_{ii}^1)^3.$$

Taking the sum of  $i$  from 2 to 4, we have

$$\begin{aligned}
(3.27) \quad f_5 &= \frac{1}{4} e_1(f_4) - \lambda g_4 - c f_3 \\
&= \frac{1}{24} T'''' - \frac{5}{6} (\lambda^2 + c) T'' + 2\lambda \lambda''' + \frac{5}{3} \lambda' \lambda'' - \frac{26}{3} \lambda^3 \lambda' - \frac{47}{6} c \lambda \lambda' \\
&\quad - \frac{25}{12} \lambda \lambda' T' - \frac{1}{12} (13\lambda \lambda'' + \lambda'^2 - 12\lambda^4 - 24c\lambda^2 - 12c^2) T.
\end{aligned}$$

□

Following the argument of Fu-Hong-Zhan in [FHZ], we can prove

**Lemma 3.4.** *The function  $T$  satisfies  $e_i(T) = 0$  for  $i = 2, 3, 4$ .*

*Proof.* Assume that  $T \neq 0$ . Substituting (3.17) into (3.13) and (3.14), we have

$$\begin{aligned}
(3.28) \quad &9c^2 + 10cT^2 + T^4 - 48c\lambda^2 - 26T^2\lambda^2 + 39\lambda^4 \\
&- 10cT' - 6T^2T' + 26\lambda^2T' + 3T'^2 + 58T\lambda\lambda' - 12\lambda'^2 \\
&+ 4TT'' - 24\lambda\lambda'' - T''' = 0,
\end{aligned}$$



$$\begin{aligned}
(3.29) \quad & 36c^2T + 40cT^3 + 4T^5 - 48cT\lambda^2 - 80T^3\lambda^2 - 84T\lambda^4 \\
& -20cTT' - 20T^3T' - 20T\lambda^2T' - 172c\lambda\lambda' + 120T^2\lambda\lambda' + 568\lambda^3\lambda' \\
& +170\lambda T'\lambda' + 2T\lambda'^2 - 10cT'' + 10T^2T'' + 50\lambda^2T'' + 10T'T'' \\
& +26T\lambda\lambda'' - 40\lambda'\lambda'' - 48\lambda\lambda''' - T'''' = 0.
\end{aligned}$$

Differentiating (3.28) with respect to  $e_1$ , we have

$$\begin{aligned}
(3.30) \quad & 20cTT' + 4T^3T' - 52T\lambda^2T' - 12TT'^2 - 96c\lambda\lambda' - 52T^2\lambda\lambda' \\
& +156\lambda^3\lambda' + 110\lambda T'\lambda' + 58T\lambda'^2 - 10cT'' - 6T^2T'' + 26\lambda^2T'' \\
& +10T'T'' + 58T\lambda\lambda'' - 48\lambda'\lambda'' + 4TT''' - 24\lambda\lambda''' - T'''' = 0.
\end{aligned}$$

From (3.29) and (3.30), eliminating  $T''''$  we have

$$\begin{aligned}
(3.31) \quad & -36c^2T - 40cT^3 - 4T^5 + 48cT\lambda^2 + 80T^3\lambda^2 + 84T\lambda^4 + 40cTT' \\
& +24T^3T' - 32T\lambda^2T' - 12TT'^2 + 76c\lambda\lambda' - 172T^2\lambda\lambda' - 412\lambda^3\lambda' \\
& -60\lambda T'\lambda' + 56T\lambda'^2 - 16T^2T'' - 24\lambda^2T'' + 32T\lambda\lambda'' - 8\lambda'\lambda'' \\
& +4TT''' + 24\lambda\lambda''' = 0.
\end{aligned}$$

From (3.28) and (3.31), eliminating  $T''''$  we have

$$\begin{aligned}
(3.32) \quad & -36cT\lambda^2 - 6T^3\lambda^2 + 60T\lambda^4 + 18T\lambda^2T' + 19c\lambda\lambda' + 15T^2\lambda\lambda' \\
& -103\lambda^3\lambda' - 15\lambda T'\lambda' + 2T\lambda'^2 - 6\lambda^2T'' - 16T\lambda\lambda'' - 2\lambda'\lambda'' + 6\lambda\lambda''' = 0.
\end{aligned}$$

Differentiating (3.32) with respect to  $e_1$ , we have

$$\begin{aligned}
(3.33) \quad & -36c\lambda^2T' - 18T^2\lambda^2T' + 60\lambda^4T' + 18\lambda^2T'^2 - 72cT\lambda\lambda' - 12T^3\lambda\lambda' \\
& +240T\lambda^3\lambda' + 66T\lambda T'\lambda' + 19c\lambda'^2 + 15T^2\lambda'^2 - 309\lambda^2\lambda'^2 - 13T'\lambda'^2 \\
& +18T\lambda^2T'' - 27\lambda\lambda'T'' + 19c\lambda\lambda'' + 15T^2\lambda\lambda'' - 103\lambda^3\lambda'' - 31\lambda T'\lambda'' \\
& -12T\lambda'\lambda'' - 2\lambda''^2 - 6\lambda^2T''' - 16T\lambda\lambda''' + 4\lambda'\lambda''' + 6\lambda\lambda'''' = 0.
\end{aligned}$$

From (3.28) and (3.33), eliminating  $T''''$  we have

$$\begin{aligned}
(3.34) \quad & 54c^2\lambda^2 + 60cT^2\lambda^2 + 6T^4\lambda^2 - 288c\lambda^4 - 156T^2\lambda^4 + 234\lambda^6 \\
& -24c\lambda^2T' - 18T^2\lambda^2T' + 96\lambda^4T' + 72cT\lambda\lambda' + 12T^3\lambda\lambda' + 108T\lambda^3\lambda' \\
& -66T\lambda T'\lambda' - 19c\lambda'^2 - 15T^2\lambda'^2 + 237\lambda^2\lambda'^2 + 13T'\lambda'^2 + 6T\lambda^2T'' \\
& +27\lambda\lambda'T'' - 19c\lambda\lambda'' - 15T^2\lambda\lambda'' - 41\lambda^3\lambda'' + 31\lambda T'\lambda'' + 12T\lambda'\lambda'' \\
& +2\lambda''^2 + 16T\lambda\lambda''' - 4\lambda'\lambda''' - 6\lambda\lambda'''' = 0.
\end{aligned}$$

From (3.32) and (3.34), eliminating  $T''''$  we have

$$(3.35) \quad a_1T' - a_1T^2 + a_2T + a_3 = 0,$$

where

$$\begin{aligned}
a_1 &= 62\lambda^2\lambda'' - 109\lambda\lambda'^2 + 192\lambda^5 - 48c\lambda^3, \\
a_2 &= 44\lambda^2\lambda''' - 124\lambda\lambda'\lambda'' + 550\lambda^4\lambda' + 18\lambda'^3 - 142c\lambda^2\lambda', \\
a_3 &= -12\lambda^2\lambda'''' + 46\lambda\lambda'\lambda''' - 82\lambda^4\lambda'' + 4\lambda\lambda''^2 \\
&\quad - 18\lambda'^2\lambda'' - 453\lambda^3\lambda'^2 + 468\lambda^7 + 108c^2\lambda^3 - 576c\lambda^5 + 133c\lambda\lambda'^2 - 38c\lambda^2\lambda''.
\end{aligned}$$

**Case (i):**  $a_1 = 0, a_2 \neq 0$ . (3.35) becomes

$$a_2T + a_3 = 0.$$

so we have  $T = -\frac{a_3}{a_2}$ , then  $e_i(T) = 0$  for  $i = 2, 3, 4$ .

