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Cascade Takagi-Sugeno Fuzzy Observer Design for Nonlinear Uncertain Systems with Unknown Inputs: A Sliding Mode Approach

Cuong M. Nguyen*, Anh-Tu Nguyen, and Sébastien Delprat

Abstract

This paper investigates the design of Takagi-Sugeno (TS) fuzzy model-based observers for nonlinear systems with parametric uncertainties and unknown inputs. To address this challenging problem, two observers are constructed in cascade. Based on the sliding mode technique, the first observer allows to examine a new system whose both state and output equations are subject to uncertainties but without unknown inputs. The second Luenberger-type observer is designed for the new system where the effects of uncertainties on the estimation error can be canceled. The TS fuzzy observer design is recast as optimization problems under linear matrix inequalities, which can be effectively solved using convex optimization technique. The new cascade observer structure enables a *simultaneous* estimation of the system states, the unknown inputs and the uncertainties of the original nonlinear system. The effectiveness and advantage of the proposed estimation method is demonstrated via two numerical examples including a nonlinear vehicle application.

Index Terms

Takagi-Sugeno fuzzy models, cascaded observers, sliding mode technique, nonlinear observer, unknown input, uncertainty.

I. INTRODUCTION

Physical systems usually involve complex nonlinear dynamics. Takagi-Sugeno (TS) fuzzy model-based approaches have become one of the most popular and promising tools to deal with such nonlinear systems [1]. For smooth nonlinear systems, TS fuzzy modeling provides a systematic framework to obtain *exact* representations in a compact set or approximations with any degree of accuracy [2]. In particular, TS fuzzy model-based conditions for stability analysis and controller/observer design of nonlinear systems can be expressed in terms of linear matrix inequalities (LMIs), which can be effectively solved using convex optimization techniques [3]. Therefore, TS fuzzy methods have been successfully applied to many engineering applications, see for instance [4]–[8].

Nonlinear observer design is a fundamental concern in control theory, which has attracted lots of attention [9]. Specifically, many works on TS fuzzy model-based observer design for nonlinear systems have been reported [10]–[13]. Since physical systems are inherently subject to uncertainties [9], the class of nonlinear uncertain systems is of crucial importance in control theory. The system uncertainties can emerge from modeling errors and/or uncertain parameters. Accordingly, a considerable effort has been devoted to develop robust observers for nonlinear systems where modeling uncertainties are taken into account [11], [14]–[16]. In addition to uncertainties, unknown inputs (UIs) are another important issue in robust control since UIs can represent faults, disturbances, or unmeasurable inputs in dynamical systems [17]–[19]. Therefore, many observer designs for uncertain systems with UIs have been developed in the literature. Using a sliding mode approach, the problem of fault estimation for nonlinear uncertain systems is studied in [20], where the estimation error depends on the bound of the uncertainty. A robust sliding mode observer is developed in [21] to estimate both states and disturbances of uncertain systems using a descriptor system approach. Fault reconstruction for a class of linear parameter varying (LPV) systems is investigated in [22] by designing sliding mode observers (SMOs). A SMO scheme for a class of affine LPV systems is proposed in [23] where the assumption on factorization of the fault distribution matrix in [22] is removed. A sensor fault estimation scheme for uncertain TS fuzzy systems is proposed in [24], which is robust to external disturbances. In [25], an optimal reset law is used to design unknown input observers for a class of nonlinear uncertain systems where states and faults can be simultaneously estimated. Despite recent advances in observer design for nonlinear uncertain systems with UIs, this research topic still remains challenging, especially for TS fuzzy systems with unmeasured nonlinearities [17]. To our knowledge, there is no existing work addressing the problem of simultaneously estimating system states, uncertainties and UIs of nonlinear systems.

Sliding mode technique has been widely used as an effective approach to deal with systems subject to UIs [26]–[31]. Due to its robustness with respect to modeling uncertainties and disturbances, this technique has been popularly used for both control

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and estimation of uncertain dynamical systems [32]–[37]. In sliding mode observers, UIs can be estimated by constructing discontinuous signals, which allows a finite-time convergence of output estimation errors [38]. This has motivated us to exploit SMO structure to solve the observer design problem for TS fuzzy models with uncertainties and UIs.

This paper presents a new method to design TS fuzzy sliding mode based observers for nonlinear systems with UIs, parametric uncertainties and unmeasured nonlinearities. Using the sliding mode technique, two cascaded observers are constructed. A SMO is first designed which allows to consider a new system without UIs but both state and output equations are subject to uncertainties. Then, a Luenberger-type observer is designed where the effects of the uncertainties on the estimation error can be canceled. This allows the estimations of the states of the original systems as well as the modeling uncertainties and the UIs. The main contributions of the paper can be summarized as follows.

- We propose a new sliding mode based observer design for nonlinear systems with uncertainties and UIs, where two TS fuzzy observers are constructed in cascade interconnection. Based on Lyapunov stability theory, the observer design is reformulated as optimization problems under LMI constraints, effectively solved with numerical solvers.
- Compared to relevant existing results [20], [22], [23], [38], the main advantage of the proposed cascade observer scheme is the ability to *simultaneously* estimate the state vector, the unknown input and the uncertainty. Note that the results in [20] can only provide the estimation of the state and the unknown input. Meanwhile, the uncertainty is not considered in [22], [23], [38]. Hence, the results in [22], [23], [38] can only provide the estimation of the state and the sum of disturbance and uncertainty. Note also that we consider a class of nonlinear uncertain systems with a state-dependent dynamic matrix as well as unmeasured nonlinearities. However, the system dynamic matrix in [20] is constant, and there is no unmeasured nonlinearity in [22], [23]. In [38], neither time-varying system matrix nor nonlinearity is investigated.
- The effectiveness and advantage of the new cascade observer scheme is demonstrated via two numerical examples including a practical application on the estimation of vehicle nonlinear dynamics.

The paper is organized as follows. Section II formulates the observer design problem for a class of nonlinear systems subject to both UIs and parametric uncertainties. Based on Lyapunov stability arguments, Section III presents the LMI-based design of two TS fuzzy observers in cascade interconnection. The sliding motion analysis is also provided. In Section IV, we demonstrate the effectiveness and advantage of the new results through two examples. Section V provides some concluding remarks and related future works.

Notation. For a positive integer r , we denote $\mathcal{I}_r = \{1, 2, \dots, r\}$. For $i \in \mathcal{I}_r$, we denote $\sigma_r(i) = [0, \dots, 0, \overset{\text{ith}}{1}, 0, \dots, 0]^\top \in \mathbb{R}^r$ a vector of the canonical basis of \mathbb{R}^r . For two vectors $x, y \in \mathbb{R}^n$, x_i denotes its i th entry, and $\text{column}\{x, y\} = [x^\top \ y^\top]^\top$. For a matrix X , X^\top denotes its transpose, $X \succ 0$ means X is positive definite, and $\text{He}(X) = X + X^\top$. $\text{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1, X_2 . I denotes an identity matrix of appropriate dimension, and I_n denotes an identity matrix of dimension n . In block matrices, the symbol \star stands for the terms deduced by symmetry. Arguments are omitted when their meanings are clear.

II. PROBLEM FORMULATION

Consider the following nonlinear system with unknown inputs and parametric uncertainties:

$$\begin{aligned} \dot{x} &= (A(\zeta) + \Delta A(\zeta))x + f(u, y) + Gg(x) + Dd, \\ y &= Cx, \quad x \in \mathcal{D}_x, \end{aligned} \quad (1)$$

where \mathcal{D}_x is a state-space compact set, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output vector, and $\zeta \in \mathbb{R}^{n_\zeta}$ is the vector of measurable premise variables. The unknown input $d \in \mathbb{R}^q$ is bounded, *i.e.*, $\|d\| \leq \Lambda_d$, for some $\Lambda_d > 0$. The constant matrices $G \in \mathbb{R}^{n \times n_g}$, $D \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$ are known; $f(u, y) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n_g}$ are nonlinear functions; and $A(\zeta) \in \mathbb{R}^{n \times n}$ is the nominal system matrix. The parametric uncertainty matrix $\Delta A(\zeta)$ is represented as

$$\Delta A(\zeta) = M(\zeta)F(t)N(\zeta), \quad (2)$$

where $M(\zeta)$ and $N(\zeta)$ are known, and the unknown time-varying matrix $F(t)$ satisfies $F^\top F \leq I$. Without loss of generality, for observer design we assume that the nonlinear system (1) is stable or can be stabilized by a controller.

Remark 1. Physical systems often operate in bounded regions [2]. Thus, it is reasonable to assume that the state vector x belongs to the compact set \mathcal{D}_x , see for instance the vehicle nonlinear dynamics [39] discussed in Section IV.

Let us denote $w = FN(\zeta)x \in \mathbb{R}^{n_w}$, then system (1) can be rewritten as

$$\begin{aligned} \dot{x} &= A(\zeta)x + M(\zeta)w + f(u, y) + Gg(x) + Dd, \\ y &= Cx, \end{aligned} \quad (3)$$

where w represents a bounded uncertainty, *i.e.*, $\|w\| \leq \Lambda_w$, for some positive scalar $\Lambda_w > 0$.

