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Observer design with exponential time-varying gain for Takagi-Sugeno fuzzy systems with continuous and aperiodic sampled outputs

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Abstract

In this paper, an observer with exponential time-varying gain is proposed for a class of nonlinear systems in Takagi-Sugeno form with both continuous-time and aperiodic sampled outputs. The proposed observer consists of two parts: one related to the continuous-time outputs and one related to the discrete outputs. Sufficient conditions involving a tuning parameter and the maximum allowed sampling time are derived in terms of **quasi linear matrix inequality with respect to a scalar** to ensure asymptotic convergence of the observer on the basis of Lyapunov-Krasovskii functional and H_∞ approach. Numerical simulations are provided to show the benefits of the proposed observer. Some comparisons with the high gain continuous-discrete observer and fuzzy observer with constant gain are provided to illustrate the interest of the proposed scheme.

Keywords: Takagi-Sugeno fuzzy observer, Continuous and aperiodic sampled measurements, Lyapunov-Krasovskii functional, H_∞ approach, Exponential time-varying gain.

1. Introduction

Designing observers for nonlinear dynamical systems is a topic that has attracted considerable attention from researchers over the last decades. Existing works in the literature can be categorized into two separate directions depending on whether the measurements are in continuous-time [1, 2, 3] or in discrete-time [4, 5]. Concerning sampled data based observers, their design was first based on the transformation of the continuous-time model into a discrete-time one, assuming that the measurements are available periodically [4, 5]. However, in practice, the sampling times are often aperiodic. For instance, in networked multi-agent systems, asynchronous and aperiodic transmission of data is inevitable due to communication constraints [6]. For the control of the autonomous train, an unbiased sampled position data can be obtained from beacons installed on the rail, leading to aperiodic measurements [7]. Furthermore, in some applications, different output measurements are not available at the same time instants, leading to a multi-rate system [8, 9, 10]. However, in most of these works (for instance those which use hybrid formalism), the sampling period must be lower bounded by a positive constant to avoid Zeno behavior preventing consideration of continuous-time outputs. Indeed, sometimes, one could either consider some measurements as continuous with respect to time and others as aperiodic sampled measurements. Such scenario may arise in the case of local intermittent information transmissions in multi-agent systems [11], estimation of in-train forces using information from

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GPS sensors and beacons [7], etc. This motivates the study of designing observers for a class of nonlinear systems with both continuous-time and aperiodic sampled outputs.

Several works have proposed sampled-data based observers. For instance, continuous-discrete observers, involving an open-loop continuous-time dynamics between sampling time instants and a discrete-time feedback correction at these instants, have been designed using an output predictor approach [12] or high-gain observers [13]. In [14], an exponential time-varying high-gain observer has been proposed for a class of triangular systems with Lipschitz nonlinearities. This work has been improved by reducing the effect of measurement noise in [15]. The use of exponential time-varying high-gain observers has been further extended for a class of triangular systems with multi-rate sampled and delayed outputs in [16]. In [17], an adaptive mechanism has been added to consider uncertainties. In [6], exponential time-varying high-gain observers has been applied for second order multi-agent systems with aperiodic and asynchronous sampled exchanged data. A limitation in these works concerns the requirement that the system has a triangular form. Moreover, continuous and discrete-time outputs are not considered together.

Other approaches have been proposed without assuming triangular systems. An output predictor has been introduced in [18] where the state is reset to the value of the system output at the sampling times. A new class of hybrid observers for aperiodic sampled data systems has been proposed in [19] using two output injection terms (one acting at the sampling instants and one providing an intersample injection). This idea has been expanded to multi-rate sampled systems in [8]. Exponential time-varying gains for sampled data estimation have been investigated in [20] using Lyapunov-Krasovskii functionals. Nevertheless, in these works, nonlinearities are not taken into account or lead to restrictive conditions.

Another solution for designing controller/observers for nonlinear systems is to use the Takagi-Sugeno (T-S) fuzzy model-based technique [21]. Indeed, T-S fuzzy modeling is an effective way to express exactly nonlinear systems in a compact set of the state variables [22, 23, 24, 25] using for instance sector nonlinearity approach [26]. In [22], the nonfragile H_∞ filtering problem for T-S fuzzy systems has been investigated using an extension of bounded real lemma. In [27], a robust sensor and actuator fault detection observer has been presented for uncertain and disturbed fuzzy systems via a descriptor formulation approach. Many existing works on the observer design for continuous-time T-S fuzzy systems rely on the continuous time measurement of the system [28, 29, 30]. However, they cannot be applied when discrete-time measurements are considered, especially when the measurement period is large. To overcome this problem, various approaches have been considered including discretization methods [31] which gives rather conservative conditions and is only applicable to constant sampling periods and time delay conversion [18, 32] which can be applied to variable sampling periods. In [33], a robust static output H_∞ feedback control for linear systems with polytopic uncertainties has been designed for discrete-time systems without considering aperiodic sampled outputs. In the time delay conversion approach, the discrete-time measurement is converted into a delayed continuous-time output. Then, the stability is analyzed using Lyapunov-Krasovskii functionals. The controller design for T-S systems with sampled-data outputs has been considered in [34, 35]. A sampled-data based fuzzy observer has been designed for nonlinear systems represented in T-S form [36, 37] under aperiodic sampling time instants on the basis of Lyapunov-Krasovskii functionals. In these works, the observer gain is constant and does not depend on the difference between two sampling times. However, it is known that the use of an exponential time-varying gain enables to significantly improve the performance compared to the constant gain case [20].

Motivated by the above practical and theoretical issues, in this paper, a new observer with exponential time-varying gain is proposed for a class of systems in T-S form where both continuous-

time and aperiodic sampled outputs are considered simultaneously. The main challenges of the investigated problem are:

- Disturbances are present in the studied system,
- Aperiodic sampled measurements are considered,
- Both continuous-time and discrete-time outputs are considered.

Due to these challenges, existing schemes cannot be directly applied. The main contributions of this paper are:

- A T-S fuzzy observer is derived to provide an estimate of the system state while considering aperiodic sampled outputs. Contrary to some exiting works, there is no severe assumption concerning the considered class of systems (no severe constraints on the nonlinearities, no need to be in triangular form).
- The proposed observer consists of two parts: one related to the continuous-time outputs and one with an exponential time-varying gain related to the discrete outputs. This allows to increase the performance compared to the use of only discrete-time outputs or the use of constant gain.
- Sufficient conditions involving a tuning parameter and the maximum allowed sampling time are derived in terms of **quasi** linear matrix inequality to ensure asymptotic convergence of the observer based on Lyapunov-Krasovskii functional and H_∞ approach.

This paper is organized as follows. In Section 2, the considered class of systems is described where both continuous-time and aperiodic sampled outputs are considered. Section 3 presents the proposed observer and the convergence analysis. Simulation results using the developed observer are given in Section 4 in order to show the benefits of the proposed approach. A practical example is considered in Section 5. The proposed T-S fuzzy observer with exponential time-varying gain is used to estimate the longitudinal train speed under variable adhesion condition. Some comparisons with the high gain continuous-discrete observer and fuzzy observer with constant gain are provided to illustrate the interest of the proposed scheme.