**Case (ii):**  $a_1 = 0, a_2 = 0$ . We have

$$(3.36) \quad 62\lambda\lambda'' - 109\lambda'^2 + 192\lambda^4 - 48c\lambda^2 = 0,$$

$$(3.37) \quad 44\lambda^2\lambda''' - 124\lambda\lambda'\lambda'' + 550\lambda^4\lambda' + 18\lambda'^3 - 142c\lambda^2\lambda' = 0.$$

Differentiating (3.36) with respect to  $e_1$ , from (3.37) we have

$$(3.38) \quad -1145c\lambda^2 + 77\lambda^4 + 279\lambda'^2 - 206\lambda\lambda'' = 0.$$

From (3.36) and (3.38), eliminating  $\lambda''$  we have

$$(3.39) \quad -40439c\lambda^2 + 22163\lambda^4 - 2578\lambda'^2 = 0.$$

Differentiating (3.39) with respect to  $e_1$ , we have

$$(3.40) \quad -40439c\lambda + 44326\lambda^3 - 2578\lambda'' = 0.$$

Substituting (3.39) and (3.40) into (3.36), we have

$$1776889c + 827421\lambda^2 = 0,$$

which is a contradiction.

**Case (iii):**  $a_1 \neq 0$ . Differentiating (3.35) with respect to  $e_1$ , we have

$$(3.41) \quad \begin{aligned} & 96cT\lambda^3T' - 384T\lambda^5T' + 324c^2\lambda^2\lambda' + 144cT^2\lambda^2\lambda' - 2880c\lambda^4\lambda' \\ & - 960T^2\lambda^4\lambda' + 3276\lambda^6\lambda' - 286c\lambda^2T'\lambda' + 1510\lambda^4T'\lambda' - 284cT\lambda\lambda'^2 \\ & + 2200T\lambda^3\lambda'^2 + 218T\lambda T'\lambda'^2 + 133c\lambda'^3 + 109T^2\lambda'^3 - 1359\lambda^2\lambda'^3 \\ & - 91T'\lambda'^3 - 48c\lambda^3T'' + 192\lambda^5T'' - 109\lambda\lambda'^2T'' - 142cT\lambda^2\lambda'' \\ & + 550T\lambda^4\lambda'' - 124T\lambda^2T'\lambda'' + 190c\lambda\lambda'\lambda'' + 94T^2\lambda\lambda'\lambda'' - 1234\lambda^3\lambda'\lambda'' \\ & - 218\lambda T'\lambda'\lambda'' - 70T\lambda'^2\lambda'' + 62\lambda^2T''\lambda'' - 124T\lambda\lambda''^2 - 32\lambda'\lambda''^2 \\ & - 38c\lambda^2\lambda''' - 62T^2\lambda^2\lambda''' - 82\lambda^4\lambda''' + 106\lambda^2T'\lambda''' - 36T\lambda\lambda'\lambda''' \\ & + 28\lambda'^2\lambda''' + 54\lambda\lambda''\lambda''' + 44T\lambda^2\lambda'''' + 22\lambda\lambda'\lambda'''' - 12\lambda^2\lambda'''' = 0. \end{aligned}$$

From (3.32) and (3.41), eliminating  $T''$  we have

$$(3.42) \quad \begin{aligned} & 1728c^2T\lambda^4 + 288cT^3\lambda^4 - 9792cT\lambda^6 - 1152T^3\lambda^6 \\ & + 11520T\lambda^8 - 288cT\lambda^4T' + 1152T\lambda^6T' + 1032c^2\lambda^3\lambda' \\ & + 144cT^2\lambda^3\lambda' - 8688c\lambda^5\lambda' - 2880T^2\lambda^5\lambda' - 120\lambda^7\lambda' \\ & - 996c\lambda^3T'\lambda' + 6180\lambda^5T'\lambda' + 2124cT\lambda^2\lambda'^2 + 654T^3\lambda^2\lambda'^2 \\ & + 7044T\lambda^4\lambda'^2 - 654T\lambda^2T'\lambda'^2 - 1273c\lambda\lambda'^3 - 981T^2\lambda\lambda'^3 \\ & + 3073\lambda^3\lambda'^3 + 1089\lambda T'\lambda'^3 - 218T\lambda^4 - 2316cT\lambda^3\lambda'' \\ & - 372T^3\lambda^3\lambda'' + 3948T\lambda^5\lambda'' + 372T\lambda^3T'\lambda'' + 2414c\lambda^2\lambda'\lambda'' \\ & + 1494T^2\lambda^2\lambda'\lambda'' - 14174\lambda^4\lambda'\lambda'' - 2238\lambda^2T'\lambda'\lambda'' + 1448T\lambda\lambda'^2\lambda'' \\ & + 218\lambda'^3\lambda'' - 1736T\lambda^2\lambda''^2 - 316\lambda\lambda'\lambda''^2 - 516c\lambda^3\lambda''' \\ & - 372T^2\lambda^3\lambda''' + 660\lambda^5\lambda''' + 636\lambda^3T'\lambda''' - 216T\lambda^2\lambda'\lambda''' \\ & - 486\lambda\lambda'^2\lambda''' + 696\lambda^2\lambda''\lambda''' + 264T\lambda^3\lambda'''' + 132\lambda^2\lambda'\lambda'''' - 72\lambda^3\lambda'''' = 0. \end{aligned}$$