Remark 2. Our study is motivated by the vehicle nonlinear dynamics, described by the form (1)–(2). This form can be rewritten as in (3), which is the class of systems we consider in this paper. In other words, we deal with any system which can be represented in the form (3), and the specific uncertain system structure (1)–(2) is only one source for (3).

Applying the sector nonlinearity approach [2, Chapter 2] with the vector of premise variables ζ and $x \in \mathcal{D}_x$, the nonlinear system (3) can be *equivalently* represented by the following TS fuzzy model with nonlinear consequents:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^N h_i(\zeta)(A_i x + M_i w) + f(u, y) + Gg(x) + Dd, \\ y &= Cx, \quad N = 2^{n_\zeta}, \end{aligned} \quad (4)$$

where the local matrices A_i and M_i are constant. For $x \in \mathcal{D}_x$, the membership functions $h_i(\zeta)$ satisfy the convex sum property

$$\sum_{i=1}^N h_i(\zeta) = 1, \quad 0 \leq h_i(\zeta) \leq 1, \quad \forall i \in \mathcal{I}_N. \quad (5)$$

Furthermore, we have

$$[A(\zeta) \quad M(\zeta)] = \sum_{i=1}^N h_i(\zeta) [A_i \quad M_i].$$

Remark 3. For $\zeta = [\zeta_1 \dots \zeta_j \dots \zeta_{n_\zeta}] \in \mathbb{R}^{n_\zeta}$, since system (1) operates in a compact set, we have $\zeta_j \in [\underline{\zeta}_j, \bar{\zeta}_j], \forall j \in \mathcal{I}_{n_\zeta}$. Let us define the following membership grades $\mu_j^0(\zeta_j)$ and $\mu_j^1(\zeta_j)$ of ζ_j :

$$\mu_j^0(\zeta_j) = \frac{\bar{\zeta}_j - \zeta_j}{\bar{\zeta}_j - \underline{\zeta}_j}, \quad \mu_j^1(\zeta_j) = \frac{\zeta_j - \underline{\zeta}_j}{\bar{\zeta}_j - \underline{\zeta}_j}.$$

Note that

$$\mu_j^0(\zeta_j) + \mu_j^1(\zeta_j) = 1, \quad \zeta_j = \underline{\zeta}_j \mu_j^0(\zeta_j) + \bar{\zeta}_j \mu_j^1(\zeta_j).$$

We name $\mu_j^0(\zeta_j)$ as “ j -negative” (or \mathcal{M}_j^0) and $\mu_j^1(\zeta_j)$ as “ j -positive” (or \mathcal{M}_j^1). The membership functions $h_i(\zeta)$ and the local constant matrices A_i, M_i can be determined via $N = 2^{n_\zeta}$ fuzzy IF-THEN rules as follows (\mathcal{M}_j^i can be either \mathcal{M}_j^0 or \mathcal{M}_j^1 , $\forall j \in \mathcal{I}_{n_\zeta}, i \in \mathcal{I}_N$):

$$\begin{aligned} \text{RULE } i: & \text{ IF } \zeta_1 \text{ is } \mathcal{M}_1^i \text{ and } \dots \zeta_j \text{ is } \mathcal{M}_j^i \text{ and } \dots \zeta_{n_\zeta} \text{ is } \mathcal{M}_{n_\zeta}^i \\ \text{THEN } & \begin{cases} \dot{x} = A_i x + M_i w + f(u, y) + Gg(x) + Dd, \\ y = Cx, \end{cases} \end{aligned} \quad (6)$$

where $A_i = A(\zeta_1^i, \dots, \zeta_j^i, \dots, \zeta_{n_\zeta}^i)$, $M_i = M(\zeta_1^i, \dots, \zeta_j^i, \dots, \zeta_{n_\zeta}^i)$, and

$$h_i(\zeta) = \prod_{j=1}^{n_\zeta} \mu_j^i(\zeta_j). \quad (7)$$

Note that, if ζ_j is \mathcal{M}_j^0 : $\zeta_j^i = \underline{\zeta}_j$, $\mu_j^i(\zeta_j) = \mu_j^0(\zeta_j)$; if ζ_j is \mathcal{M}_j^1 : $\zeta_j^i = \bar{\zeta}_j$, $\mu_j^i(\zeta_j) = \mu_j^1(\zeta_j)$. Using the center-of-gravity method for defuzzification [2], the TS fuzzy system (6) can be represented in the compact form (4). For $h_i(\zeta)$ given by (7), it can be verified that $\sum_{i=1}^N h_i(\zeta) = 1$. Furthermore, (4) is an *equivalent* TS fuzzy model of system (3). See the examples in Section IV for more details.

For observer design, we consider the following assumptions for system (1), equivalently represented in the TS fuzzy form (4) within the state-space compact set \mathcal{D}_x .

Assumption 1. The following rank conditions hold:

$$\begin{aligned} \text{rank}(CD) &= \text{rank}(D), \\ \text{rank}(C) &= p > q = \text{rank}(D). \end{aligned} \quad (8)$$

Assumption 2. The nonlinear function $g(x)$ is differentiable with respect to x and satisfies

$$\underline{g}_{ij} \leq \frac{\partial g_i}{\partial x_j}(x) \leq \bar{g}_{ij}, \quad \forall i \in \mathcal{I}_{n_g}, \forall j \in \mathcal{I}_n, \quad (9)$$

with

$$\underline{g}_{ij} = \min_{x \in \mathcal{D}_x} \left(\frac{\partial g_i}{\partial x_j}(x) \right), \quad \bar{g}_{ij} = \max_{x \in \mathcal{D}_x} \left(\frac{\partial g_i}{\partial x_j}(x) \right).$$

Remark 4. Note that under condition (8) in Assumption 1, there exists a coordinate transformation \mathcal{T} for system (1) such that C and D have the following form [38]:

$$C = [0 \quad I_p], \quad D = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \begin{array}{l} \updownarrow n-p \\ \updownarrow p \end{array}. \quad (10)$$

We further assume that D_2 can be partitioned as

$$D_2 = \begin{bmatrix} 0 \\ D_{22} \end{bmatrix} \begin{array}{l} \updownarrow p-q \\ \updownarrow q \end{array}. \quad (11)$$

To simplify the presentation, hereafter we consider system (1) where C and D have the form (10)–(11). Note that condition $p > q$ in Assumption 1 will allow us to have the partition (29) where $\nu_{eq1} \in \mathbb{R}^{p-q}$. Then, we can consider system (32) for the second observer design of the proposed cascade observer scheme. Note that Assumption 2 is useful to deal with the mismatching nonlinear term caused by the unmeasured nonlinearity $g(x)$ in the estimation error using the differential mean value theorem [10], [40].

Remark 5. In the following, for brevity we will use $A(\zeta)$ and $\sum_{i=1}^N h_i(\zeta)A_i$ interchangeably as $A(\zeta) = \sum_{i=1}^N h_i(\zeta)A_i$. Similarly, we will use $M(\zeta)$ and $\sum_{i=1}^N h_i(\zeta)M_i$ interchangeably as $M(\zeta) = \sum_{i=1}^N h_i(\zeta)M_i$.

Study objective: Based on a cascade observer approach, this paper aims at designing an observer scheme to simultaneously estimate the state vector x , the unknown input d and the uncertainty w of system (1), equivalently represented by the TS fuzzy model (4).

III. CASCADE TAKAGI-SUGENO FUZZY OBSERVER DESIGN

This section presents the design of two TS fuzzy observers, whose cascade interconnection is depicted in Fig. 1.

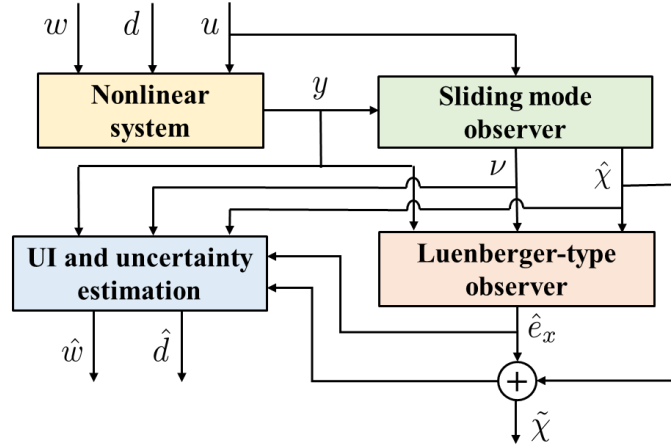


Fig. 1. TS fuzzy cascade observer structure.