2. Problem statement

Consider the following T-S fuzzy model-based nonlinear system, which is described by the following rules:

$$\begin{aligned} &\text{RULE } \mathcal{R}_i : \text{ if } z_1(t) \text{ is } \mathcal{M}_1^i \text{ and } \dots \text{ and } z_p(t) \text{ is } \mathcal{M}_p^i \\ &\text{THEN } \left\{ \begin{array}{l} \dot{x}(t) = A_i x(t) + B_i u(t) + G_i w(t) \\ y_1(t_k) = C_1 x(t_k) \\ y_2(t) = C_2 x(t) + D v(t) \end{array} \right. \quad (1) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^{n_w}$ and $v(t) \in \mathbb{R}^{n_v}$ denotes the state vector, system input, system disturbance and measurement disturbance, respectively. A_i , B_i and G_i are known constant matrices with appropriate dimensions. The fuzzy set for premise variable z_j is denoted by \mathcal{M}_j^i in the i th rule where $i = 1, \dots, N$ and $j = 1, \dots, p$ (N is the number of inference rules) and is

measurable. The output consists of a continuous-time part $y_2(t) \in \mathbb{R}^{p_2}$ and a discrete-time part $y_1(t_k) \in \mathbb{R}^{p_1}$ with $p_1 + p_2 \leq n$. Hence, it is assumed that y_1 is only available at aperiodic time instance t_k ($k \in \mathbb{N}$). The sequence of time instances are strictly increasing and such that $t_k \rightarrow \infty$ when $k \rightarrow \infty$. Matrices $C_1 \in \mathbb{R}^{p_1 \times n}$, $C_2 \in \mathbb{R}^{p_2 \times n}$ and $D \in \mathbb{R}^{p_2 \times n_v}$ are constant.

Assumption 1. Let $\tau_k = t_k - t_{k-1}$ be the sampling period for the discrete-time measurements. The sequence of time instants $(t_k)_{k \in \mathbb{N}}$ satisfies $\tau_m < \tau_k < \tau_M$ for all $k \in \mathbb{N}$ where constants $\tau_m > 0$ and $\tau_M > 0$ are respectively the minimum and the maximum sampling period.

Using the center-of-gravity method for defuzzification, the T-S fuzzy system (1) can be written in the compact form

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^N h_i(z) [A_i x(t) + B_i u(t) + G_i w(t)] \\ y_1(t_k) &= C_1 x(t_k) \\ y_2(t) &= C_2 x(t) + D v(t) \end{cases} \quad (2)$$

where $h_i(z)$, $i = 1, \dots, N$ is the membership function given by

$$h_i(z) = \frac{\prod_{j=1}^p \mu_j^i(z_j)}{\sum_{i=1}^N \prod_{j=1}^p \mu_j^i(z_j)}$$

with $\mu_j^i(z_j)$ the membership grade of z_j in the fuzzy set \mathcal{M}_j^i . The membership functions satisfy the convex sum property

$$\begin{aligned} h_i(z(t)) &\geq 0, \\ \sum_{i=1}^N h_i(z(t)) &= 1. \end{aligned} \quad (3)$$

For the design of the T-S observer, the following assumption is considered.

Assumption 2. The pairs $(A_i, [C_1^T \ C_2^T]^T)$ are observable for all $i = 1, \dots, N$. Nonetheless, it is worth noting that pairs (A_i, C_1) and (A_i, C_2) are not supposed to be observable.

The objective of this paper is to design a fuzzy observer that provides a continuous-time estimation of the system state using both continuous-time measurements that are corrupted by a disturbance and aperiodically sampled measurements with a bounded sampling period. Sufficient conditions involving the maximum allowed sampling time are derived in terms of **quasi-linear matrix inequality (quasi-LMI)** to ensure asymptotic convergence of the observer based on Lyapunov-Krasovskii functionals and H_∞ approach.

3. Fuzzy observer design for T-S system with continuous-time and sampled outputs

Using both continuous-time measurement (i.e., $y_2(t)$) and discrete-time measurement (i.e., $y_1(t_k)$), a candidate T-S observer for system (2) is defined as $\forall t \in [t_k, t_{k+1})$,

$$\dot{\hat{x}}(t) = \sum_{i=1}^N h_i(z(t_k)) \left[A_i \hat{x}(t) + B_i u(t) - K_{i2} (C_2 \hat{x}(t) - y_2(t)) - e^{-\eta(t-t_k)} K_{i1} (C_1 \hat{x}(t_k) - y_1(t_k)) \right] \quad (4)$$

where $\hat{x} \in \mathbb{R}^n$ is the state estimate, $K_{i2}, i = 1, \dots, N$ is the constant observer gain matrix related to the continuous-time measurements, and $e^{-\eta(t-t_k)}K_{i1}$ is the exponential time-varying gain related to the aperiodic sampled measurements with $\eta > 0$.

Remark 1. It is worth noting that since pairs (A_i, C_1) and (A_i, C_2) are not supposed to be observable, both continuous-time and discrete-time measurements are needed. The proposed observer (4) uses continuous-time measurements through the term $K_{i2}(C_2\hat{x}(t) - y_2(t))$ where K_{i2} is a suitable output error injection gain and aperiodically sampled measurements through the term $e^{-\eta(t-t_k)}K_{i1}(C_1\hat{x}(t_k) - y_1(t_k))$ with $\eta > 0$. The discrete correction part of the observer involves a time-varying gain $e^{-\eta(t-t_k)}$ with $\eta > 0$. This gain enables to significantly improve the performance compared to the constant gain case (i.e., $\eta = 0$). It can be seen as an output predictor: open-loop prediction when no measurement is available (i.e., between two sampling times) and impulsive correction once a new measurement is available. Such time-varying gain is particularly useful when dealing with aperiodically sampled measurements with large maximum sampling period.

Let us define the observation error as follows

$$e(t) = \hat{x}(t) - x(t). \quad (5)$$

The dynamics of the observation error can be written as, $\forall t \in [t_k, t_{k+1})$

$$\begin{aligned} \dot{e}(t) = & \sum_{i=1}^N h_i(z(t_k)) \left[(A_i - K_{i1}C_1 - K_{i2}C_2)e(t) + K_{i2}Dv(t) + K_{i1}C_1e(t) - e^{-\eta(t-t_k)}K_{i1}C_1e(t_k) \right] \\ & - \sum_{i=1}^N [h_i(z(t)) - h_i(z(t, t_k))] (A_i x(t) + B_i u(t)) - \sum_{i=1}^N h_i(z(t))G_i \omega(t). \end{aligned} \quad (6)$$

Let us introduce the following change of variable:

$$\varepsilon(t) = e^{\eta t} e(t) \quad (7)$$

with $\eta > 0$ a constant. From this change of variable, one can obtain $\forall t \in [t_k, t_{k+1})$

$$\begin{aligned} \dot{\varepsilon}(t) = & \sum_{i=1}^N h_i(z(t_k)) [(\eta I_n + A_i - K_{i1}C_1 - K_{i2}C_2)\varepsilon(t)] + \sum_{i=1}^N h_i(z(t_k))K_{i2}D\bar{v}(t) \\ & + \sum_{i=1}^N h_i(z(t_k))K_{i1}C_1\tilde{\varepsilon}(t, t_k) - \sum_{i=1}^N \tilde{h}_i(t, t_k) (A_i \bar{x}(t) + B_i \bar{u}(t)) - \sum_{i=1}^N h_i(z(t))G_i \bar{\omega}(t) \end{aligned} \quad (8)$$

with $\tilde{\varepsilon}(t, t_k) = \varepsilon(t) - \varepsilon(t_k)$, $\tilde{h}_i(t, t_k) = h_i(z(t)) - h_i(z(t_k))$, $\bar{v}(t) = e^{\eta t} v(t)$, $\bar{x}(t) = e^{\eta t} x(t)$, $\bar{u}(t) = e^{\eta t} u(t)$ and $\bar{\omega}(t) = e^{\eta t} \omega(t)$. It can also be written as follows:

$$\begin{aligned} \dot{\varepsilon}(t) = & \sum_{i=1}^N h_i(z(t_k)) [(\eta I_n + A_i - K_{i1}C_1 - K_{i2}C_2)\varepsilon(t_k)] + \sum_{i=1}^N h_i(z(t_k))K_{i2}D\bar{v}(t) \\ & + \sum_{i=1}^N h_i(z(t_k))(\eta I_n + A_i - K_{i2}C_2)\tilde{\varepsilon}(t, t_k) - \sum_{i=1}^N \tilde{h}_i(t, t_k) (A_i \bar{x}(t) + B_i \bar{u}(t)) - \Delta_G \bar{\omega}(t) \end{aligned} \quad (9)$$

with $\Delta_G = \sum_{i=1}^N h_i(z(t))G_i$.