From (3.35) and (3.42), eliminating  $T'$  we have

$$(3.43) \quad b_1 T + b_2 = 0,$$

where

$$\begin{aligned} b_1 = & -6480c^3\lambda^6 + 63936c^2\lambda^8 - 204336c\lambda^{10} + 209088\lambda^{12} \\ & -40350c^2\lambda^4\lambda'^2 + 237750c\lambda^6\lambda'^2 - 309288\lambda^8\lambda'^2 + 4812c\lambda^2\lambda'^4 \\ & -227013\lambda^4\lambda'^4 + 520\lambda'^6 + 20898c^2\lambda^5\lambda'' - 125856c\lambda^7\lambda'' \\ & +174078\lambda^9\lambda'' - 25773c\lambda^3\lambda'^2\lambda'' + 302157\lambda^5\lambda'^2\lambda'' - 975\lambda\lambda'^4\lambda'' \\ & -5622c\lambda^4\lambda'^2 - 7830\lambda^6\lambda''^2 + 1350\lambda^2\lambda'^2\lambda''^2 - 13640\lambda^3\lambda''^3 \\ & +19719c\lambda^4\lambda'\lambda''' - 89523\lambda^6\lambda'\lambda''' - 717\lambda^2\lambda'^3\lambda''' + 18354\lambda^3\lambda'\lambda''\lambda''' \\ & -3498\lambda^4\lambda''^2 - 2016c\lambda^5\lambda''^3 + 8064\lambda^7\lambda''^3 - 4578\lambda^3\lambda'^2\lambda''^3 \\ & +2604\lambda^4\lambda''\lambda''^3, \\ b_2 = & 7254c^3\lambda^5\lambda' - 78246c^2\lambda^7\lambda' + 295434c\lambda^9\lambda' - 364410\lambda^{11}\lambda' \\ & -4566c^2\lambda^3\lambda'^3 - 11349c\lambda^5\lambda'^3 + 361623\lambda^7\lambda'^3 - 760c\lambda\lambda'^5 \\ & +19795\lambda^3\lambda'^5 + 18996c^2\lambda^4\lambda'\lambda'' - 66342c\lambda^6\lambda'\lambda'' - 146838\lambda^8\lambda'\lambda'' \\ & -3926c\lambda^2\lambda'^3\lambda'' + 120509\lambda^4\lambda'^3\lambda'' - 520\lambda'^5\lambda'' + 10472c\lambda^3\lambda'\lambda''^2 \\ & -143462\lambda^5\lambda'\lambda''^2 + 415\lambda\lambda'^3\lambda''^2 - 1330\lambda^2\lambda'\lambda''^3 - 5490c^2\lambda^5\lambda''' \\ & +29448c\lambda^7\lambda''' - 21366\lambda^9\lambda''' + 5100c\lambda^3\lambda'^2\lambda''' - 20178\lambda^5\lambda'^2\lambda''' \\ & +360\lambda\lambda'^4\lambda''' - 5154c\lambda^4\lambda''\lambda''' + 28338\lambda^6\lambda''\lambda''' + 1050\lambda^2\lambda'^2\lambda''\lambda''' \\ & +5076\lambda^3\lambda''^2\lambda''' - 3657\lambda^3\lambda'\lambda''^2 - 2286c\lambda^4\lambda'\lambda''^3 + 12438\lambda^6\lambda'\lambda''^3 \\ & -165\lambda^2\lambda'^3\lambda''^3 - 2334\lambda^3\lambda'\lambda''\lambda''^3 + 954\lambda^4\lambda''\lambda''^3 + 432c\lambda^5\lambda''^3 \\ & -1728\lambda^7\lambda''^3 + 981\lambda^3\lambda'^2\lambda''^3 - 558\lambda^4\lambda''\lambda''^3. \end{aligned}$$

If  $b_1 \neq 0$ , then  $T = -\frac{b_2}{b_1}$ , and  $e_i(T) = 0$  for  $i = 2, 3, 4$ . If  $b_1 = 0$ , then  $b_2 = 0$ . Using the similar technique in [FHZ] and (3.15), (3.16), we can eliminate  $\lambda''''$ ,  $\lambda'''$ ,  $\lambda''$ ,  $\lambda'$  and get a nontrivial polynomial of  $\lambda$  with constant coefficients. Thus  $\lambda$  is a constant, which is a contradiction.  $\square$

Following the argument of Fu-Hong-Zhan in [FHZ], we can get

**Lemma 3.5.** *Suppose  $M^4$  has four distinct principal curvatures, then  $e_j(\lambda_i) = 0$  for  $2 \leq i, j \leq 4$ .*

Moreover, from (3.8) we have  $\omega_{ii}^j = 0$  for  $2 \leq i, j \leq 4$ .

#### 4. PROOF OF THEOREM 1.1

We need the following two lemmas proved by Fu and Hong in [FH].

**Lemma 4.1.** ([FH])

$$(4.1) \quad \omega_{23}^4(\lambda_3 - \lambda_4) = \omega_{32}^4(\lambda_2 - \lambda_4) = \omega_{43}^2(\lambda_3 - \lambda_2),$$

$$(4.2) \quad \omega_{23}^4\omega_{32}^4 + \omega_{34}^2\omega_{43}^2 + \omega_{24}^3\omega_{42}^3 = 0,$$

$$(4.3) \quad \omega_{23}^4 (\omega_{33}^1 - \omega_{44}^1) = \omega_{32}^4 (\omega_{22}^1 - \omega_{44}^1) = \omega_{43}^2 (\omega_{33}^1 - \omega_{22}^1).$$

**Lemma 4.2.** (*[FH]*)

$$(4.4) \quad \omega_{22}^1 \omega_{33}^1 - 2\omega_{23}^4 \omega_{32}^4 = -\lambda_2 \lambda_3 - c,$$

$$(4.5) \quad \omega_{22}^1 \omega_{44}^1 - 2\omega_{24}^3 \omega_{42}^3 = -\lambda_2 \lambda_4 - c,$$

$$(4.6) \quad \omega_{33}^1 \omega_{44}^1 - 2\omega_{34}^2 \omega_{43}^2 = -\lambda_3 \lambda_4 - c.$$

**The proof of Theorem 1.1:**

Let  $M^4$  be a biharmonic hypersurface in  $\mathbb{R}^5(c)$  ( $c \neq 0$ ). There exists a smooth function  $a$  such that

$$(4.7) \quad \omega_{23}^4 = a(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4).$$

From (4.1) and (4.7) we have

$$(4.8) \quad \omega_{34}^2 = a(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_2),$$

$$(4.9) \quad \omega_{42}^3 = a(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3).$$

Taking  $X = e_1, Y = e_2, Z = e_3$  in Gauss equation (2.4), we have

$$(4.10) \quad \begin{aligned} e_1(a) = & -\frac{a}{3(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \{ (5\lambda_2^2 + \lambda_2\lambda_3 - \lambda_3^2 + \lambda_2\lambda_4 - 5\lambda_3\lambda_4 - \lambda_4^2)\omega_{22}^1 \\ & + (-\lambda_2^2 - 4\lambda_2\lambda_3 + 4\lambda_3\lambda_4 + \lambda_4^2)\omega_{33}^1 \\ & + (-\lambda_2^2 + \lambda_3^2 - 4\lambda_2\lambda_4 + 4\lambda_3\lambda_4)\omega_{44}^1 \}. \end{aligned}$$

Taking  $X = e_1, Y = e_3, Z = e_2$  in Gauss equation (2.4), we have

$$(4.11) \quad \begin{aligned} e_1(a) = & -\frac{a}{3(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)} \{ (4\lambda_2\lambda_3 + \lambda_3^2 - 4\lambda_2\lambda_4 - \lambda_4^2)\omega_{22}^1 \\ & + (\lambda_2^2 - \lambda_2\lambda_3 - 5\lambda_3^2 + 5\lambda_2\lambda_4 - \lambda_3\lambda_4 + \lambda_4^2)\omega_{33}^1 \\ & + (-\lambda_2^2 + \lambda_3^2 - 4\lambda_2\lambda_4 + 4\lambda_3\lambda_4)\omega_{44}^1 \}. \end{aligned}$$

Taking  $X = e_1, Y = e_4, Z = e_2$  in Gauss equation (2.4), we have

$$(4.12) \quad \begin{aligned} e_1(a) = & -\frac{a}{3(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \{ (4\lambda_2\lambda_3 + \lambda_3^2 - 4\lambda_2\lambda_4 - \lambda_4^2)\omega_{22}^1 \\ & + (\lambda_2^2 + 4\lambda_2\lambda_3 - 4\lambda_3\lambda_4 - \lambda_4^2)\omega_{33}^1 \\ & + (-\lambda_2^2 - 5\lambda_2\lambda_3 - \lambda_3^2 + \lambda_2\lambda_4 + \lambda_3\lambda_4 + 5\lambda_4^2)\omega_{44}^1 \}. \end{aligned}$$