A. First Design: TS Fuzzy Sliding Mode Observer

Consider the following TS fuzzy SMO for system (1):

$$\dot{\hat{x}} = \sum_{i=1}^N h_i(\zeta)(A_i \hat{x} + L_i e_y) + f(u, y) + Gg(\bar{x}) - J\nu, \quad (12)$$

where \hat{x} is the estimate of x , the observer gains L_i , for $i \in \mathcal{I}_N$, are to be designed, and

$$\hat{\chi} = \begin{bmatrix} I_{n-p} & 0 \end{bmatrix} \hat{x}, \quad e_y = y - C\hat{x}, \\ \bar{x} = \begin{bmatrix} \hat{\chi} \\ y \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

The discontinuous signal $\nu(t)$ is defined as

$$\nu = \begin{cases} -\mu \frac{P_y^{-1} e_y}{\|e_y\|}, & e_y \neq 0, \\ 0, & \text{otherwise} \end{cases} \quad (13)$$

where $\mu > 0$ and $P_y \in \mathbb{R}^{p \times p}$ are to be determined. From (1) and (12), the dynamics of the state estimation error $e = x - \hat{x}$ is defined as

$$\dot{e} = (A(\zeta) - L(\zeta)C)e + M(\zeta)w + G(g(x) - g(\bar{x})) + Dd + J\nu, \quad (14)$$

where $L(\zeta) = \sum_{i=1}^N h_i(\zeta)L_i$. The following differential mean value theorem (DMVT) is useful to deal with the mismatching nonlinear term $g(x) - g(\bar{x})$.

Lemma 1 ([40]). Let $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n_g}$ and $a, b \in \mathbb{R}^n$. If $g(x)$ is differentiable with respect to x on the convex hull of a and b , then there exist vectors $c_i = \alpha_i a + (1 - \alpha_i)b$, with $\alpha_i \in (0, 1)$, $\forall i \in \mathcal{I}_{n_g}$, such that

$$g(a) - g(b) = \left(\sum_{i=1}^{n_g} \sum_{j=1}^n \sigma_{n_g}(i) \sigma_n^\top(j) \frac{\partial g_i}{\partial x_j}(c_i) \right) (a - b).$$

Under Assumption 2 and by Lemma 1, we can rewrite the nonlinear term $g(x) - g(\bar{x})$ as

$$g(x) - g(\bar{x}) = \mathcal{A}_g(x - \bar{x}) = \mathcal{A}_g \begin{bmatrix} e_x \\ 0 \end{bmatrix},$$

where $e_x = [I_{n-p} \ 0]e$, and

$$\mathcal{A}_g = \sum_{i=1}^{n_g} \sum_{j=1}^n \sigma_{n_g}(i) \sigma_n^\top(j) \frac{\partial g_i}{\partial x_j}(c_i),$$

with $c_i = \alpha_i x + (1 - \alpha_i)\bar{x}$ and $\alpha_i \in (0, 1)$. Note that \mathcal{A}_g is a matrix with parameters varying in a bounded convex set \mathcal{H}_g , whose the set of vertices is given by

$$\mathcal{V}_{\mathcal{H}_g} = \left\{ \rho = (\rho_{ij}) \in \mathbb{R}^{n_g \times n} : \rho_{ij} \in \left\{ \underline{g}_{ij}, \bar{g}_{ij} \right\} \right\},$$

where \underline{g}_{ij} and \bar{g}_{ij} are determined as in (9). Let us denote

$$\mathcal{A}_g^1 = \mathcal{A}_g [I_{n-p} \ 0]^\top, \quad \mathcal{A}_g^{10} = [\mathcal{A}_g^1 \ 0].$$

Then, it follows that $g(x) - g(\bar{x}) = \mathcal{A}_g^{10} e$. Note that the matrix \mathcal{A}_g^1 (respectively \mathcal{A}_g^{10}) is with parameters varying in a bounded convex set \mathcal{H}_g^1 (respectively \mathcal{H}_g^{10}), whose the set of vertices is $\mathcal{V}_{\mathcal{H}_g^1}$ (respectively $\mathcal{V}_{\mathcal{H}_g^{10}}$). As a result, the error dynamics (14) can be rewritten as

$$\dot{e} = (A(\zeta) + G\mathcal{A}_g^{10} - L(\zeta)C)e + M(\zeta)w + Dd + J\nu. \quad (15)$$

Based on the \mathcal{H}_∞ filtering approach, we design the observer gain $L(\zeta)$ such that the following condition holds:

$$\dot{V}(e) + e^\top e - \lambda^2 w^\top w < 0, \quad (16)$$

where $V(e) = e^\top P e$, with $P \succ 0$, is a Lyapunov function candidate, and $\lambda > 0$ is the level of disturbance attenuation. The following theorem provides sufficient conditions to guarantee condition (16) with a minimal value of λ .

Theorem 1. Consider the error dynamics (15). Condition (16) is guaranteed with a minimal λ if there exist matrices of appropriate dimensions $P_1 \succ 0$, $P_y \succ 0$, X_i , for $i \in \mathcal{I}_N$, and a scalar $\bar{\lambda} > 0$ that solve the convex optimization problem

$$\begin{aligned} & \min(\bar{\lambda}) \text{ subject to} \\ & \Omega_i = \begin{bmatrix} \text{He} [P(A_i + G\mathcal{A}_g^{10}) - X_i C] + I & P M_i \\ \star & -\bar{\lambda} I \end{bmatrix} \prec 0, \end{aligned} \quad (17)$$

for $\forall i \in \mathcal{I}_N$, $\forall \mathcal{A}_g^{10} \in \mathcal{V}_{\mathcal{H}_g^{10}}$, with $P = \text{diag}(P_1, P_y)$, and the positive scalar μ in (13) satisfies

$$\mu \geq \|P_y D_2\| \Lambda_d. \quad (18)$$

Furthermore, the observer gain matrices in (12) are obtained as $L_i = P^{-1} X_i$, and $\lambda = \sqrt{\bar{\lambda}}$.

Proof. The time-derivative of $V(e)$ along the solution of system (15) is given by

$$\begin{aligned} \dot{V}(e) &= 2e^\top P \left((A(\zeta) + G\mathcal{A}_g^{10} - L(\zeta)C)e + M(\zeta)w \right) + 2e_y^\top P_y D_2 d - 2\mu \|e_y\| \\ &\leq 2e^\top P \left((A(\zeta) + G\mathcal{A}_g^{10} - L(\zeta)C)e + M(\zeta)w \right) - 2\|e_y\|(\mu - \|P_y D_2\| \Lambda_d). \end{aligned} \quad (19)$$

Since $X_i = PL_i$, for $i \in \mathcal{I}_N$ and $\bar{\lambda} = \lambda^2$, it follows from (16), (18) and (19) that

$$\begin{aligned} \dot{V}(e) + e^\top e - \lambda^2 w^\top w &\leq e^\top \{ \text{He} [P(A(\zeta) + G_2 \mathcal{A}_g^{10}) - X(\zeta)C] + I \} e + 2e^\top PM(\zeta)w - \bar{\lambda} w^\top w \\ &= \sum_{i=1}^N h_i(\zeta) [e^\top \quad w^\top] \Omega_i \begin{bmatrix} e \\ w \end{bmatrix}. \end{aligned} \quad (20)$$

Due to the convex sum property (5) of the membership functions $h_i(\zeta)$, for $i \in \mathcal{I}_N$, it follows from inequality (20) that condition (16) holds if (17) holds. Moreover, λ is minimized by the minimization of $\bar{\lambda}$. \square

Remark 6. Note that if condition (16) holds, then e is bounded, i.e., $\|e(t)\| \leq \Lambda_e$, for $\forall t \geq 0$. This means that \hat{x} only estimates x within a bound. The second observer will be designed in Section III-C to improve the estimation precision of x .

Remark 7. The differential mean value theorem has been shown as an effective tool to deal with unmeasured nonlinearities in various estimation problems, see for instance [7], [17], [40] and related references. Recently, an alternative approach to DMVT has been proposed in [41], [42]. However, the approach in [41], [42] requires the derivation of the polynomial forms and their corresponding TS models, which could be possibly confuse the readers if integrated into our TS fuzzy cascaded observer scheme. As a result, we prefer to use the classical DMVT approach to ensure the readability of this paper.

B. Sliding Motion Analysis

We partition e , G , L_i , A_i and M_i , for $i \in \mathcal{I}_N$, as follows:

$$e = \begin{bmatrix} e_x \\ e_y \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}, \quad M_i = \begin{bmatrix} M_{i1} \\ M_{i2} \end{bmatrix}, \quad (21)$$

where $e_y \in \mathbb{R}^p$, $L_{i2} \in \mathbb{R}^{p \times p}$, $G_2 \in \mathbb{R}^{p \times n_g}$, $A_{i22} \in \mathbb{R}^{p \times p}$, and $M_{i2} \in \mathbb{R}^{p \times n_w}$. Then, it follows from (15) and (21) that

$$\dot{e}_x = (A_{11}(\zeta) + G_1 \mathcal{A}_g^1) e_x + (A_{12}(\zeta) - L_1(\zeta)) e_y + M_1(\zeta) w, \quad (22)$$

$$\dot{e}_y = (A_{21}(\zeta) + G_2 \mathcal{A}_g^1) e_x + (A_{22}(\zeta) - L_2(\zeta)) e_y + M_2(\zeta) w + D_2 d + \nu, \quad (23)$$

with

$$\begin{bmatrix} A_{11}(\zeta) & A_{12}(\zeta) & L_1(\zeta) & M_1(\zeta) \\ A_{21}(\zeta) & A_{22}(\zeta) & L_2(\zeta) & M_2(\zeta) \end{bmatrix} = \sum_{i=1}^N h_i(\zeta) \begin{bmatrix} A_{i11} & A_{i12} & L_{i1} & M_{i1} \\ A_{i21} & A_{i22} & L_{i2} & M_{i2} \end{bmatrix}.$$

The following theorem provides a sufficient condition for the occurrence of the sliding motion.