Remark 2. The considered class of systems given in (1) includes disturbances in the continuous-time measurements (i.e., $v(t)$) and in the system dynamics (i.e., $w(t)$). It is worth noting that the presence of aperiodic sampled measurements (i.e., $y_1(t_k)$) prevents the application of existing classical results. In the following, the unknown terms due to all these factors are grouped into $W(t) = [e^{\eta t} w(t)^T, e^{\eta t} v(t)^T]^T$.

To analyze the stability of system (9), the following lemma will be useful.

Lemma 1. [38] Given any function $\chi(t)$ and any differentiable function $\varepsilon : [r_1, r_2] \rightarrow \mathbb{R}^n$ ($r_1 < r_2$), any matrix N , and positive definite matrix Q with appropriate dimensions, then the following inequality is satisfied:

$$-2\chi^T(t)N \int_{r_1}^{r_2} \varepsilon(s)ds \leq (r_2 - r_1)\chi^T(t)NQ^{-1}N^T\chi(t) + \int_{r_1}^{r_2} \varepsilon^T(s)Q\varepsilon(s)ds. \quad (10)$$

The result concerning the observer convergence can now be established.

Theorem 1. Assuming that Assumptions 1-2 are fulfilled. For a given scalar $\eta > 0$, if there exist a scalar $\sigma_1 > 0$, some symmetric positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$, some matrices $M \in \mathbb{R}^{n \times n}$, $N_i^T = [N_{i1}^T \ N_{i2}^T \ N_{i3}^T \ N_{i4}^T]$ where $N_{ij} \in \mathbb{R}^{n \times n}$, for $i = 1, \dots, N$ and $j = 1, \dots, 3$ and $N_{i4} \in \mathbb{R}^{(n_v \times n)}$, for $i = 1, \dots, N$, $w_{ik} \in \mathbb{R}^{n \times p_k}$, $i = 1, \dots, N$, $k = 1, 2$ such that the following **quasi-LMI** optimization problem is solved:

$$\min \gamma^2 \quad (11)$$

such that

$$\begin{bmatrix} \Phi_i & * \\ N_i^T & -\frac{1}{\tau_M}Q \\ & M & -\Sigma^{-1} \end{bmatrix} < 0 \quad (12)$$

where

$$\Phi_i = \begin{bmatrix} \phi_{11} & * & * & * \\ \phi_{21} & \tau_M Q - \text{sym}(M) & * & * \\ \phi_{31} & \phi_{32} & \phi_{33} & * \\ D^T w_{i2}^T & D^T w_{i2}^T & D^T w_{i2}^T & -\gamma^2 I_{n_v} \end{bmatrix}$$

with

$$\begin{aligned} \phi_{11} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i1}C_1 - w_{i2}C_2] + \sigma_1^{-1}\mu^2 I_n + I_n, \\ \phi_{21} &= P - M + M^T(\eta I_n + A_i) - w_{i1}C_1 - C_2^T w_{i2}^T, \\ \phi_{31} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i2}C_2] - w_{i1}C_1 + N_{i1}^T + \sigma_1^{-1}\mu^2 I_n + I_n, \\ \phi_{32} &= P - M^T + (\eta I_n + A_i)^T M - C_2^T w_{i2}^T + N_{i2}^T, \\ \phi_{33} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i2}C_2 + N_{i3}^T] + \sigma_1^{-1}\mu^2 I_n + I_n, \end{aligned}$$

$$M = \begin{bmatrix} M & M & M & 0_{n \times n_v} \\ M & M & M & 0_{n \times n_v} \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_1^{-1} I_n & 0_n \\ 0_n & \frac{\gamma^2}{\lambda_{\max}(\Delta_G^T(t)\Delta_G(t))} I_n \end{bmatrix}.$$

$\lambda_{\max}(\cdot)$ is the maximum eigenvalue and γ the attenuation level for the H_∞ performance. Then, the observer error system (9) is asymptotically stable if $W(t) = \begin{bmatrix} \bar{\omega}(t) \\ \bar{v}(t) \end{bmatrix} = 0$ and $\|\varepsilon(t)\| \leq \gamma \|W(t)\|$ for $W(t) \neq 0$. Furthermore, the observer gains are given by $K_{ij} = (M^T)^{-1} w_{ij}$, $i = 1, \dots, N$, $j = 1, 2$.

PROOF OF THEOREM. Let us consider the following Lyapunov-Krasovskii candidate functional $V(t) = V_1(t) + V_2(t)$ with

$$V_1(t) = \varepsilon^T(t) P \varepsilon(t) \quad (13)$$

$$V_2(t) = (t_{k+1} - t) \int_{t_k}^t \dot{\varepsilon}^T(s) Q \dot{\varepsilon}(s) ds \quad (14)$$

with $P = P^T > 0$, $Q = Q^T > 0$ and $t \in [t_k, t_{k+1})$.

Then, the time derivative of (13) along the system trajectories is as follows

$$\begin{aligned} \dot{V}_1(t) &= \dot{\varepsilon}^T(t) P \varepsilon(t) + \varepsilon^T(t) P \dot{\varepsilon}(t) \\ &= \dot{\varepsilon}^T(t) P (\tilde{\varepsilon}(t, t_k) + \varepsilon(t_k)) + (\tilde{\varepsilon}(t, t_k) + \varepsilon(t_k))^T P \dot{\varepsilon}(t) \\ &= \begin{bmatrix} \varepsilon^T(t_k) \\ \dot{\varepsilon}^T(t) \\ \tilde{\varepsilon}^T(t, t_k) \\ \bar{v}^T(t) \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & P & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon(t_k) \\ \dot{\varepsilon}(t) \\ \tilde{\varepsilon}(t, t_k) \\ \bar{v}(t) \end{bmatrix} \end{aligned} \quad (15)$$

with $\varepsilon(t) = \tilde{\varepsilon}(t, t_k) + \varepsilon(t_k)$.

From the error system (9), the following equality is always satisfied for any nonsingular matrix $M \in \mathbb{R}^{n \times n}$

$$\begin{aligned} 0 &= 2(M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k))^T [-\dot{\varepsilon}(t) + \sum_{i=1}^N h_i(z(t_k))[(\eta I_n + A_i - K_{i1}C_1 - K_{i2}C_2)\varepsilon(t_k) + K_{i2}D\bar{v}(t) \\ &\quad + (\eta I_n + A_i - K_{i2}C_2)\tilde{\varepsilon}(t, t_k)] - \sum_{i=1}^N \tilde{h}_i(t, t_k) (A_i \bar{x}(t) + B_i \bar{u}(t))] - \Delta_G \bar{\omega}(t)]. \end{aligned} \quad (16)$$