(4.10)+(4.11)+(4.12) implies

$$(4.13) \quad e_1(a) = \frac{a}{9(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} (k_2\omega_{22}^1 - k_3\omega_{33}^1 + k_4\omega_{44}^1),$$

where

$$k_2 = (\lambda_3 - \lambda_4)(-13\lambda_2^2 + 2\lambda_3^2 + 7\lambda_3\lambda_4 + 2\lambda_4^2 + \lambda_2\lambda_3 + \lambda_2\lambda_4),$$

$$k_3 = (\lambda_2 - \lambda_4)(2\lambda_2^2 - 13\lambda_3^2 + \lambda_3\lambda_4 + 2\lambda_4^2 + \lambda_2\lambda_3 + 7\lambda_2\lambda_4),$$

$$k_4 = (\lambda_2 - \lambda_3)(2\lambda_2^2 + 7\lambda_2\lambda_3 + 2\lambda_3^2 + \lambda_2\lambda_4 + \lambda_3\lambda_4 - 13\lambda_4^2).$$

Taking  $X = e_2, Y = e_4, Z = e_1$  in Gauss equation (2.4), we have

$$(4.14) \quad a\{(\lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1\} = 0.$$

We can rewrite the biharmonic equation (2.7) as

$$(4.15) \quad -e_1 e_1(\lambda_1) + e_1(\lambda_1)(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) + \lambda_1(8c + 4\lambda_1^2 - R) = 0.$$

Taking  $X = e_2, Y = e_4, Z = e_2$  in Gauss equation (2.4), we have  $e_2(a) = 0$  and

$$(4.16) \quad \omega_{22}^1 \omega_{44}^1 + 2a^2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)^2(\lambda_3 - \lambda_4) + \lambda_2 \lambda_4 + c = 0.$$

By symmetry, we have  $e_3(a) = e_4(a) = 0$  and

$$(4.17) \quad \omega_{33}^1 \omega_{44}^1 - 2a^2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)^2 + \lambda_3 \lambda_4 + c = 0,$$

$$(4.18) \quad \omega_{22}^1 \omega_{33}^1 + \lambda_2 \lambda_3 - 2a^2(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4) + c = 0.$$

Here we introduce the new variables  $y_1, y_2, y_3$  by

$$(4.19) \quad y_1 = \lambda_2 + \lambda_3 + \lambda_4 = -3\lambda_1,$$

$$(4.20) \quad y_2 = \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4,$$

$$(4.21) \quad y_3 = \lambda_2 \lambda_3 \lambda_4.$$

Next we consider two cases.

**Case A:**  $a \neq 0$ . From (4.14) we have

$$(4.22) \quad (\lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1 = 0.$$

From (4.1) and (4.3), there exists smooth functions  $\kappa$  and  $\tau$  such that

$$(4.23) \quad \omega_{ii}^1 = \kappa \lambda_i + \tau, \quad i = 2, 3, 4.$$

From (2.13) and (2.14), we have

$$(4.24) \quad e_1(\kappa) = -\frac{1}{3}(1 + \kappa^2)(\lambda_2 + \lambda_3 + \lambda_4) + \kappa\tau,$$

$$(4.25) \quad e_1(\tau) = c - \frac{1}{3}\kappa\tau(\lambda_2 + \lambda_3 + \lambda_4) + \tau^2.$$

(4.16)+(4.17)+(4.18) implies

$$(4.26) \quad 3c + (1 + \kappa^2)y_2 + 2\kappa\tau y_1 + 3\tau^2 = 0.$$

From (4.26) we can solve for  $y_2$

$$(4.27) \quad y_2 = -\frac{1}{1 + \kappa^2}(3c + 2\kappa\tau y_1 + 3\tau^2).$$

From (2.13), (2.14), (4.23), we have

$$(4.28) \quad e_1(y_1) = \frac{4}{3}\kappa y_1^2 - 2\kappa y_2 + 2\tau y_1.$$

From (4.24), (4.25), (4.27), (4.28), we have

$$(4.29) \quad e_1(y_1^2 - 2y_2) = \frac{4}{3(1 + \kappa^2)^2}(2y_1^3 \kappa(1 + \kappa^2)^2 + 15c y_1(\kappa + \kappa^3) + 9c(1 + 2\kappa^2)\tau + y_1^2(1 + \kappa^2)(2 + 13\kappa^2)\tau + 3y_1 \kappa(7 + 9\kappa^2)\tau^2 + 9(1 + 2\kappa^2)\tau^3).$$

From (2.13), (2.14), (4.23), we have

$$(4.30) \quad e_1(\lambda_2^2 + \lambda_3^2 + \lambda_4^2) = \frac{2}{3}(4y_1^3 \kappa - 11y_1 y_2 \kappa + 9y_3 \kappa + 4y_1^2 \tau - 6y_2 \tau).$$

From (4.29) and (4.30), we have

$$(4.31) \quad -cy_1 - 3y_3 - cy_1\kappa^2 - 6y_3\kappa^2 - 3y_3\kappa^4 + 6c\kappa\tau - y_1\tau^2 + 3y_1\kappa^2\tau^2 + 6\kappa\tau^3 = 0.$$

From (4.31), we can solve for  $y_3$

$$(4.32) \quad y_3 = \frac{1}{3(1+\kappa^2)^2}(-c(1+\kappa^2)y_1 + 6c\kappa\tau + (3\kappa^2 - 1)\tau^2 y_1 + 6\kappa\tau^3).$$

Using (4.23), (4.15) can be rewritten as

$$(4.33) \quad 27cy_1 - 7y_1^3 + 12y_1y_2 + 8y_1^3\kappa^2 - 27y_1y_2\kappa^2 + 27y_3\kappa^2 + 6y_1^2\kappa\tau - 18y_2\kappa\tau = 0.$$

Differentiating (4.33) with respect to  $e_1$ , we have

$$(4.34) \quad \begin{aligned} & -684cy_1^2\kappa - 100y_1^4\kappa + 2106c^2\kappa^3 + 324cy_1^2\kappa^3 \\ & -120y_1^4\kappa^3 + 1008cy_1^2\kappa^5 + 60y_1^4\kappa^5 + 80y_1^4\kappa^7 \\ & -324cy_1\tau - 108y_1^3\tau + 648cy_1\kappa^2\tau - 618y_1^3\kappa^2\tau \\ & +4806cy_1\kappa^4\tau + 330y_1^3\kappa^4\tau + 840y_1^3\kappa^6\tau - 1062y_1^2\kappa\tau^2 \\ & +5346c\kappa^3\tau^2 + 558y_1^2\kappa^3\tau^2 + 3240y_1^2\kappa^5\tau^2 - 486y_1\tau^3 \\ & +324y_1\kappa^2\tau^3 + 5400y_1\kappa^4\tau^3 + 3240\kappa^3\tau^4 = 0. \end{aligned}$$