Theorem 2. Consider system (22)–(23). If the positive scalar μ in (13) is selected as

$$\begin{aligned} \mu = & \left\{ \max_{1 \leq i \leq N} \|P_y A_{21i}\| + \max_{\mathcal{A}_g^1 \in \mathcal{V}_{\mathcal{A}_g^1}} \|P_y G_2 \mathcal{A}_g^1\| + \max_{1 \leq i \leq N} \|P_y (A_{22i} - L_{2i})\| \right\} \Lambda_e \\ & + \max_{1 \leq i \leq N} \|P_y M_{2i}\| \Lambda_w + \|P_y D_2\| \Lambda_d + \bar{\mu}, \end{aligned} \quad (24)$$

with $\bar{\mu} > 0$. Then, the sliding motion occurs on the surface $\mathcal{S} = \{e : Ce = 0\}$, where $e_y = 0$ and $\dot{e}_y = 0$.

Proof. For sliding motion analysis, we consider the Lyapunov function candidate $V_y(e_y) = e_y^\top P_y e_y$. Differentiating the function $V_y(e_y)$ along the solution of system (23), it follows that

$$\begin{aligned} \dot{V}_y(e_y) &= 2e_y^\top P_y \{ (A_{21}(\zeta) + G_2 \mathcal{A}_g^1) e_x + (A_{22}(\zeta) - L_2(\zeta)) e_y + M_2(\zeta) w + D_2 d \} - 2\mu \|e_y\| \\ &\leq -2\|e_y\| (\mu - \|P_y A_{21}(\zeta)\| \|e_x\| - \|P_y G_2 \mathcal{A}_g^1\| \|e_x\| - \|P_y (A_{22}(\zeta) - L_2(\zeta))\| \|e_y\| \\ &\quad - \|P_y M_2(\zeta)\| \|w\| - \|P_y D_2\| \|d\|). \end{aligned} \quad (25)$$

If condition (24) holds, then it follows from (25) that

$$\dot{V}_y(e_y) \leq -2\bar{\mu} \|e_y\| \leq -\frac{2\bar{\mu}}{\sqrt{\lambda_{\max}(P_y)}} \sqrt{V_y(e_y)}, \quad (26)$$

where $\lambda_{\max}(P_y)$ is the maximal eigenvalue of P_y . Note that (26) is the *reachability condition*, hence the sliding motion occurs on $\mathcal{S} = \{e : Ce = 0\}$, where $e_y = 0$ and $\dot{e}_y = 0$. \square

Remark 8. If the unknown input d is bounded by a known time-varying function instead of a known constant, i.e., $\|d\| \leq \varphi(t, u, y)$, where $\varphi(t, u, y) > 0$, for $\forall(t, u, y)$, the sliding motion on the surface $\mathcal{S} = \{e : Ce = 0\}$ is guaranteed by choosing a time-varying parameter $\mu(t, u, y)$ instead of a constant scalar μ as follows:

$$\mu(t, u, y) = \left\{ \max_{1 \leq i \leq N} \|P_y A_{21i}\| + \max_{\mathcal{A}_g^1 \in \mathcal{V}_{\neq 0}^1} \|P_y G_2 \mathcal{A}_g^1\| + \max_{1 \leq i \leq N} \|P_y (A_{22i} - L_{2i})\| \right\} \Lambda_e + \max_{1 \leq i \leq N} \|P_y M_{2i}\| \Lambda_w + \|P_y D_2\| \varphi(t, u, y) + \bar{\mu}.$$

This can be proved in a similar manner to the proof of Theorem 2. It is important to note that a constant bound for the unknown input d is preferred over a time-varying bound as it simplifies the design of μ , thus the overall observer design.

Under the sliding motion, it follows from (22) and (23) that

$$\dot{e}_x = (A_{11}(\zeta) + G_1 \mathcal{A}_g^1) e_x + M_1(\zeta) w, \quad (27)$$

$$0 = (A_{21}(\zeta) + G_2 \mathcal{A}_g^1) e_x + M_2(\zeta) w + D_2 d + \nu_{eq}, \quad (28)$$

where ν_{eq} is the equivalent control.

Remark 9. The stability of system (27)-(28) is guaranteed by Theorem 1. Indeed, condition (17) implies that

$$P_1 (A_{11}(\zeta) + G_1 \mathcal{A}_g^1) + (A_{11}(\zeta) + G_1 \mathcal{A}_g^1)^\top P_1 \prec 0.$$

We further partition ν_{eq} , G_2 , A_{i21} , M_{i2} , for $i \in \mathcal{I}_N$, as follows:

$$\nu_{eq} = \begin{bmatrix} \nu_{eq1} \\ \nu_{eq2} \end{bmatrix}, \quad G_2 = \begin{bmatrix} G_{21} \\ G_{22} \end{bmatrix}, \quad A_{i21} = \begin{bmatrix} A_{i211} \\ A_{i212} \end{bmatrix}, \quad M_{i2} = \begin{bmatrix} M_{i21} \\ M_{i22} \end{bmatrix}, \quad (29)$$

where $\nu_{eq2} \in \mathbb{R}^q$, $G_{22} \in \mathbb{R}^{q \times n_g}$, $A_{i212} \in \mathbb{R}^{q \times (n-p)}$ and $M_{i22} \in \mathbb{R}^{q \times n_w}$. Then, it follows from (28) and (29) that

$$\nu_{eq1} = -(A_{211}(\zeta) + G_{21} \mathcal{A}_g^1) e_x - M_{21}(\zeta) w, \quad (30)$$

$$\nu_{eq2} = -(A_{212}(\zeta) + G_{22} \mathcal{A}_g^1) e_x - M_{22}(\zeta) w - D_{22} d, \quad (31)$$

with

$$\begin{bmatrix} A_{211}(\zeta) & M_{21}(\zeta) \\ A_{212}(\zeta) & M_{22}(\zeta) \end{bmatrix} = \sum_{i=1}^N h_i(\zeta) \begin{bmatrix} A_{i211} & M_{i21} \\ A_{i212} & M_{i22} \end{bmatrix}.$$

From (27) and (30), let us consider the following system:

$$\begin{aligned} \dot{e}_x &= A_{11}(\zeta) e_x + G_1 (g(x) - g(\bar{x}) + M_1(\zeta) w), \\ \nu_{eq1} &= -A_{211}(\zeta) e_x - G_{21} (g(x) - g(\bar{x})) - M_{21}(\zeta) w, \end{aligned} \quad (32)$$

where e_x is the state vector, and ν_{eq1} is the measured output.

Remark 10. With the sliding mode observer (12), the estimation error e_x is only guaranteed to be bounded, i.e., \hat{x} only estimates x within a bound. However, the SMO design allows to consider system (32) with the following properties:

- the unknown input d is not present in (32),
- both state and output equations of system (32) are subject to the uncertainty w .

System (32) enables the design of the second observer to \mathcal{H}_∞ asymptotically estimate e_x , i.e., the effect of w on the estimation error e can be minimized. Moreover, the uncertainty w and the unknown input d can also be recovered under an extra rank condition stated in Assumption 3.

C. Second Design: Luenberger-Type TS Fuzzy Observer

Consider the following TS fuzzy Luenberger-type observer structure for system (32):

$$\dot{\hat{e}}_x = A_{11}(\zeta) \hat{e}_x + \sum_{j=1}^N h_j(\zeta) K_j (\nu_{eq1} + A_{211}(\zeta) \hat{e}_x) + \left(G_1 + \sum_{j=1}^N h_j(\zeta) K_j G_{21} \right) (g(\bar{x} + \bar{e}_x) - g(\bar{x})), \quad (33)$$

where \hat{e}_x is the estimate of e_x , the observer gains K_j , for $j \in \mathcal{I}_N$, are to be determined, and $\bar{e}_x = \text{column}\{\hat{e}_x, 0\}$. The dynamics of the estimation error $\tilde{e}_x = e_x - \hat{e}_x$ is given by

$$\dot{\tilde{e}}_x = ((A_{11}(\zeta) + G_1 \mathcal{A}_g^1) + K(\zeta) (A_{211}(\zeta) + G_{21} \mathcal{A}_g^1)) \tilde{e}_x + (M_1(\zeta) + K(\zeta) M_{21}(\zeta)) w, \quad (34)$$

where $K(\zeta) = \sum_{j=1}^N h_j(\zeta)K_j$. Hereafter, the observer gain $K(\zeta)$ is designed such that the following condition holds:

$$\dot{V}_e(\tilde{e}_x) + \tilde{e}_x^\top \tilde{e}_x - \gamma^2 w^\top w < 0, \quad (35)$$

where $V_e(\tilde{e}_x) = \tilde{e}_x^\top Q \tilde{e}_x$, with $Q \succ 0$, is a Lyapunov function candidate, and $\gamma > 0$ is the level of disturbance attenuation to be minimized.