By adding this null term to $\dot{V}_1(t)$, one obtains

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^N h_i(z(t_k)) \begin{bmatrix} \varepsilon^T(t_k) \\ \dot{\varepsilon}^T(t) \\ \tilde{\varepsilon}^T(t, t_k) \\ \bar{v}^T(t) \end{bmatrix}^T \Theta_i \begin{bmatrix} \varepsilon(t_k) \\ \dot{\varepsilon}(t) \\ \tilde{\varepsilon}(t, t_k) \\ \bar{v}(t) \end{bmatrix} + 2(M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k))^T \\ &\quad \left[- \sum_{i=1}^N \tilde{h}_i(t, t_k) (A_i \bar{x}(t) + B_i \bar{u}(t))] - \Delta_G \bar{\omega}(t) \right] \end{aligned} \quad (17)$$

where

$$\Theta_i = \begin{bmatrix} \theta_{11} & * & * & * \\ \theta_{21} & -\text{sym}[M] & * & * \\ \theta_{31} & \theta_{32} & \theta_{33} & * \\ D^T w_{i2}^T & D^T w_{i2}^T & D^T w_{i2}^T & 0 \end{bmatrix}$$

with

$$\begin{aligned} \theta_{11} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i1}C_1 - w_{i2}C_2], \\ \theta_{21} &= P - M + M^T(\eta I_n + A_i) - w_{i1}C_1 - w_{i2}C_2, \\ \theta_{31} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i2}C_2] - w_{i1}C_1, \\ \theta_{32} &= P - M^T + (\eta I_n + A_i)^T M - C_2^T w_{i2}^T, \\ \theta_{33} &= \text{sym}[M^T(\eta I_n + A_i) - w_{i2}C_2], \end{aligned}$$

where $w_{ij} = M^T K_{ij}$ with $i = 1, \dots, N$ and $j = 1, 2$. Let us recall the well-known Jensen's inequality

$$X^T Y + Y^T X \leq \sigma X^T X + \sigma^{-1} Y^T Y \quad (18)$$

in which σ is a positive scalar, and X and Y are any matrices. Furthermore, similarly to [39], there exists $\mu > 0$ such that the following inequality is satisfied for all $i = 1, \dots, N$

$$\|\tilde{h}_i(t, t_k) [A_i \bar{x}(t) + B_i \bar{u}(t)]\| \leq \mu \|\varepsilon(t)\|. \quad (19)$$

Applying inequalities (18)-(19) yield

$$\begin{aligned} & 2(M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k))^T \left[-\sum_{i=1}^N \tilde{h}_i(t, t_k) (A_i \bar{x}(t) + B_i \bar{u}(t)) - \Delta_G(t) \bar{\omega}(t) \right] \\ & \leq (M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k))^T \begin{bmatrix} \sigma_1 I_n & 0 \\ 0 & \sigma_2 I_n \end{bmatrix} (M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k)) \\ & + \sigma_1^{-1} \varepsilon^T(t) \mu^2 \varepsilon(t) + \sigma_2^{-1} \bar{\omega}^T(t) \Delta_G^T(t) \Delta_G(t) \bar{\omega}(t). \end{aligned} \quad (20)$$

Moreover, after some simple manipulations and setting $\chi(t) = [\varepsilon^T(t_k) \quad \dot{\varepsilon}^T(t) \quad \tilde{\varepsilon}^T(t, t_k) \quad \bar{v}^T(t)]^T$, one can obtain

$$\begin{aligned} & (M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k))^T \begin{bmatrix} \sigma_1 I_n & 0 \\ 0 & \sigma_2 I_n \end{bmatrix} (M\varepsilon(t_k) + M\dot{\varepsilon}(t) + M\tilde{\varepsilon}(t, t_k)) \\ & = \chi^T(t) \begin{bmatrix} M & M & M & 0_{n \times n_v} \\ M & M & M & 0_{n \times n_v} \end{bmatrix}^T \begin{bmatrix} \sigma_1 I_n & 0 \\ 0 & \sigma_2 I_n \end{bmatrix} \begin{bmatrix} M & M & M & 0_{n \times n_v} \\ M & M & M & 0_{n \times n_v} \end{bmatrix} \chi(t) \\ & = \chi^T(t) \mathcal{M}^T \Sigma \mathcal{M} \chi(t). \end{aligned} \quad (21)$$

Hence, the time derivative of V_1 along the system trajectories can be bounded as follows:

$$\dot{V}_1(t) \leq \sum_{i=1}^N h_i(z(t_k)) \chi^T(t) \left[\Theta_i + \mathcal{M}^T \Sigma \mathcal{M} \right] \chi(t) + \sigma_1^{-1} \varepsilon^T(t) \mu^2 \varepsilon(t) + \sigma_2^{-1} \bar{\omega}^T(t) \Delta_G^T(t) \Delta_G(t) \bar{\omega}(t). \quad (22)$$

The time derivative of (14) is as follows

$$\dot{V}_2(t) = (t_{k+1} - t)\dot{\varepsilon}^T(t)Q\dot{\varepsilon}(t) - \int_{t_k}^t \dot{\varepsilon}^T(s)Q\dot{\varepsilon}(s)ds \quad (23)$$

Considering the following null term:

$$0 = 2 \sum_{i=1}^N h_i(z(t_k)) \left[\chi^T(t)N_i \left(\tilde{\varepsilon}(t, t_k) - \int_{t_k}^t \dot{\varepsilon}(s)ds \right) \right]$$

where $N_i = \begin{bmatrix} N_{i1}^T & N_{i2}^T & N_{i3}^T & N_{i4}^T \end{bmatrix}^T$, with N_{ij} free matrices with appropriate dimensions, $i = 1, \dots, N$ and $j = 1, 2$ and using Lemma 1, Equation (23) becomes

$$\dot{V}_2(t) \leq (t_{k+1} - t)\dot{\varepsilon}^T(t)Q\dot{\varepsilon}(t) + 2 \sum_{i=1}^N h_i(z(t_k)) [\chi^T(t)N_i\tilde{\varepsilon}(t, t_k) + (t - t_k)\chi^T(t)N_iQ^{-1}N_i^T\chi(t)] \quad (24)$$

or equivalently

$$\dot{V}_2(t) \leq \sum_{i=1}^N h_i(z(t_k))\chi^T(t) \left[\psi_i + \tau_M N_i Q^{-1} N_i^T \right] \chi(t) \quad (25)$$

with

$$\psi_i = \begin{bmatrix} 0 & * & * & * \\ 0 & \tau_M Q & * & * \\ N_{i1}^T & N_{i2}^T & \text{sym}(N_{i3}) & * \\ 0 & 0 & N_{i4}^T & 0 \end{bmatrix}.$$

Considering (22) and (25), the time derivative of $V(t)$ is as follows

$$\begin{aligned} \dot{V}(t) \leq & \sum_{i=1}^N h_i(z(t_k))\chi^T(t) \left[\Phi_i + \tau_M N_i Q^{-1} N_i^T + M^T \Sigma M \right] \chi(t) + \sigma_1^{-1} \varepsilon^T(t) \mu^2 \varepsilon(t) \\ & + \sigma_2^{-1} \bar{\omega}^T(t) \Delta_G^T(t) \Delta_G(t) \bar{\omega}(t) \end{aligned} \quad (26)$$

where $\Phi_i = \Theta_i + \psi_i$.

From **quasi-LMI** (12), when the unknown term $W(t) = \begin{bmatrix} \bar{\omega}(t) \\ \bar{v}(t) \end{bmatrix} = 0$, the observer error (9) tends to zero asymptotically.

Using an H_∞ criterion, one investigates the stability analysis when $\|\varepsilon(t)\| \leq \gamma \|W(t)\|$ for $W(t) \neq 0$ where $\gamma > 0$ is the attenuation level to be determined.