Differentiating (4.34) with respect to  $e_1$ , we have

$$(4.35) \quad \begin{aligned} & -972c^2y_1 + 360cy_1^3 + 100y_1^5 - 29970c^2y_1\kappa^2 \\ & -14454cy_1^3\kappa^2 - 1040y_1^5\kappa^2 + 15390c^2y_1\kappa^4 - 18684cy_1^3\kappa^4 \\ & -3000y_1^5\kappa^4 + 44388c^2y_1\kappa^6 + 7434cy_1^3\kappa^6 - 1760y_1^5\kappa^6 \\ & +11304cy_1^3\kappa^8 + 820y_1^5\kappa^8 + 720y_1^5\kappa^{10} - 5832c^2\kappa\tau \\ & -24732cy_1^2\kappa\tau - 2652y_1^4\kappa\tau + 62694c^2\kappa^3\tau - 77112cy_1^2\kappa^3\tau \\ & -18294y_1^4\kappa^3\tau + 137538c^2\kappa^5\tau + 59022cy_1^2\kappa^5\tau - 16992y_1^4\kappa^5\tau \\ & +111402cy_1^2\kappa^7\tau + 10290y_1^4\kappa^7\tau + 11640y_1^4\kappa^9\tau - 7290cy_1\tau^2 \\ & -1206y_1^3\tau^2 - 77436cy_1\kappa^2\tau^2 - 35964y_1^3\kappa^2\tau^2 + 158922cy_1\kappa^4\tau^2 \\ & -58194y_1^3\kappa^4\tau^2 + 333396cy_1\kappa^6\tau^2 + 48564y_1^3\kappa^6\tau^2 + 72000y_1^3\kappa^8\tau^2 \\ & -14580c\kappa\tau^3 - 29268y_1^2\kappa\tau^3 + 136566c\kappa^3\tau^3 - 89154y_1^2\kappa^3\tau^3 \\ & +302778c\kappa^5\tau^3 + 115074y_1^2\kappa^5\tau^3 + 213840y_1^2\kappa^7\tau^3 - 7290y_1\tau^4 \\ & -54270y_1\kappa^2\tau^4 + 144180y_1\kappa^4\tau^4 + 304560y_1\kappa^6\tau^4 - 8748\kappa\tau^5 \\ & +73872\kappa^3\tau^5 + 165240\kappa^5\tau^5 = 0. \end{aligned}$$

From (4.33) and (4.34), eliminating  $\tau$  we have

$$(4.36) \quad \sum_{m=0}^8 P_{2m}\kappa^{2m} = 0,$$

where

$$\begin{aligned}
P_0 &= -164025c^3y_1^4 - 149445c^2y_1^6 - 17739cy_1^8 - 567y_1^{10}, \\
P_2 &= -157464c^4y_1^2 + 988524c^3y_1^4 - 1879848c^2y_1^6 + 449388cy_1^8 - 18072y_1^{10}, \\
P_4 &= 6114852c^4y_1^2 + 8627715c^3y_1^4 - 4223745c^2y_1^6 + 79353cy_1^8 - 8223y_1^{10}, \\
P_6 &= 6141096c^5 - 46189440c^4y_1^2 + 84187836c^3y_1^4 - 36738684c^2y_1^6 \\
&\quad + 4146300cy_1^8 - 172964y_1^{10}, \\
P_8 &= 6141096c^5 - 223047756c^4y_1^2 + 137387340c^3y_1^4 - 83992140c^2y_1^6 \\
&\quad + 12535884cy_1^8 - 488104y_1^{10}, \\
P_{10} &= -35311302c^5 + 54154494c^4y_1^2 - 49487436c^3y_1^4 - 69538500c^2y_1^6 \\
&\quad + 11334546cy_1^8 - 413882y_1^{10}, \\
P_{12} &= -18423288c^5 + 1192107456c^4y_1^2 - 355203792c^3y_1^4 - 6360768c^2y_1^6 \\
&\quad + 3415752cy_1^8 - 121088y_1^{10}, \\
P_{14} &= 55269864c^5 + 1533226968c^4y_1^2 - 345382704c^3y_1^4 + 18635184c^2y_1^6 \\
&\quad + 46152cy_1^8 - 15496y_1^{10}, \\
P_{16} &= 587865600c^4y_1^2 - 105629184c^3y_1^4 + 5505408c^2y_1^6 - 36864cy_1^8 - 2432y_1^{10}.
\end{aligned}$$

From (4.33) and (4.35), eliminating  $\tau$  we have

$$(4.37) \quad \sum_{m=0}^{13} Q_{2m}\kappa^{2m} = 0,$$

where

$$\begin{aligned}
Q_0 &= 89813529c^4y_1^4 + 623321244c^3y_1^6 + 1108270998c^2y_1^8 \\
&\quad + 92935836cy_1^{10} + 1996569y_1^{12}, \\
Q_2 &= 7620155352c^5y_1^2 + 18958390038c^4y_1^4 - 2051878392c^3y_1^6 \\
&\quad + 18386833140c^2y_1^8 - 3132184464cy_1^{10} + 199117734y_1^{12}, \\
Q_4 &= 2550916800c^6 - 3713237316c^5y_1^2 + 396683144775c^4y_1^4 \\
&\quad - 281478327804c^3y_1^6 + 162873940914c^2y_1^8 - 23218046136cy_1^{10} \\
&\quad + 1170436927y_1^{12}, \\
Q_6 &= 21533989320c^6 - 253030869900c^5y_1^2 + 2983350761604c^4y_1^4 \\
&\quad - 2233514241576c^3y_1^6 + 897578511552c^2y_1^8 - 110846416140cy_1^{10} \\
&\quad + 5032123316y_1^{12}, \\
Q_8 &= 169801776792c^6 - 4722853102668c^5y_1^2 + 19054644126723c^4y_1^4 \\
&\quad - 11292218972532c^3y_1^6 + 3695744541258c^2y_1^8 - 455003760600cy_1^{10} \\
&\quad + 19351326643y_1^{12}, \\
Q_{10} &= 554978521890c^6 - 30549126042468c^5y_1^2 + 83834398712052c^4y_1^4 \\
&\quad - 42196006098576c^3y_1^6 + 11288209375170c^2y_1^8 - 1281849786108cy_1^{10} \\
&\quad + 48941467416y_1^{12},
\end{aligned}$$

$$\begin{aligned}
Q_{12} &= -310363669764c^6 - 57933464951484c^5y_1^2 + 211026987943857c^4y_1^4 \\
&\quad - 105270974689716c^3y_1^6 + 23637191018346c^2y_1^8 - 2258485653816cy_1^{10} \\
&\quad + 72974080657y_1^{12}, \\
Q_{14} &= -4681635955884c^6 + 135848247734196c^5y_1^2 + 275115121235820c^4y_1^4 \\
&\quad - 165192051148632c^3y_1^6 + 32239821679308c^2y_1^8 - 2470452821820cy_1^{10} \\
&\quad + 62750560596y_1^{12}, \\
Q_{16} &= -5901463112004c^6 + 844079357467764c^5y_1^2 + 106282529065260c^4y_1^4 \\
&\quad - 157860516846024c^3y_1^6 + 27762314926644c^2y_1^8 - 1657847338092cy_1^{10} \\
&\quad + 29261950884y_1^{12}, \\
Q_{18} &= 6145346701314c^6 + 1705659405062004c^5y_1^2 - 177158182581522c^4y_1^4 \\
&\quad - 87108572161368c^3y_1^6 + 14676006828366c^2y_1^8 - 656114701212cy_1^{10} \\
&\quad + 5566963010y_1^{12}, \\
Q_{20} &= 17073860098680c^6 + 1765021213832760c^5y_1^2 - 262032349848480c^4y_1^4 \\
&\quad - 25663081979952c^3y_1^6 + 4778135352216c^2y_1^8 - 147573429768cy_1^{10} \\
&\quad - 751835760y_1^{12}, \\
Q_{22} &= 9338536521864c^6 + 941182952949648c^5y_1^2 - 130448854754568c^4y_1^4 \\
&\quad - 5532982699104c^3y_1^6 + 1153495908792c^2y_1^8 - 23152890672cy_1^{10} \\
&\quad - 611376440y_1^{12}, \\
Q_{24} &= 205652969940096c^5y_1^2 - 16668899694720c^4y_1^4 - 2570855428224c^3y_1^6 \\
&\quad + 271365697152c^2y_1^8 - 3749437440cy_1^{10} - 144665088y_1^{12}, \\
Q_{26} &= 3762339840000c^4y_1^4 - 676026777600c^3y_1^6 + 35234611200c^2y_1^8 \\
&\quad - 235929600cy_1^{10} - 15564800y_1^{12}.
\end{aligned}$$