Remark 11. Note from (35) that if γ is minimized to be arbitrarily small, *i.e.*, $\gamma \approx 0$, then the effect of w on the estimation error \tilde{e}_x can be negligible. Using the \mathcal{H}_∞ filtering based approach, condition (35) guarantees the \mathcal{H}_∞ asymptotic stability of system (34) without requiring to solve a set of algebraic equations to cancel the effect of w on \tilde{e}_x .

The following lemma is useful to design the TS fuzzy observer (33).

Lemma 2 ([43]). Consider the inequality

$$\sum_{i=1}^N \sum_{j=1}^N h_i(\zeta)h_j(\zeta)\Psi_{ij} < 0, \quad (36)$$

where Ψ_{ij} , for $i, j \in \mathcal{I}_N$, are symmetric matrices of appropriate dimensions, and the functions $h_i(\zeta)$ satisfy the convex sum property (5). Condition (36) holds if

$$\begin{aligned} \Psi_{ii} < 0, \quad \forall i \in \mathcal{I}_N, \\ \frac{2}{N-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} < 0, \quad \forall i, j \in \mathcal{I}_N, i \neq j. \end{aligned}$$

The following theorem provides sufficient conditions to design the observer (33) while minimizing the value of γ in (35).

Theorem 3. Consider the error estimation dynamics (34). Condition (35) is guaranteed with a minimal γ if there exist matrices of appropriate dimensions $Q \succ 0$, Y_i ($1 \leq i \leq N$), and a scalar $\bar{\gamma} > 0$ that solve the convex optimization problem

$\min(\bar{\gamma})$ subject to

$$\Phi_{ii} < 0, \quad \forall i \in \mathcal{I}_N, \quad (37)$$

$$\frac{2}{N-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0, \quad \forall i, j \in \mathcal{I}_N, i \neq j, \quad (38)$$

with

$$\Phi_{ij} = \begin{bmatrix} \text{He} [Q(A_{11i} + G_1 \mathcal{A}_g^1) + Y_j(A_{211i} + G_{21} \mathcal{A}_g^1)] + I & QM_{1i} + Y_j M_{21i} \\ \star & -\bar{\gamma}I \end{bmatrix}, \quad \text{for } \forall \mathcal{A}_g^1 \in \mathcal{V}_{\mathcal{H}_g^1}.$$

Moreover, the observer gains in (33) are obtained as $K_j = Q^{-1}Y_j$, for $j \in \mathcal{I}_N$, and $\gamma = \sqrt{\bar{\gamma}}$.

Proof. Since $Y(\zeta) = QK(\zeta)$ and $\bar{\gamma} = \gamma^2$, we have

$$\begin{aligned} \dot{V}_e(\tilde{e}_x) + \tilde{e}_x^\top \tilde{e}_x - \gamma^2 w^\top w \\ = \tilde{e}_x^\top \{ \text{He} [Q(A_{11}(\zeta) + G_1 \mathcal{A}_g^1) + Y(\zeta)(A_{211}(\zeta) + G_{21} \mathcal{A}_g^1)] + I \} \tilde{e}_x \\ + 2\tilde{e}_x^\top (QM_1(\zeta) + Y(\zeta)M_{21}(\zeta))w - \bar{\gamma}w^\top w \\ = \sum_{i=1}^N \sum_{j=1}^N h_i(\zeta)h_j(\zeta) [\tilde{e}_x^\top \ w^\top] \Phi_{ij} \begin{bmatrix} \tilde{e}_x \\ w \end{bmatrix}. \end{aligned} \quad (39)$$

Applying Lemma 2, it follows that condition (35), or equivalently (39), is verified if conditions (37) and (38) hold. Moreover, the value of $\gamma = \sqrt{\bar{\gamma}}$ is minimized by the minimization of $\bar{\gamma}$. \square

Remark 12. Since $y = [0 \ I_p]x$ is already available from the output, we only need to estimate $\chi = [I_{n-p} \ 0]x$. We have $\chi = \hat{\chi} + e_x$, where e_x is bounded. If \hat{e}_x can be designed to \mathcal{H}_∞ asymptotically estimate e_x , then χ is \mathcal{H}_∞ asymptotically estimated by

$$\tilde{\chi} = \hat{\chi} + \hat{e}_x. \quad (40)$$

Note that, if γ can be minimized to be sufficiently small $\gamma \approx 0$, the estimation of χ by $\tilde{\chi}$ is similar to asymptotic estimation.

The following assumption is necessary for the estimation of the uncertainty w and the unknown input d .

Assumption 3. The matrix $M_{21}(\zeta)$, obtained from the partitions (21) and (29) of the uncertainty structure matrix $M(\zeta)$ in (2), is full column rank.

Remark 13. Note from (2), (21) and (29) that $M_{21}(\zeta)$ depends on the uncertainty structure of a given system. Hence, Assumption 3 should be checked on a case-by-case basis to be able to estimate both w and d .

Let us denote $M_{21}^\dagger(\zeta) = (M_{21}(\zeta)^\top M_{21}(\zeta))^{-1} M_{21}(\zeta)^\top$. It follows from (30) that

$$w = -M_{21}^\dagger(\zeta) (\nu_{eq1} + A_{211}(\zeta)e_x + G_{21}(g(x) - g(\bar{x}))). \quad (41)$$

From (41), we can estimate the uncertainty w as

$$\hat{w} = -M_{21}^\dagger(\zeta) (\nu_{eq1} + A_{211}(\zeta)\hat{e}_x + G_{21}\Delta_{\hat{g}}), \quad (42)$$

with $\Delta_{\hat{g}} = g(\tilde{x}) - g(\bar{x})$ and $\tilde{x} = \text{column}\{\tilde{\chi}, y\}$. Similarly, it follows from (31) that

$$d = -D_{22}^{-1} (\nu_{eq2} + A_{212}(\zeta)e_x + G_{22}(g(x) - g(\bar{x})) + M_{22}(\zeta)w). \quad (43)$$

From (43), the unknown input d can be estimated by

$$\hat{d} = -D_{22}^{-1} (\nu_{eq2} + A_{212}(\zeta)\hat{e}_x + G_{22}\Delta_{\hat{g}} + M_{22}(\zeta)\hat{w}). \quad (44)$$

Since $\tilde{\chi}$ is an \mathcal{H}_∞ asymptotic estimation of χ , from (41) and (42), \hat{w} provides an \mathcal{H}_∞ asymptotic estimation for w . A similar remark can be done for the estimation of d with (44). Note that, if the minimal γ is sufficiently small $\gamma \approx 0$, the estimations of w and d are similar to asymptotic estimation.

Remark 14. When $g(x) \equiv 0$, by denoting $f_i = [w \ d]^\top$, and $\mathcal{D}(\zeta) = [M(\zeta) \ D]$, system (3) can be rewritten into a similar form with system (1)-(2) in [22], [23]. Note that, it is assumed in [22] that $\mathcal{D}(\zeta)$ can be factorized into a constant and a varying component. This assumption is necessary for the existence of the coordinate transformation in the next step to transform (1)-(2) in [22] to the required form. If $\mathcal{D}(\zeta)$ cannot be factorized into a constant and a varying component, the result in [22] as well as its direct extension cannot be applied. Similarly, the result in [20] as well as its direct extension cannot be applied in this case. However, the proposed observer scheme in this paper is still applicable. On the other hand, the SMO scheme in [23] would strictly require $p > q + n_w$ while our observer scheme only requires $p > q$. Thus, the scheme in this paper still works in many cases where the scheme in [23] as well as its direct extension is not applicable. The advantage of our result over the results in [20], [22], [23] is illustrated in Example 2 in Section IV.

The proposed cascade TS fuzzy observer design is summarized in Algorithm 1.

Algorithm 1: Cascade Observer Design Procedure

Input: Nonlinear uncertain system (1).

Output: Cascade TS fuzzy observer (12), (33) and (40), UI estimator (44), and uncertainty estimator (42).

- 1 Verify the conditions in Assumptions 1 and 2.
 - If YES, then go to Step 2.
 - If NO, then the proposed method is unapplicable to the considered system.
 - 2 Transform system (1) into the desired form (10)–(11) (if necessary), and obtain the equivalent TS fuzzy model (4).
 - 3 Solve the optimization problem in Theorem 1 to compute the gain $L(\zeta)$ for the sliding mode observer (12), and determine P_y for the discontinuous signal ν .
 - 4 Use condition (24) to determine μ for ν .
 - 5 Solve the optimization problem in Theorem 3 to compute the gain $K(\zeta)$ for the Luenberger observer (33).
 - 6 Estimate the unmeasurable state x with the expression of $\tilde{\chi}$ in (40).
 - 7 Verify the rank condition in Assumption 3.
 - If YES, then go to Step 8.
 - If NO, then we cannot estimate the UI and the uncertainty.
 - 8 Estimate the uncertainty w with (42), and the unknown input d with (44).
-

IV. ILLUSTRATIVE EXAMPLES

To illustrate the effectiveness and advantage of the proposed results over existing results in the literature, we will consider two numerical examples including a practical example of vehicle nonlinear dynamics and an academic example. Simulations of both examples are carried out via MATLAB with a sampling time of 0.001 [s]. All the convex optimization problems are solved using MATLAB Robust Control Toolbox.