Adding $\varepsilon^T(t)\varepsilon(t) - \gamma^2 W^T(t)W(t)$ into $\dot{V}(t)$ results in

$$\begin{aligned} \dot{V}(t) + \varepsilon^T(t)\varepsilon(t) - \gamma^2 W^T(t)W(t) \leq & \sum_{i=1}^N h_i(z(t_k))\chi^T(t) \left[\Phi_i + \tau_M N_i Q^{-1} N_i^T + M^T \Sigma M \right] \chi(t) + \varepsilon^T(t)\varepsilon(t) \\ & + \sigma_1^{-1} \varepsilon^T(t) \mu^2 \varepsilon(t) - \gamma^2 \bar{v}(t)^T \bar{v}(t) + \bar{\omega}^T(t) (\sigma_2^{-1} \Delta_G^T(t) \Delta_G(t) - \gamma^2) \bar{\omega}(t) \end{aligned} \quad (27)$$

One can set σ_2 as follows

$$\sigma_2 = \frac{\lambda_{\max}(\Delta_G^T(t) \Delta_G(t))}{\gamma^2}. \quad (28)$$

Hence, one can rewrite (27) as follows

$$\dot{V}(t) + \varepsilon^T(t)\varepsilon(t) - \gamma^2 W(t) \leq \sum_{i=1}^N h_i(z(t_k)) \chi^T(t) \Psi \chi(t) \quad (29)$$

where

$$\Psi = \left[\Phi_i + \tau_M N_i Q^{-1} N_i^T + \mathcal{M}^T \Sigma \mathcal{M} + \begin{bmatrix} \sigma_1^{-1} \mu^2 I_n + I_n & 0 & \sigma_1^{-1} \mu^2 I_n + I_n & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_1^{-1} \mu^2 I_n + I_n & 0 & \sigma_1^{-1} \mu^2 I_n + I_n & 0 \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} \end{bmatrix} \right] \quad (30)$$

Hence, to guarantee asymptotic stability of the observer error (9) when $\|\varepsilon(t)\| \leq \gamma \|W(t)\|$ for $W(t) \neq 0$, it is sufficient to have

$$\Phi_i + \tau_M N_i Q^{-1} N_i^T + \mathcal{M}^T \Sigma \mathcal{M} + \begin{bmatrix} \sigma_1^{-1} \mu^2 I_n + I_n & 0 & \sigma_1^{-1} \mu^2 I_n + I_n & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_1^{-1} \mu^2 I_n + I_n & 0 & \sigma_1^{-1} \mu^2 I_n + I_n & 0 \\ 0 & 0 & 0 & -\gamma^2 I_{n_v} \end{bmatrix} < 0. \quad (31)$$

Applying the Schur-complement to (31), one obtains the **quasi-LMI** (12).

□

Remark 3. If all the system measurements are available in continuous-time (i.e. $C_1 = 0$) and (A_i, C_2) is observable, observer (4) can be reduced to a T-S Luenberger observer by setting $K_{i1} = 0$ and $L = K_{i2}$, i.e.,

$$\dot{\hat{x}}(t) = \sum_{i=1}^N h_i(z(t_k)) [A_i \hat{x}(t) + B_i u(t) - L C_2 (\hat{x}(t) - x(t))].$$

Theorem 1, in this case, is reduced to the computation of gain K_{i2} via **quasi-LMI** (12) and ensures local asymptotic convergence of the estimation error.

Remark 4. If all the system measurements are aperiodically sampled (i.e. $C_2 = 0$) and the pair (A_i, C_1) is observable, observer (4) can be reduced to a sampled data based fuzzy observer by setting $K_{i2} = 0$, i.e.,

$$\dot{\hat{x}}(t) = \sum_{i=1}^N h_i(z(t_k)) [A_i \hat{x}(t) + B_i u(t) - K_{i1} e^{-\eta(t-t_k)} C_1 (\hat{x}(t_k) - x(t_k))]. \quad (32)$$

For comparison purposes, some specific cases could be considered:

- Considering $N = 1$ and the case without sensor disturbance (i.e., $\omega(t) = 0$), observer (32) can be reduced to an exponential time-varying high gain observer similar to [14] by setting $\eta = \theta \geq 1$ and $K_{i1} = \theta \Delta_\theta^{-1} L$ where $\Delta_\theta = \text{diag} \left[1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}} \right]$ and L is such that $(A - LC_1)$ is Hurwitz, i.e.,

$$\dot{\hat{x}}(t) = A \hat{x}(t) + B u(t) - \theta \Delta_\theta^{-1} L e^{-\theta L_1(t-t_k)} C_1 (\hat{x}(t_k) - x(t_k)). \quad (33)$$

- Considering $\eta = 0$, observer (32) can be reduced to a fuzzy observer with constant observer gain similarly to [36, 37], i.e.,

$$\hat{x}(t) = \sum_{i=1}^N h_i(z(t_k)) [A_i \hat{x}(t) + B_i u(t) - K_{i1} C_1 (\hat{x}(t_k) - x(t_k))]. \quad (34)$$

However, it is known that the use of an exponential time-varying gain enables to significantly improve the performance compared to the constant gain case.

Remark 5. To ensure stability of the observation error using the fuzzy observer with exponentially time varying gain (4), parameter η should be appropriately selected. It is worth noting that this parameter mainly depends on the maximum sampling period τ_M . Roughly speaking, when τ_M increases, η should be decreased. **The design conditions in Theorem 1 are expressed in terms of quasi-LMI condition with a respect to the scalar η . Note that, in the literature, several solutions are available to deal with this problem, notably using the general eigenvalue problems (GEVP) [40]. A gridding method can also be performed to search for η by constituting a grid of values around a fixed value, e.g., one [41], or fixing them previously [42, 43, 44, 45, 46]. Then, the optimization problem (11) under LMI constraint (12) can be effectively solved with available numerical solvers.**

4. Numerical example

The performance of the proposed observer will be illustrated through the following example

$$\begin{cases} \dot{x}(t) &= A(x(t))x(t) + Bu(t) \\ y_1(t_k) &= C_1 x(t_k) \\ y_2(t) &= C_2 x(t) + Dv(t) \end{cases} \quad (35)$$

where $x(t) = [x_1 \ x_2 \ x_3 \ x_4]^T \in \mathbb{R}^4$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $y_1(t_k) \in \mathbb{R}$ and $y_2(t) \in \mathbb{R}$ denotes respectively the discrete-time and the continuous-time outputs, and

$$A(x(t)) = \begin{bmatrix} -1.25 & -48.6 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -19.5 & -19.5 & -3.3(\sin(x_1(t)) + x_2^2(t)) \end{bmatrix},$$

$$B = [21.6 \ 0 \ 0 \ 0]^T,$$

$$C_1 = [1 \ 0 \ 0 \ 0],$$

$$C_2 = [0 \ 1 \ 0 \ 0],$$

$$D = 1.$$

It is worth noting that example (35) includes nonlinearities (i.e., $\sin(x_1)$ and x_2^2). The measurement fault $v(t) \in \mathbb{R}$ is bounded. Furthermore, it is obvious that the function $\sin(y_1(t))$ is bounded

as $y_{1min} \leq \sin(y_1(t)) \leq y_{1max}$ with $y_{1min} = -1$ and $y_{1max} = 1$. Moreover, one supposes that the continuous-time output is bounded such that $y_{2min} \leq y_2(t) \leq y_{2max}$ with $y_{2min} = -4$ and $y_{2max} = 4$. Hence, system (35) can be written in the form of (2), i.e.,

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^2 h_i(z(t)) [A_i x(t) + B_i u(t) + G_i w(t)] \\ y_1(t_k) &= C_1 x(t_k) \\ y_2(t) &= C_2 x(t) + Dv(t) \end{cases} \quad (36)$$

where $z(t) = [y_1(t) \ y_2(t)]$ is the measurable premise variable. The membership functions ($N = 2$) which satisfy the convex sum property (3), are given by

$$h_1(z(t)) = \frac{(y_{1max} + y_{2max}) - [\sin(y_1(t)) + y_2^2(t)]}{(y_{1max} + y_{2max}) - (y_{1min} + y_{2min})}$$

$$h_2(z(t)) = 1 - \frac{(y_{1max} + y_{2max}) - [\sin(y_1(t)) + y_2^2(t)]}{(y_{1max} + y_{2max}) - (y_{1min} + y_{2min})}$$

and

$$A_1 = \begin{bmatrix} -1.25 & -48.6 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -19.5 & -19.5 & -3.3(y_{1min} + y_{2min}) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.25 & -48.6 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -19.5 & -19.5 & -3.3(y_{1max} + y_{2max}) \end{bmatrix},$$

$B_1 = B_2 = B$, $G_1 = G_2 = [0 \ 0 \ 0 \ 1]^T$ and $w(t) = -3.3(v^2(t) - 2y_2(t)v(t))$.