From (4.36), (4.37), eliminating  $\kappa$  we get a polynomial of  $y_1$  with constant coefficients of degree 428 and then  $y_1$  is a constant, which is a contradiction.

**Case B:**  $a = 0$ . (4.4)-(4.6) becomes

$$(4.38) \quad \omega_{22}^1 \omega_{33}^1 = -\lambda_2 \lambda_3 - c,$$

$$(4.39) \quad \omega_{22}^1 \omega_{44}^1 = -\lambda_2 \lambda_4 - c,$$

$$(4.40) \quad \omega_{33}^1 \omega_{44}^1 = -\lambda_3 \lambda_4 - c.$$

From the assumption  $c \neq 0$  and (4.38)-(4.40), we can get  $\omega_{ii}^1 \neq 0$  and  $\lambda_i \lambda_j + c \neq 0$  for  $2 \leq i, j \leq 4, i \neq j$ , then

$$(4.41) \quad \omega_{33}^1 = -\frac{\lambda_2 \lambda_3 + c}{\omega_{22}^1},$$

$$(4.42) \quad \omega_{44}^1 = -\frac{\lambda_2 \lambda_4 + c}{\omega_{22}^1},$$



$$(4.43) \quad (\omega_{22}^1)^2 = -\frac{(\lambda_2\lambda_3 + c)(\lambda_2\lambda_4 + c)}{\lambda_3\lambda_4 + c}.$$

From (2.13), (2.14), (4.41)-(4.43), we have

$$(4.44) \quad e_1(y_1) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(6c^2y_1 + 5cy_1y_2 - 9cy_3 + 4y_1^2y_3 - 6y_2y_3),$$

$$(4.45) \quad e_1(y_2) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(2c^2y_1^2 + 6c^2y_2 + cy_1^2y_2 + 6cy_2^2 - 3cy_1y_3 + 5y_1y_2y_3 - 9y_3^2),$$

$$(4.46) \quad e_1(y_3) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(c^2y_1y_2 + 9c^2y_3 + 2cy_1^2y_3 + 6cy_2y_3 + 6y_1y_3^2).$$

Using (4.41)-(4.43), (4.15) can be rewritten as

$$(4.47) \quad \begin{aligned} & 27c^4y_1 - 7c^3y_1^3 + 36c^3y_1y_2 - 7c^2y_1^3y_2 + 7c^2y_1y_2^2 + 27c^3y_3 \\ & + 27c^2y_1^2y_3 - 7cy_1^4y_3 + 45c^2y_2y_3 + cy_1^2y_2y_3 + 24cy_2^2y_3 + 54cy_1y_3^2 \\ & - 15y_1^3y_3^2 + 39y_1y_2y_3^2 - 27y_3^3 = 0. \end{aligned}$$

Differentiating (4.47) with respect to  $e_1$ , we have

$$(4.48) \quad \begin{aligned} & 162c^6y_1 - 54c^5y_1^3 - 14c^4y_1^5 + 594c^5y_1y_2 - 182c^4y_1^3y_2 - 14c^3y_1^5y_2 \\ & + 567c^4y_1y_2^2 - 132c^3y_1^3y_2^2 + 143c^3y_1y_2^3 + 900c^4y_1^2y_3 - 238c^3y_1^4y_3 - 14c^2y_1^6y_3 \\ & + 351c^4y_2y_3 + 1395c^3y_1^2y_2y_3 - 328c^2y_1^4y_2y_3 + 765c^3y_2^2y_3 + 420c^2y_1^2y_2^2y_3 + 390c^2y_3^2y_3^2 \\ & + 513c^3y_1y_3^2 + 444c^2y_1^3y_3^2 - 214cy_1^5y_3^2 + 1890c^2y_1y_2y_3^2 - 23cy_1^3y_2y_3^2 + 1269cy_1y_2^2y_3^2 \\ & - 1620c^2y_3^3 + 981cy_1^2y_3^3 - 360y_1^4y_3^3 - 1593cy_2y_3^3 + 1089y_1^2y_2y_3^3 - 234y_2^2y_3^3 - 837y_1y_3^4 = 0. \end{aligned}$$

Differentiating (4.48) with respect to  $e_1$ , we have

$$(4.49) \quad \begin{aligned} & 972c^8y_1 + 216c^7y_1^3 - 784c^6y_1^5 - 28c^5y_1^7 \\ & + 7938c^7y_1y_2 - 1416c^6y_1^3y_2 - 1802c^5y_1^5y_2 - 28c^4y_1^7y_2 \\ & + 17091c^6y_1y_2^2 - 4395c^5y_1^3y_2^2 - 1026c^4y_1^5y_2^2 + 13836c^5y_1y_2^3 \\ & - 2715c^4y_1^3y_2^3 + 3679c^4y_1y_2^4 - 1458c^7y_3 + 19926c^6y_1^2y_3 \\ & - 2736c^5y_1^4y_3 - 2000c^4y_1^6y_3 - 28c^3y_1^8y_3 - 1053c^6y_2y_3 \\ & + 65034c^5y_1^2y_2y_3 - 13479c^4y_1^4y_2y_3 - 2266c^3y_1^6y_2y_3 + 11610c^5y_2^2y_3 \\ & + 67221c^4y_1^2y_2^2y_3 - 11346c^3y_1^4y_2^2y_3 + 19611c^4y_2^3y_3 + 21341c^3y_1^2y_2^3y_3 \\ & + 8502c^3y_2^4y_3 - 10287c^5y_1y_3^2 + 40437c^4y_1^3y_3^2 - 11912c^3y_1^5y_3^2 \\ & - 1276c^2y_1^7y_3^2 + 13716c^4y_1y_2y_3^2 + 72372c^3y_1^3y_2y_3^2 - 17465c^2y_1^5y_2y_3^2 \\ & + 64944c^3y_1y_2^2y_3^2 + 28074c^2y_1^3y_2^2y_3^2 + 39249c^2y_1y_2^3y_3^2 - 51516c^4y_3^3 \\ & + 3348c^3y_1^2y_3^3 + 13011c^2y_1^4y_3^3 - 9008cy_1^6y_3^3 - 115587c^3y_2y_3^3 \\ & + 78183c^2y_1^2y_2y_3^3 - 304cy_1^4y_2y_3^3 - 80649c^2y_2^2y_3^3 + 68562cy_1^2y_2^2y_3^3 \\ & - 14634cy_2^3y_3^3 - 94203c^2y_1y_3^4 + 28710cy_1^3y_3^4 - 12240y_1^5y_3^4 \\ & - 113724cy_1y_2y_3^4 + 42399y_1^3y_2y_3^4 - 19620y_1y_2^2y_3^4 + 21870cy_3^5 \\ & - 33237y_1^2y_3^5 + 9234y_2y_3^5 = 0. \end{aligned}$$