Example 1. We consider the estimation problem of the vehicle nonlinear dynamics. Under normal driving conditions without longitudinal slip and with small angles assumption, the vehicle dynamics can be described as follows [7], [39]:

$$\begin{aligned}\dot{v}_y &= -\frac{2(C_f + C_r)v_y}{m_v v_x} + \frac{2(C_r l_r - C_f l_f)y_r}{m_v v_x} - y_r v_x + \frac{2C_f \delta}{m_v} - \frac{C_{dy} \rho_a A_{fy} v_y^2}{2m_v}, \\ \dot{y}_r &= \frac{2(C_r l_r - C_f l_f)v_y}{I_z v_x} - \frac{2(C_f l_f^2 + C_r l_r^2)y_r}{I_z v_x} + \frac{2C_f l_f \delta}{I_z}, \\ \dot{v}_x &= y_r v_y - \frac{C_{dx} \rho_a A_{fx} v_x^2}{2m_v} + \frac{T_w}{m_v R_t},\end{aligned}\quad (45)$$

where the vehicle nomenclature is given in Table I. Here, the numerical parameters are taken from the INSA autonomous vehicle in our LAMIH-CNRS laboratory.

TABLE I
VEHICLE PARAMETERS.

Description	Value	
v_y	Lateral speed	–
v_x	Longitudinal speed	–
y_r	Yaw rate	–
δ	Front wheel steering angle	–
T_w	Longitudinal wheel torque force	–
m_v	Vehicle mass	1077 [kg]
C_f	Front cornering stiffness	47135 [N/rad]
C_r	Rear cornering stiffness	56636 [N/rad]
l_f	Distance between front axle and gravity center	1.08 [m]
l_r	Distance between rear axle and gravity center	1.24 [m]
I_z	Yaw moment of inertia	1442 [kgm ²]
R_t	Tire radius	0.26 [m]
ρ_a	Air density	1.23 [kg/m ³]
C_{dy}	Lateral drag coefficient	0.35 [–]
C_{dx}	Longitudinal drag coefficient	0.32 [–]
A_{fy}	Lateral frontal area	2.01 [m ²]
A_{fx}	Longitudinal frontal area	1.97 [m ²]

Note that the yaw rate y_r and the longitudinal speed v_x can be measured online. However, the measurement of the lateral speed v_y is not available due to cost reasons. The steering angle $u = \delta$ is considered as a known control input, while the wheel torque $d = T_w$ is an unknown input. Furthermore, it is not possible to perfectly identify the tire cornering stiffness in practice [6]. For illustrations, we assume that the rear cornering stiffness C_r is an uncertain parameter, *i.e.*, $C_r = C_{r0} + \Delta C_r$, where $C_{r0} = 56636$ is the nominal value and ΔC_r represents the parametric uncertainty: $\Delta C_r = C_{r\Delta} C_{r0} F(t)$, with $C_{r\Delta} \in (0, 1)$ and $|F(t)| \leq 1$. In this paper, we consider the case with 10% of parametric uncertainty, *i.e.*, $C_{r\Delta} = 0.1$, and we assume that $F(t) = \cos(t)$ for simulation purposes. The vehicle system (45) can be reformulated in the form (3) where $x = [v_y \ y_r \ v_x]^\top$, $\zeta = [y_r \ \frac{1}{v_x}]^\top$, and

$$\begin{aligned}A(\zeta) &= \begin{bmatrix} -\frac{2(C_f + C_{r0})}{m_v v_x} & \frac{2(C_{r0} l_r - C_f l_f)}{m_v v_x} & -y_r \\ \frac{2(C_{r0} l_r - C_f l_f)}{I_z v_x} & -\frac{2(C_f l_f^2 + C_{r0} l_r^2)}{I_z v_x} & 0 \\ y_r & 0 & 0 \end{bmatrix}, \quad f(u, y) = \begin{bmatrix} \frac{2C_f \delta}{m_v} \\ \frac{2C_f l_f \delta}{I_z} \\ -\frac{C_{dy} \rho_a A_{fy} v_y^2}{2m_v} \end{bmatrix}, \\ G &= \begin{bmatrix} -\frac{C_{dy} \rho_a A_{fy}}{2m_v} \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_v R_t} \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} \frac{2C_{r0} C_{r\Delta}}{m_v v_x} \\ -\frac{2C_{r0} C_{r\Delta} l_r}{I_z v_x} \\ 0 \end{bmatrix}, \\ g(x) &= v_y^2, \quad C = [0 \ I_2], \quad N = [-1 \ l_r \ 0].\end{aligned}\quad (46)$$

Considering the physical limitations during normal driving conditions [7], the vehicle state-space compact set is defined as

$$\mathcal{D}_x = \left\{ (v_y, y_r, v_x) \in \mathbb{R}^3 : v_y \in [\underline{v}_y, \bar{v}_y], y_r \in [\underline{y}_r, \bar{y}_r], v_x \in [\underline{v}_x, \bar{v}_x] \right\},$$

where $\underline{v}_y = -1.5$ [m/s], $\bar{v}_y = 1.5$ [m/s], $\underline{y}_r = -0.6$ [rad/s] and $\bar{y}_r = 0.6$ [rad/s], and $\underline{v}_x = 5$ [m/s], $\bar{v}_x = 30$ [m/s]. Using the sector nonlinearity method [2] with the vector of measured premise variables $\zeta = [y_r \ \frac{1}{v_x}]^\top$, an equivalent four-rule TS fuzzy

model of the vehicle nonlinear system (45) can be obtained for $x \in \mathcal{D}_x$. To this end, we define the membership functions $h_i(\zeta)$, for $i \in \mathcal{I}_4$, as

$$\begin{aligned} h_1(\zeta) &= \frac{(\bar{y}_r - y_r)(1/v_x - 1/v_x)}{(\bar{y}_r - \underline{y}_r)(1/v_x - 1/\bar{v}_x)}, & h_2(\zeta) &= \frac{(y_r - \underline{y}_r)(1/v_x - 1/v_x)}{(\bar{y}_r - \underline{y}_r)(1/v_x - 1/\bar{v}_x)}, \\ h_3(\zeta) &= \frac{(\bar{y}_r - y_r)(1/v_x - 1/\bar{v}_x)}{(\bar{y}_r - \underline{y}_r)(1/v_x - 1/\bar{v}_x)}, & h_4(\zeta) &= \frac{(y_r - \underline{y}_r)(1/v_x - 1/\bar{v}_x)}{(\bar{y}_r - \underline{y}_r)(1/v_x - 1/\bar{v}_x)}. \end{aligned}$$

It is clear that $\sum_{i=1}^4 h_i(\zeta) = 1$, $h_i(\zeta) \geq 0$, for $i \in \mathcal{I}_4$, and

$$y_r = \underline{y}_r(h_1(\zeta) + h_3(\zeta)) + \bar{y}_r(h_2(\zeta) + h_4(\zeta)), \quad (47)$$

$$\frac{1}{v_x} = \frac{h_1(\zeta) + h_2(\zeta)}{\bar{v}_x} + \frac{h_3(\zeta) + h_4(\zeta)}{v_x}, \quad (48)$$

Substituting (47) and (48) into $A(\zeta)$ and $M(\zeta)$, it follows that $A(\zeta) = \sum_{i=1}^4 h_i(\zeta)A_i$ and $M(\zeta) = \sum_{i=1}^4 h_i(\zeta)M_i$, with

$$\begin{aligned} A_1 &= A(\underline{y}_r, 1/\bar{v}_x), & A_2 &= A(\bar{y}_r, 1/\bar{v}_x), & A_3 &= A(\underline{y}_r, 1/v_x), & A_4 &= A(\bar{y}_r, 1/v_x), \\ M_1 &= M(\underline{y}_r, 1/\bar{v}_x), & M_2 &= M(\bar{y}_r, 1/\bar{v}_x), & M_3 &= M(\underline{y}_r, 1/v_x), & M_4 &= M(\bar{y}_r, 1/v_x). \end{aligned}$$

Since $\text{rank}(CD) = \text{rank}(D)$ and $\text{rank}(D) = 1 < 2 = \text{rank}(C)$, Assumption 1 holds. As shown in (46), the vehicle model (45) already has C and D in the form (10)–(11), thus a coordinate transformation is not required. Note that Assumption 2 also holds where $g(x) = v_y^2$ is differentiable with respect to v_y and $2v_y \leq \frac{\partial g}{\partial v_y} = 2v_y \leq 2\bar{v}_y$. Accordingly, the sets of vertices are obtained as $\mathcal{V}_{\mathcal{H}_g^1} = \{2v_y, 2\bar{v}_y\}$, $\mathcal{V}_{\mathcal{H}_g^{10}} = \{[2v_y \ 0 \ 0], [2\bar{v}_y \ 0 \ 0]\}$. Since $M_{21}(\zeta) = -2C_{r0}C_{r\Delta}l_r/(I_z v_x)$, where $v_x \in [5, 30]$, $M_{21}(\zeta)$ is full column rank, which verifies Assumption 3.