It can be noted that pairs $(A_i, [C_1 \ C_2])$ ($i = 1, 2$) are observable. The optimization problem (11) under **quasi-LMI** constraint (12) is solved using CVX solver within MATLAB R2022b environment. **A gridding method is performed to search for η** . The observer gains K_{i1} and K_{i2} ($i = 1, 2$) are computed according to the conditions of Theorem 1 with $\tau_M = 0.2s$ and are given as

$$K_{11} = [9.1305 \ 0.4390 \ -7.2803 \ 288.8461]^T,$$

$$K_{12} = [-48.3 \ 18.9 \ -117.5 \ 2606.1]^T,$$

$$K_{21} = [9.1304 \ 0.4509 \ -7.3666 \ 278.8907]^T,$$

$$K_{22} = [-48.3 \ 19 \ -118.2 \ 2098.7]^T,$$

$$\eta = 1.$$

In the following simulation results, one considers an aperiodic sampling time as presented in Figure 1. It is worth noting that the first sampling times, when $t < 6.4s$, fulfill the condition $t_k - t_{k+1} \leq 0.2$. Nevertheless, in order to test the performance of the proposed observer in the case of loss of information during a certain period of time, this theoretical condition on τ_k is not always verified after 6.4s.

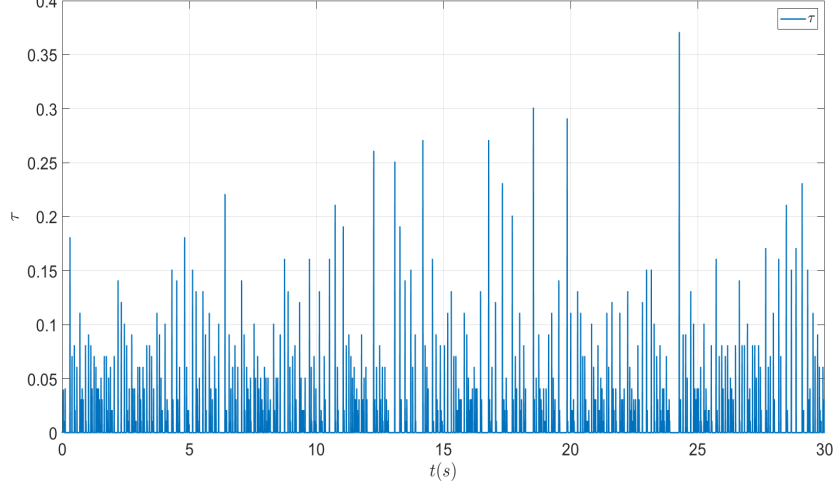


Figure 1: Aperiodic sampling period for discrete-time measurement y_1 .

The initial conditions of the system have been set all to 1 while the initial conditions of the observer have been arbitrarily set to zero.

In the following, the proposed fuzzy observer (4), labelled in the following as *FuzzyCDO*, is compared with the high gain continuous-discrete observer, labelled as *CDO*, proposed in [14] and the fuzzy observer with constant observer gain similarly to [36, 37].

Remark 6. The assumptions considered when designing the proposed fuzzy observer are similar to the assumptions of the high-gain continuous-discrete observer presented in [14], i.e.,

- The considered system is a nonlinear system with aperiodically sampled measurements with a bounded sampling period.
- The input of the system is considered bounded.
- The measurement disturbances are also considered bounded.

However, for observability reason, it is needed to consider both y_1 and y_2 . Therefore, information from the continuous-time measurement y_2 is used at the same sampling time t_k . Hence, the following output matrix is considered as $\tilde{C}_1 = [C_1^T \ C_2^T]^T$. The *CDO* is given as follows

$$\hat{\hat{x}}(t) = \tilde{A}\hat{\hat{x}}(t) + Bu(t) + \tilde{g}(\hat{\hat{x}}) - \theta\Delta_\theta^{-1}Le^{-\theta L_1(t-t_k)}\tilde{C}_1(\hat{x}(t_k) - x(t_k)) \quad (37)$$

with $\tilde{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\tilde{g}(\hat{\hat{x}}) = \begin{bmatrix} -1.25\hat{x}_1(t) - 48.6\hat{x}_2(t) \\ \hat{x}_1(t) \\ \hat{x}_2(t) \\ -19.5\hat{x}_2(t) - 19.5\hat{x}_3(t) - 3.3(\sin(\hat{x}_1(t)) + \hat{x}_2^2(t))\hat{x}_4(t) \end{bmatrix}$. The gain $L = [L_1, L_2]^T$ (with L_1 a square matrix) of the observer is chosen such that matrix $(\tilde{A} - L\tilde{C}_1)$

is Hurwitz and $\theta \geq 1$. Here, one sets $\theta = 1$ and $L = \begin{bmatrix} 0.4034 & 0 \\ 0 & 0.4020 \\ 0.0407 & 0 \\ 0 & 0.0404 \end{bmatrix}$. The high gain approach

is used to compensate for the effect of $\tilde{g}(x)$ on the dynamics of the estimation error. However, the high gain amplify the effect of measurement disturbances compared to the proposed fuzzy observer.

The proposed approach is also compared with the fuzzy observer with constant observer gain, labelled in the following *Fuzzy*, given in (34) (see [36, 37]).

Figure 2-Figure 5 show the estimation error $e = [e_1, e_2, e_3, e_4]^T = \hat{x} - x$ between the system state and the estimated state obtained from the proposed fuzzy observer (*FuzzyCDO*), from the high gain continuous-discrete observer (*CDO*) and from the fuzzy observer with constant observer gain (*Fuzzy*). It is worth noting that the estimated state quickly converges to the system state. The norm of the estimation errors is presented in Figure 6.

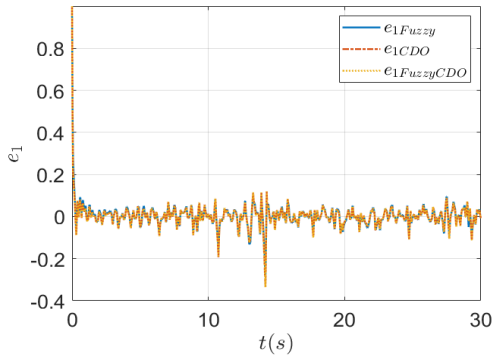


Figure 2: State estimation for x_1 .

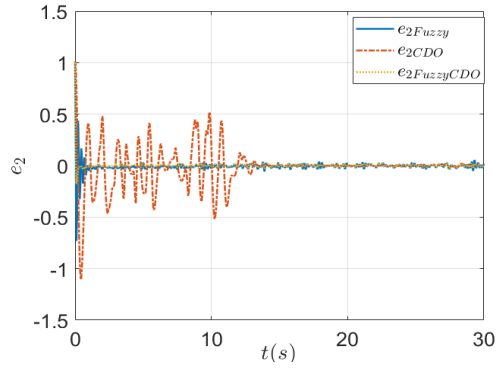


Figure 3: State estimation for x_2 .