From(4.47), (4.48), eliminating  $y_2$  we have

$$\begin{aligned}
& (cy_1 - 3y_3)(c^3 - cy_1y_3 - 2y_3^2)(961551c^7y_1^4 - 550638c^6y_1^6 \\
& + 92295c^5y_1^8 + 1078c^4y_1^{10} + 3241134c^6y_1^3y_3 \\
& - 3718062c^5y_1^5y_3 + 852336c^4y_1^7y_3 - 16548c^3y_1^9y_3 \\
& + 2071089c^5y_1^2y_3^2 - 10454265c^4y_1^4y_3^2 + 4078449c^3y_1^6y_3^2 \\
(4.50) \quad & - 228795c^2y_1^8y_3^2 - 4234032c^4y_1y_3^3 - 9157698c^3y_1^3y_3^3 \\
& + 9454374c^2y_1^5y_3^3 - 621684cy_1^7y_3^3 - 4185918c^3y_3^4 \\
& + 4435965c^2y_1^2y_3^4 + 10034766cy_1^4y_3^4 - 479115y_1^6y_3^4 \\
& + 7050888cy_1y_3^5 + 4106700y_1^3y_3^5 + 2217618y_3^6) = 0.
\end{aligned}$$

From(4.47), (4.49), eliminating  $y_2$  we have

$$\begin{aligned}
& (cy_1 - 3y_3)(c^3 - cy_1y_3 - 2y_3^2)(121234158c^{12}y_1^5 + 4502127582c^{11}y_1^7 \\
& - 4130694792c^{10}y_1^9 + 1312681986c^9y_1^{11} - 138793970c^8y_1^{13} \\
& - 1131900c^7y_1^{15} - 6493539798c^{11}y_1^4y_3 + 17508226059c^{10}y_1^6y_3 \\
& - 26579271393c^9y_1^8y_3 + 12168575541c^8y_1^{10}y_3 - 1901077395c^7y_1^{12}y_3 \\
& + 41618150c^6y_1^{14}y_3 - 19265759766c^{10}y_1^3y_3^2 + 4495598658c^9y_1^5y_3^2 \\
& - 42049903410c^8y_1^7y_3^2 + 42284538216c^7y_1^9y_3^2 - 10550910060c^6y_1^{11}y_3^2 \\
& + 560212114c^5y_1^{13}y_3^2 - 6970891914c^9y_1^2y_3^3 - 13811443002c^8y_1^4y_3^3 \\
& + 83680475085c^7y_1^6y_3^3 + 21298486113c^6y_1^8y_3^3 - 21213196659c^5y_1^{10}y_3^3 \\
(4.51) \quad & + 1830863937c^4y_1^{12}y_3^3 + 30101526828c^8y_1y_3^4 + 45869158224c^7y_1^3y_3^4 \\
& + 279229797288c^6y_1^5y_3^4 - 232982629074c^5y_1^7y_3^4 + 23242316040c^4y_1^9y_3^4 \\
& - 602724402c^3y_1^{11}y_3^4 + 23531498892c^7y_3^5 + 95793204717c^6y_1^2y_3^5 \\
& - 35556908661c^5y_1^4y_3^5 - 505869269463c^4y_1^6y_3^5 + 173198646333c^3y_1^8y_3^5 \\
& - 12359609514c^2y_1^{10}y_3^5 - 28067406876c^5y_1y_3^6 - 471693791502c^4y_1^3y_3^6 \\
& - 188063792496c^3y_1^5y_3^6 + 305129824146c^2y_1^7y_3^6 - 20136670848cy_1^9y_3^6 \\
& - 99047493522c^4y_3^7 - 128261179269c^3y_1^2y_3^7 + 309915643131c^2y_1^4y_3^7 \\
& + 254417953509cy_1^6y_3^7 - 9828590625y_1^8y_3^7 + 92615758344c^2y_1y_3^8 \\
& + 252963396576cy_1^3y_3^8 + 90957709080y_1^5y_3^8 \\
& + 45154416006cy_3^9 + 25402919166y_1^2y_3^9) = 0.
\end{aligned}$$

Next we check three subcases.

**Case B.1:**  $cy_1 - 3y_3 = 0$ . Substituting  $y_3 = \frac{cy_1}{3}$  into (4.47), (4.48) gives

$$(4.52) \quad (9c + 4y_1^2 + 9y_2)(12c - 3y_1^2 + 5y_2) = 0,$$

$$(4.53) \quad (9c + 4y_1^2 + 9y_2)(162c^2 + 171cy_1^2 - 94y_1^4 + 549cy_2 + 19y_1^2y_2 + 273y_2^2) = 0.$$

From (4.52), (4.53), we consider the two following subcases:

- (i)  $9c + 4y_1^2 + 9y_2 \neq 0$ , or
- (ii)  $9c + 4y_1^2 + 9y_2 = 0$ .

If (i) holds, by eliminating  $y_2$  we can get that  $y_1$  satisfies a polynomial with constant coefficients and  $y_1$  is a constant. If (ii) holds, taking  $9c + 4y_1^2 + 9y_2 = 0$  into (4.44) we get  $e_1(y_1) = 0$  and  $y_1$  is also a constant. So in both subcases we get that  $y_1$  is a constant, which is a contradiction.

**Case B.2:**  $c^3 - cy_1y_3 - 2y_3^2 = 0$ . Since  $c \neq 0$ , we solve for  $y_1$  in terms of  $y_3$ .

Substituting  $y_1 = \frac{c^3 - 2y_3^2}{cy_3}$  into (4.47) and (4.48), we have

$$(4.54) \quad (2c^3 + c^2y_2 - y_3^2)(7c^9 - 65c^6y_3^2 - 7c^5y_2y_3^2 + 89c^3y_3^4 - 10c^2y_2y_3^4 + 8y_3^6) = 0,$$

$$(4.55) \quad \begin{aligned} & (2c^3 + c^2y_2 - y_3^2)(14c^{15} + 106c^{12}y_3^2 + 132c^{11}y_2y_3^2 - 2004c^9y_3^4 \\ & - 1493c^8y_2y_3^4 - 143c^7y_2^2y_3^4 + 5885c^6y_3^6 + 2429c^5y_2y_3^6 \\ & - 104c^4y_2^2y_3^6 - 3842c^3y_3^8 - 68c^2y_2y_3^8 + 192y_3^{10}) = 0. \end{aligned}$$

From (4.54), (4.55), we consider the two following subcases:

(i)  $2c^3 + c^2y_2 - y_3^2 \neq 0$ , or

(ii)  $2c^3 + c^2y_2 - y_3^2 = 0$ .