Note that, since $p = 2 = q + n_w$, the SMO scheme in [23] as well as its direct extension are not applicable to system (45). Solving the convex optimization problem in Theorem 1, we obtain the observer gain for the sliding mode observer (12) as

$$\begin{aligned} L_1 &= \begin{bmatrix} 0.7978 & -8.7115 \\ 70.6866 & -0.0003 \\ -0.0000 & 2.3496 \end{bmatrix}, & L_2 &= \begin{bmatrix} 0.7978 & 8.7115 \\ 70.6866 & 0.0003 \\ 0.0000 & 2.3496 \end{bmatrix}, \\ L_3 &= \begin{bmatrix} -20.1846 & -8.6860 \\ 61.3977 & 0.0064 \\ 0.0002 & 2.3496 \end{bmatrix}, & L_4 &= \begin{bmatrix} -20.1846 & 8.6860 \\ 61.3977 & -0.0064 \\ -0.0002 & 2.3496 \end{bmatrix}, \end{aligned}$$

with the attenuation level $\lambda = 0.0781$. With regard to the discontinuous signal ν in (13), the matrix P_y is computed as

$$P_y = \begin{bmatrix} 0.0449 & 0.0000 \\ 0.0000 & 1.4127 \end{bmatrix},$$

and μ is computed by (24) as $\mu = 23$. For the observer implementation, to avoid the chattering, the discontinuous signal ν is approximated by

$$\nu_\delta = -\mu \frac{P_y^{-1}e_y}{\|e_y\| + \delta}, \quad \delta = 0.3.$$

Solving the convex optimization problem in Theorem 3, we can compute the observer gain for observer (33) as $K_i = 1.08$, for $i \in \mathcal{I}_4$, with the attenuation level $\gamma = 8.1370 \times 10^{-8}$. e_x is estimated by \hat{e}_x , and $\chi = v_y$ is estimated by $\tilde{\chi} = \hat{\chi} + \hat{e}_x$ accordingly. Furthermore, the unknown input d is estimated with (44), and the uncertainty w is estimated with (42).

Hereafter, a realistic driving scenario is considered to demonstrate the estimation performance of the proposed cascade observer scheme. Note that due to the lack of vehicle sensors, the validation tests have been performed with the nonlinear vehicle model (45). However, for each driving scenario, the real data of the steering angle δ and the wheel torque T_w , collected from the INSA autonomous vehicle, are used as vehicle inputs. In this driving scenario, the vehicle follows a random driving trajectory, then enters and exits a roundabout. The corresponding vehicle trajectory, longitudinal speed, and steering angle are depicted in Figs. 2(a), (b) and (c), respectively. The convergence to the sliding surface of the SMO error e_y is illustrated in Fig. 2(d). The estimation performance of the proposed cascade TS fuzzy observer is demonstrated in Fig. 3. The estimation of the SMO error e_x is illustrated in Fig. 3(a). Meanwhile, Fig. 3(b) shows the estimation of the unmeasured lateral speed v_y , Fig. 3(c) shows the estimation of the wheel torque T_w considered as unknown input d , and Fig. 3(d) shows the estimation of the parametric uncertainty w . We can see that the proposed observer scheme provides a satisfactory estimation performance despite the presence of the parametric uncertainty.

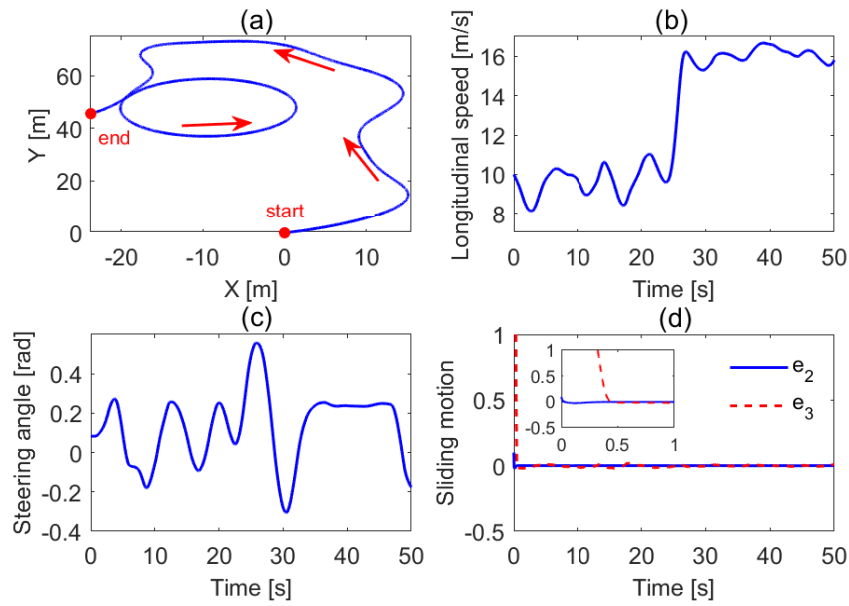


Fig. 2. Example 1: (a) vehicle trajectory X - Y , (b) longitudinal speed, (c) steering angle δ , (d) sliding motion of $e_y = [e_2 \ e_3]^T$.

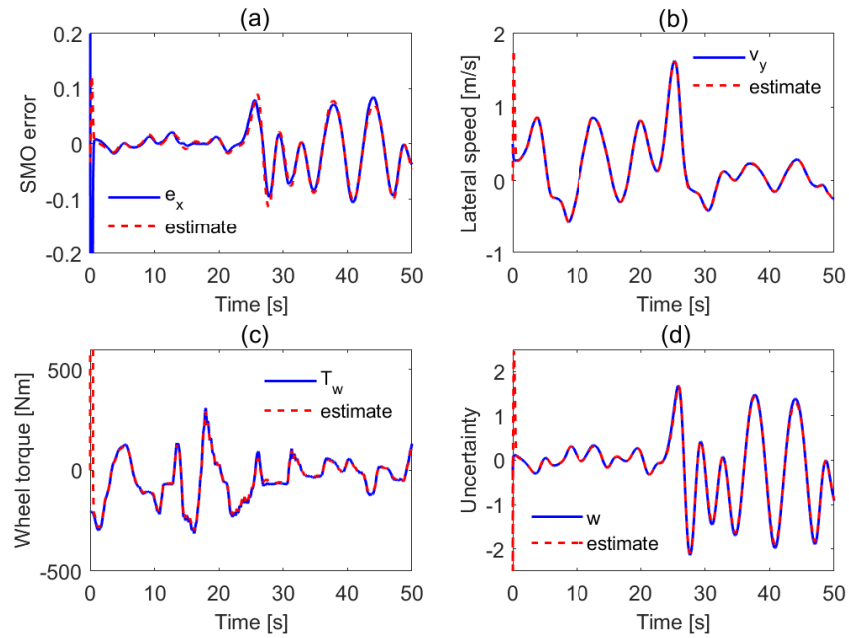


Fig. 3. Example 1: (a) estimation of the SMO error e_x , (b) estimation of the lateral speed v_y , (c) estimation of the wheel torque $d = T_w$, (d) estimation of the uncertainty w .

Example 2. Let us consider a nonlinear systems subject to uncertainty and unknown input which can be represented in the

form (3) where $x = [x_1 \ x_2 \ x_3 \ x_4]^\top$, $\zeta = [x_2 \ x_3]^\top$, and

$$\begin{aligned} A(\zeta) &= \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ -x_3 & -1 & -3 & 1 \\ x_2 & 0 & 0 & -1 \end{bmatrix}, \quad f(u, y) = \begin{bmatrix} u \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ G &= \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} -0.3 & 0.1 \\ 0.2 & 0 \\ -0.1x_3 & -0.1 \\ 0.1x_2 & 0 \end{bmatrix}, \\ g(x) &= x_1^3, \quad C = [0 \ I_3], \quad N = [I_2 \ 0]. \end{aligned} \quad (49)$$

For illustrations, let $u \equiv 1$, $d = 0.1 \sin(0.2t)$, and $F = \text{diag}(\cos(t), \sin(t))$. The state-space compact set is defined as $\mathcal{D}_x = [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1]$. Using the sector nonlinearity method [2] with the vector of measured premise variables $\zeta = [x_2 \ x_3]^\top$, an equivalent four-rule TS fuzzy model of system (49) can be obtained for $x \in \mathcal{D}_x$. The membership functions $h_i(\zeta)$ are given by

$$\begin{aligned} h_1(\zeta) &= \frac{(1-x_3)(1-x_2)}{4}, & h_2(\zeta) &= \frac{(x_3+1)(1-x_2)}{4}, \\ h_3(\zeta) &= \frac{(1-x_3)(x_2+1)}{4}, & h_4(\zeta) &= \frac{(x_3+1)(x_2+1)}{4}. \end{aligned}$$