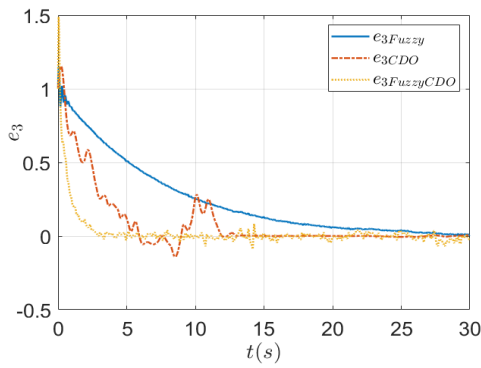


Figure 4: State estimation for x_3 .

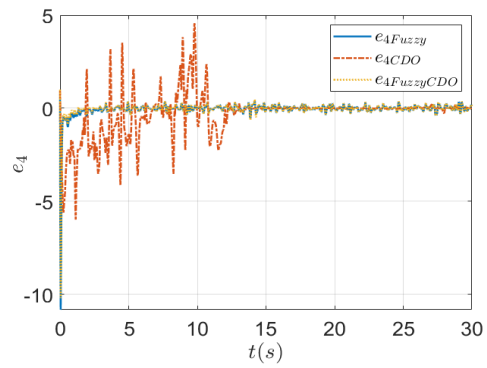


Figure 5: State estimation for x_4 .

Compared to the high gain continuous-discrete observer (*CDO*) where the decay rate of the error dynamics cannot be easily selected due to inter-dependence between observer parameters

L and θ and the maximum sampling period, the proposed fuzzy observer benefits from an easier tuning of the decay rate since the time-varying gain is simpler due to its dependence on the scalar parameter η . Indeed, it is worth noting that the decay rate of the estimation error can be easily adjusted from the **quasi-LMIs** conditions given in Theorem 1 based on the maximum allowed sampling time contrary to the high gain continuous-discrete observer (37).

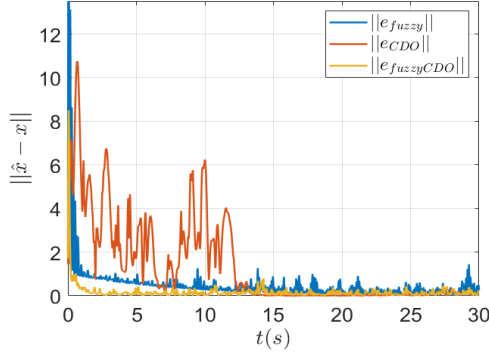


Figure 6: Estimation error norm.

Compared to the fuzzy observer (*Fuzzy*), the presence of exponential time-varying gain allows to increase the performances in terms of accuracy and convergence speed (see Fig. 4 and Fig. 5).

5. Application to longitudinal train speed estimation under variable adhesion condition

5.1. Modeling and observer design

The longitudinal train dynamics can be described as follows

$$\begin{cases} \dot{p}(t) &= v(t) \\ m\dot{v}(t) &= F(t) - F_r(v(t)) \end{cases} \quad (38)$$

where $v(t)$ and $p(t)$ represents respectively the train longitudinal velocity and position, m is the total mass of the train, $F(t)$ is the traction/braking force. $F_r(v)$ represents the basic resistance approximated by the Davis formula, i.e.,

$$F_r(v) = a + bv(t) + cv(t)^2 \quad (39)$$

where a , b and c are positive constants.

Let y_1 be the continuous-time measurement and y_2 be the discrete-time one, given as

$$\begin{cases} y_1(t_k) = p(t_k) \\ y_2(t) = v(t) + \delta_v(t) \end{cases} \quad (40)$$

where δ_v is the speed measurement bias due to variable adhesion condition and $p(t_k)$ is train position which is available only at aperiodic sampling times. The output $y_2(t)$ is bounded by known constants, i.e., $y_{2min} \leq y_2 \leq y_{2max}$.

Considering the train dynamics (38), the Davis formula (39) and the outputs (40), the considered system can be written as

$$\begin{cases} \dot{x}(t) &= A(x(t))x(t) + Bu(t) \\ y_1(t_k) &= C_1x(t_k) \\ y_2(t) &= C_2x(t) + Dv(t) \end{cases} \quad (41)$$

with $x(t) = [p(t) \quad v(t)]^T$, $u = F - a$,

$$\begin{aligned} A(x(t)) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} - \frac{c}{m}x_2(t) \end{bmatrix} \\ B &= \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix}^T \\ D &= 1 \\ C_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

System (41) can be expressed in the form of system (2) with the measurable premise variable $z = [y_1(t) \quad y_2(t)]^T$ and membership functions $h_1(z) = \frac{y_{2max} - y_2(t)}{y_{2max} - y_{2min}}$, $h_2(z) = 1 - h_1(z)$. The perturbation is given as $w(t) = -\delta_v^2(t) + 2y_2(t)\delta_v(t)$ and $G_1 = G_2 = G = [0 \quad 1]^T$. The matrices A_i are defined as $A_1 = A(y_{2min})$ and $A_2 = A(y_{2max})$ and $B_1 = B_2 = B$.

It can be noted that pairs $(A_i, [C_1 \quad C_2])$ ($i = 1, 2$) are observable.

5.2. Numerical simulations

For simulation, the parameters of the train model are: $m = 90000kg$, $a = 900N$, $b = 32N.s.m^{-1}$, $c = 1.7N.s^2.m^{-1}$. For simplicity, a parallel PI controller was applied with $K_p = 7.2 \times 10^5$ and $K_i = 4.5 \times 10^5$. In the following, $y_{2min} = 5m/s$ and $y_{2max} = 50m/s$. The observer gains $\eta = 3$, K_{i1} and K_{i2} ($i = 1, 2$) are computed according to the conditions of Theorem 1 with $\tau_M = 0.5s$. In the following simulations, one considers an aperiodic sampling time as presented in Figure 7

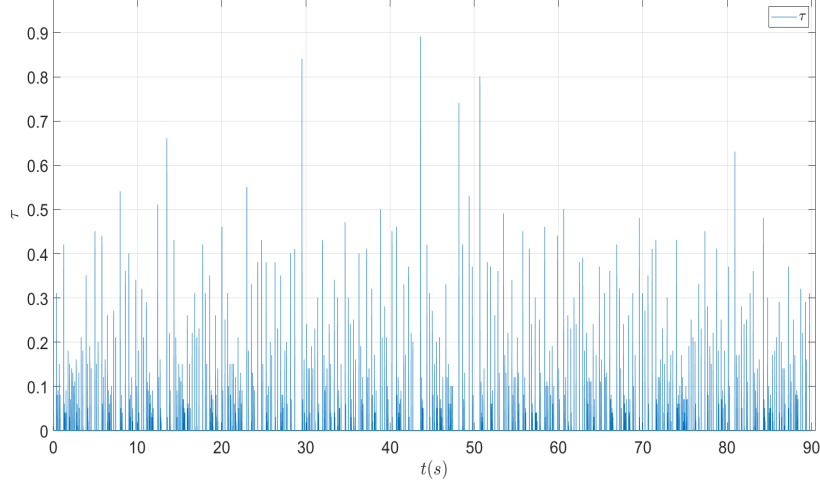


Figure 7: Aperiodic sampling period for discrete-time measurement y_1 .

As in the previous example, the observer performances are tested in the case of loss of information during a certain period of time which does not verify the condition $\tau_M \leq 0.5s$. The initial conditions of the system have been set all to 0 while the initial conditions of the observer have been arbitrarily set to $[0 \ 1]^T$.

As shown in Figure 8-Figure 11, the estimated speed converges to the actual value of the train speed with a maximum error of $0.25m/s$ in the traction and braking phases which correspond to the time instances when there is a variation of the acceleration. At time instants $t \in [10, 20]s$ and $t \in [70, 80]s$, the train crosses a low adhesion zone which causes wheel slippage that biases the speed measurement. However, the proposed observer still gives a good estimate of the longitudinal speed of the train with a maximum error of $0.35m/s$ in traction phase and $0.5m/s$ in braking phase. Therefore, it is possible to design a detector of excessive wheel slippage that corresponds to the low grip zones (see Figure 12).