If (i) holds, by eliminating  $y_2$  we can get that  $y_3$  satisfies a polynomial with constant coefficients and  $y_3$  is a constant. If (ii) holds, taking  $2c^3 + c^2y_2 - y_3^2 = 0$  into (4.46) we get  $e_1(y_3) = 0$  and  $y_3$  is also a constant. So in both subcases we get that  $y_3$  is a constant, then  $y_1$  is a constant, which is a contradiction.

**Case B.3:**  $(cy_1 - 3y_3)(c^3 - cy_1y_3 - 2y_3^2) \neq 0$ . From (4.50), (4.51), we have

$$(4.56) \quad \begin{aligned} & 961551c^7y_1^4 - 550638c^6y_1^6 + 92295c^5y_1^8 + 1078c^4y_1^{10} \\ & + 3241134c^6y_1^3y_3 - 3718062c^5y_1^5y_3 + 852336c^4y_1^7y_3 - 16548c^3y_1^9y_3 \\ & + 2071089c^5y_1^2y_3^2 - 10454265c^4y_1^4y_3^2 + 4078449c^3y_1^6y_3^2 - 228795c^2y_1^8y_3^2 \\ & - 4234032c^4y_1y_3^3 - 9157698c^3y_1^3y_3^3 + 9454374c^2y_1^5y_3^3 - 621684cy_1^7y_3^3 \\ & - 4185918c^3y_3^4 + 4435965c^2y_1^2y_3^4 + 10034766cy_1^4y_3^4 - 479115y_1^6y_3^4 \\ & + 7050888cy_1y_3^5 + 4106700y_1^3y_3^5 + 2217618y_3^6 = 0, \end{aligned}$$

$$(4.57) \quad \begin{aligned} & 121234158c^{12}y_1^5 + 4502127582c^{11}y_1^7 - 4130694792c^{10}y_1^9 \\ & + 1312681986c^9y_1^{11} - 138793970c^8y_1^{13} - 1131900c^7y_1^{15} \\ & - 6493539798c^{11}y_1^4y_3 + 17508226059c^{10}y_1^6y_3 - 26579271393c^9y_1^8y_3 \\ & + 12168575541c^8y_1^{10}y_3 - 1901077395c^7y_1^{12}y_3 + 41618150c^6y_1^{14}y_3 \\ & - 19265759766c^{10}y_1^3y_3^2 + 4495598658c^9y_1^5y_3^2 - 42049903410c^8y_1^7y_3^2 \\ & + 42284538216c^7y_1^9y_3^2 - 10550910060c^6y_1^{11}y_3^2 + 560212114c^5y_1^{13}y_3^2 \\ & - 6970891914c^9y_1^2y_3^3 - 13811443002c^8y_1^4y_3^3 + 83680475085c^7y_1^6y_3^3 \\ & + 21298486113c^6y_1^8y_3^3 - 21213196659c^5y_1^{10}y_3^3 + 1830863937c^4y_1^{12}y_3^3 \\ & + 30101526828c^8y_1y_3^4 + 45869158224c^7y_1^3y_3^4 + 279229797288c^6y_1^5y_3^4 \\ & - 232982629074c^5y_1^7y_3^4 + 23242316040c^4y_1^9y_3^4 - 602724402c^3y_1^{11}y_3^4 \\ & + 23531498892c^7y_3^5 + 95793204717c^6y_1^2y_3^5 - 35556908661c^5y_1^4y_3^5 \\ & - 505869269463c^4y_1^6y_3^5 + 173198646333c^3y_1^8y_3^5 - 12359609514c^2y_1^{10}y_3^5 \end{aligned}$$

$$\begin{aligned}
& -28067406876c^5y_1y_3^6 - 471693791502c^4y_1^3y_3^6 - 188063792496c^3y_1^5y_3^6 \\
& + 305129824146c^2y_1^7y_3^6 - 20136670848cy_1^9y_3^6 - 99047493522c^4y_3^7 \\
& - 128261179269c^3y_1^2y_3^7 + 309915643131c^2y_1^4y_3^7 + 254417953509cy_1^6y_3^7 \\
& - 9828590625y_1^8y_3^7 + 92615758344c^2y_1y_3^8 + 252963396576cy_1^3y_3^8 \\
& + 90957709080y_1^5y_3^8 + 45154416006cy_3^9 + 25402919166y_1^2y_3^9 = 0.
\end{aligned}$$

From (4.56) and (4.57), eliminating  $y_3$  we get a polynomial of  $y_1$  with constant coefficients of degree 118 and then  $y_1$  is a constant, that is,  $\lambda_1$  is constant, which is a contradiction. Therefore we complete the proof of Theorem 1.1.

## REFERENCES

- [BMO1] A. Balmus, S. Montaldo and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. 168 (2008), 201-220.
- [BMO2] A. Balmus, S. Montaldo and C. Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr. 283 (2010), no. 12, 1696-1705.
- [CMO1] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds of  $\mathbb{S}^3$* , Internat. J. Math. 12 (2001), no. 8, 867-876.
- [CMO2] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. 130 (2002), 109-123.
- [C1] B. -Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. 17 (1991) no. 2, 169-188.
- [C2] B. -Y. Chen, *Some open problems and conjectures on submanifolds of finite type: recent development*, Tamkang J. Math. 45:1 (2014), 87-108.
- [C3] B. -Y. Chen, *Total Mean curvature and submanifolds of finite type*, 2nd Edition, World Scientific, Hackensack, NJ, 2015.
- [CM] B. -Y. Chen and M. I. Munteanu, *Biharmonic ideal hypersurfaces in Euclidean spaces*, Differential Geom. Appl. 31 (2013), 1-16.
- [ES] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86 (1964), 109-160.
- [F1] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space*, Tohoku Math. J. 67 (2015), 465-479.
- [F2] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in spheres*, Math. Nachr. 288 (2015) 7, 763-774.
- [F3] Y. Fu, *On biharmonic hypersurfaces with constant scalar curvature in  $\mathbb{S}^5$* , Proc. Amer. Math. Soc. 143 (2015), 5399-5409.
- [FH] Y. Fu and M. C. Hong, *Biharmonic hypersurfaces with constant scalar curvature in space forms*, Pacific J. Math. 294 (2) (2018), 329-350.
- [FHZ] Y. Fu, M. C. Hong and X. Zhan, *On Chen's biharmonic conjecture for hypersurfaces in  $\mathbb{R}^5$* , arXiv: 2006.07612v2.
- [HV] T. Hasanis and T. Vlachos, *Hypersurfaces in  $\mathbb{E}^4$  with harmonic mean curvature vector field*, Math. Nachr. 172 (1995), 145-169.
- [J1] G. Y. Jiang, *2-Harmonic maps and their first and second variational formulas*, Chin. Ann. Math. Ser. A 7 (1986), 389-402.
- [J2] G. Y. Jiang, *Some non-existence theorems of 2-harmonic isometric immersions into Euclidean spaces*, Chin. Ann. Math. Ser. A 8 (1987), 376-383.
- [OT] Y.-L. Ou and L. Tang, *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J. 61 (2012), 531-542.
- [OC] Y.-L. Ou and B. -Y. Chen, *Biharmonic submanifolds and biharmonic maps in Riemannian geometry*, World Scientific Publishing, Hackensack, NJ, 2020.

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