It is easy to verify that $\sum_{i=1}^4 h_i(\zeta) = 1$, $h_i(\zeta) \geq 0$, $\forall i \in \mathcal{I}_4$, and

$$x_2 = -h_1(\zeta) - h_2(\zeta) + h_3(\zeta) + h_4(\zeta), \quad (50)$$

$$x_3 = -h_1(\zeta) - h_3(\zeta) + h_2(\zeta) + h_4(\zeta). \quad (51)$$

Substituting (50) and (51) into $A(\zeta)$ and $M(\zeta)$, it follows that $A(\zeta) = \sum_{i=1}^4 h_i(\zeta)A_i$ and $M(\zeta) = \sum_{i=1}^4 h_i(\zeta)M_i$, with

$$\begin{aligned} A_1 &= A(-1, -1), & A_2 &= A(-1, 1), & A_3 &= A(1, -1), & A_4 &= A(1, 1), \\ M_1 &= M(-1, -1), & M_2 &= M(-1, 1), & M_3 &= M(1, -1), & M_4 &= M(1, 1). \end{aligned}$$

As $\text{rank}(CD) = \text{rank}(D)$ and $\text{rank}(D) = 1 < 3 = \text{rank}(C)$, Assumption 1 holds. System (49) already has C and D in the form (10)–(11), thus a coordinate transformation is not required. Assumption 2 holds as $g(x) = x_1^3$ is differentiable with respect to x_1 and $0 \leq \frac{\partial g}{\partial x_1} = 3x_1^2 \leq 3$. The sets of vertices are obtained as $\mathcal{V}_{\mathcal{H}_g^1} = \{0, 3\}$, $\mathcal{V}_{\mathcal{H}_g^0} = \{[0 \ 0 \ 0 \ 0], [3 \ 0 \ 0 \ 0]\}$.

Since $M_{21}(\zeta) = \begin{bmatrix} 0.2 & 0 \\ -0.1x_3 & -0.1 \end{bmatrix}$ has non-zero determinant, $M_{21}(\zeta)$ is full column rank and Assumption 3 is verified. It is important to note that, in this example, $M(\zeta)$ and accordingly $[M(\zeta) \ D]$ cannot be factorized into a constant and a varying component. Then, the results in [20], [22] as well as their direct extensions cannot be applied. On the other hand, since $p = 3 = q + n_w$, the SMO scheme in [23] and its direct extension are not applicable to system (49).

Solving the convex optimization problem in Theorem 1, we obtain the observer gain for the sliding mode observer (12) as

$$\begin{aligned} L_1 &= \begin{bmatrix} 0.2287 & 0.2557 & -0.1939 \\ 8.0747 & -0.6136 & -0.5473 \\ -0.6415 & 2.0668 & 0.4986 \\ -0.2441 & 0.3592 & 2.9191 \end{bmatrix}, & L_2 &= \begin{bmatrix} 0.1850 & -0.4010 & -0.0951 \\ 8.0624 & -1.6174 & -0.4579 \\ -0.9885 & 2.0164 & 0.9675 \\ -0.2340 & 0.8682 & 2.8680 \end{bmatrix}, \\ L_3 &= \begin{bmatrix} 0.2254 & 0.1570 & 0.4116 \\ 8.0754 & -0.7237 & 0.4758 \\ -0.6208 & 2.0496 & 0.9564 \\ 0.1711 & 0.9317 & 2.8724 \end{bmatrix}, & L_4 &= \begin{bmatrix} 0.1814 & -0.4994 & 0.5098 \\ 8.0914 & -1.7615 & 0.6510 \\ -0.9789 & 2.0431 & 0.4828 \\ 0.1441 & 0.4175 & 2.9164 \end{bmatrix}, \end{aligned}$$

with the attenuation level $\lambda = 0.1055$. With regard to the discontinuous signal ν in (13), the matrix P_y is computed as

$$P_y = \begin{bmatrix} 0.3734 & 0.0714 & -0.0054 \\ 0.0714 & 1.0912 & -0.1638 \\ -0.0054 & -0.1638 & 1.0076 \end{bmatrix},$$

and μ is computed by (24) as $\mu = 10$. To avoid the chattering, the discontinuous signal ν is approximated by

$$\nu_\delta = -\mu \frac{P_y^{-1} e_y}{\|e_y\| + \delta}, \quad \delta = 0.03.$$

Solving the convex optimization problem in Theorem 3, we can compute the observer gain for the Luenberger-type observer (33) as $K_1 = K_3 = [1 \ 1]$, $K_2 = K_4 = [2 \ 1]$, with the attenuation level $\gamma = 3.6297 \times 10^{-7}$. Since $\gamma \approx 0$, the estimations of e_x , x_1 , d , w are similar to asymptotic estimation.

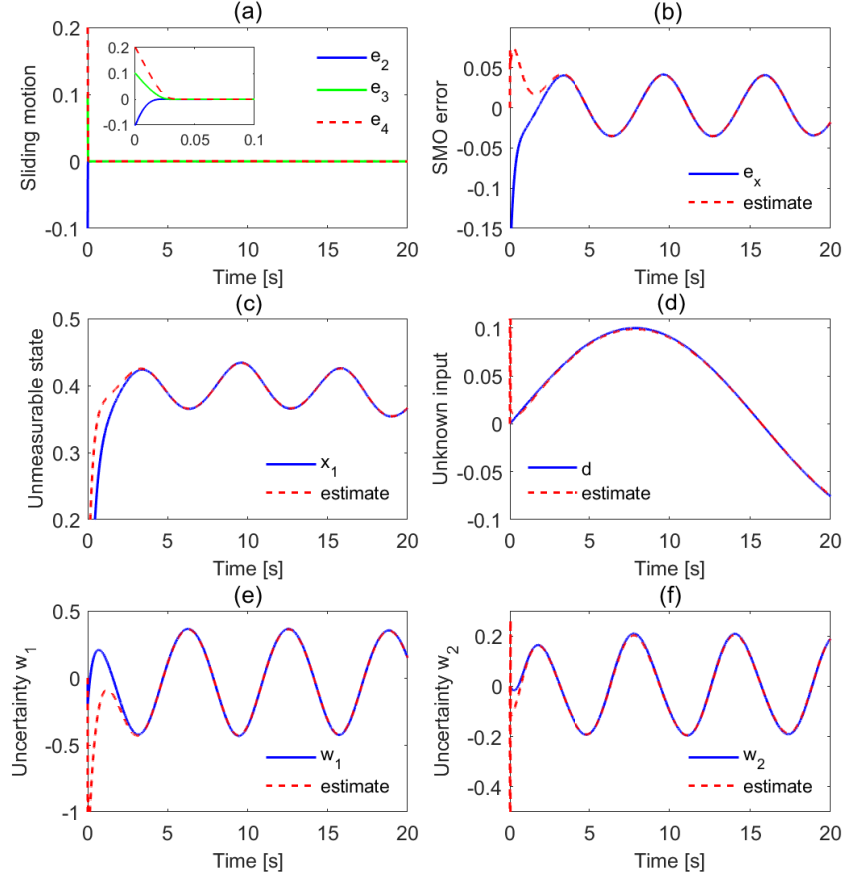


Fig. 4. Example 2: (a) sliding motion of $e_y = [e_2 \ e_3 \ e_4]^T$ (b) estimation of the SMO error e_x , (c) estimation of the unmeasurable state x_1 , (d) estimation of the unknown input d , (e) estimation of the uncertainty w_1 , (f) estimation of the uncertainty w_2 .

The convergence to the sliding surface of the SMO error e_y is illustrated in Fig. 4(a). The estimation of the SMO error e_x is illustrated in Fig. 4(b). Meanwhile, Fig. 4(c) illustrates the estimation of the unmeasured state x_1 , Fig. 4(d) illustrates the estimation of the unknown input d , Figs. 4(e) and (f) illustrate the estimation of the parametric uncertainties w_1 and w_2 . The proposed cascade observer scheme provides a satisfactory estimation performance as shown in Fig. 4.

V. CONCLUDING REMARKS

We have studied the observer design for a class of nonlinear systems with parametric uncertainties and unknown inputs based on TS fuzzy models and sliding mode technique. To this end, two cascaded observers have been constructed. A sliding mode observer is first designed which allows to examine a new system whose both state and output equations are subject to only uncertainties, without unknown inputs. Then, a Luenberger-type observer is designed for the new system where the effect of uncertainties on the estimation error can be eliminated. Accordingly, the overall cascade observer scheme can provide the estimations of the system state as well as the unknown input and the uncertainty. Two numerical examples including an application on the estimation of vehicle nonlinear dynamics have been provided to demonstrate the effectiveness and advantage of the new results. For future works, we focus on the experimental validation of the proposed method with the INSA autonomous vehicle and real vehicle sensors. Investigating systems with output measurements subject to sensor faults is also a promising extension of this paper.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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