In the following, the fuzzy observer (4) (*FuzzyCDO*), is compared with the high gain continuous-discrete observer (*CDO*) and the fuzzy observer with constant observer gain (*Fuzzy*) by imposing $\eta = 0$. Similarly to the previous section, the following output matrix is considered as $\tilde{C}_1 = [C_1^T \ C_2^T]^T$. The *CDO* has a similar structure as in (37) with $\theta = 2$ and $L = [0.9807 \ 1.3269]^T$.

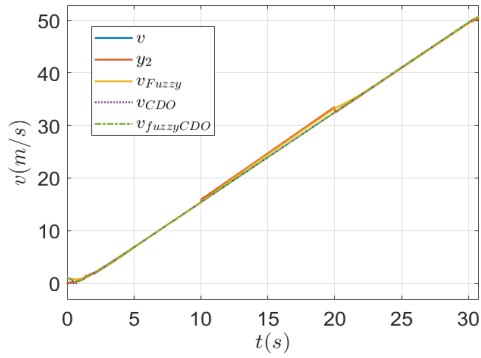


Figure 8: Train speed estimation in traction phase.

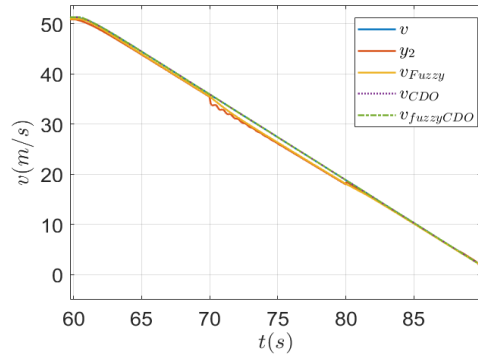


Figure 9: Train speed estimation in braking phase.

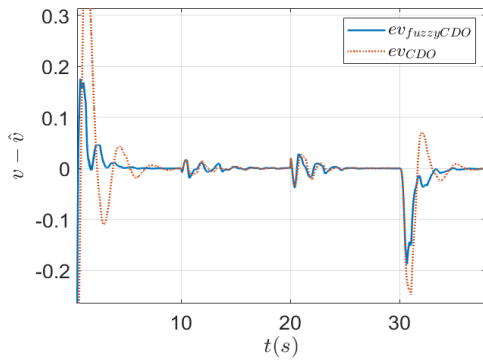


Figure 10: Estimation error in traction phase.

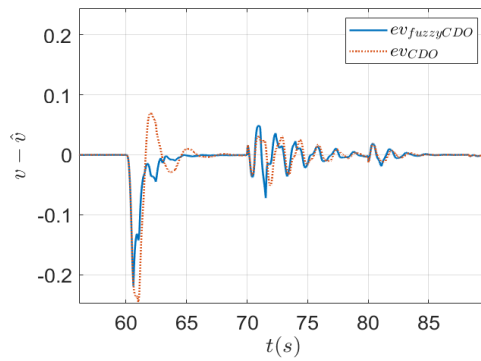


Figure 11: Estimation error in braking phase.

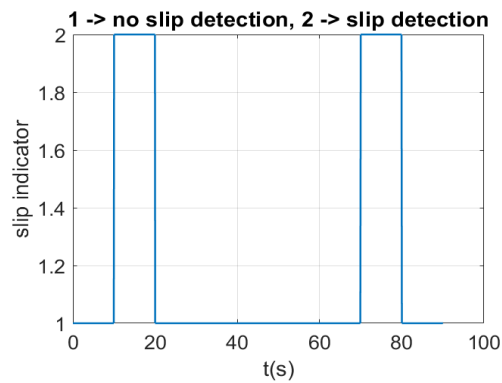


Figure 12: Slip detection using the proposed fuzzy observer.

The estimation error norm obtained using the proposed fuzzy observer *FuzzyCDO*, the *CDO* and that *Fuzzy* observer are depicted in Figure 13-Figure 16. From these figures, one can clearly

see the interest of the use of exponential time-varying gain.

For comparison purpose, one can give the Root Mean Square Error (RMSE) using the different approaches. It is equal to 0.75 for the *CDO*, 4.2886 for the *Fuzzy* observer and 0.535 for the proposed *FuzzyCDO* in the traction phase.

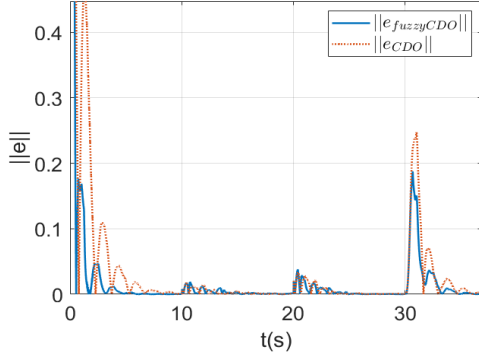


Figure 13: Estimation error norm in traction phase.

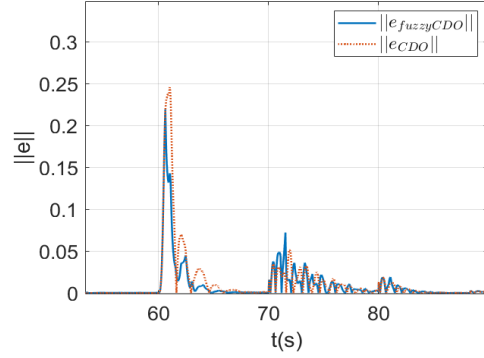


Figure 14: Estimation error norm in braking phase.

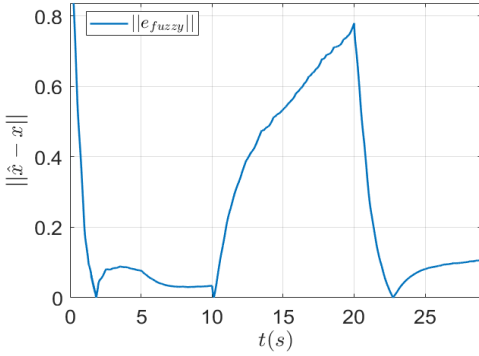


Figure 15: Estimation error norm in traction phase (fuzzy approach using constant gain).

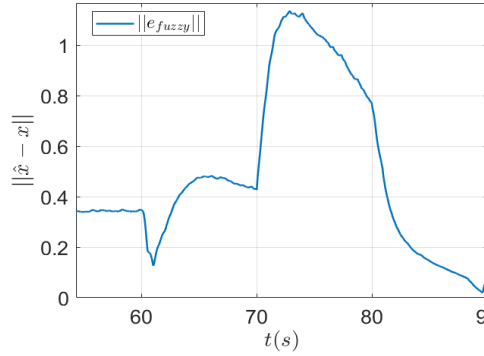


Figure 16: Estimation error norm in braking phase (fuzzy approach using constant gain).

6. Conclusion

This paper proposed a fuzzy observer for the state estimation of a class of nonlinear systems with both continuous-time and aperiodic sampled outputs. The proposed observer with exponential time-varying gains consists of two parts: one related to the continuous-time outputs and the other one related to the discrete outputs. It is shown that this allows to increase the performance compared to the use of only discrete-time outputs. Sufficient tractable conditions involving a tuning parameter and the maximum allowed sampling time have been derived in terms of **quasi-LMI with respect to a scalar number** to ensure asymptotic convergence of the observer based on Lyapunov-Krasovskii functional and H_∞ approach. Numerical simulations were provided to

show the benefits of the proposed observer. In the future, it is expected to consider a larger class of systems with output matrices depending on the premise variables.

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