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A model reconstruction approach for control synthesis of Takagi-Sugeno fuzzy systems

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Abstract

This paper investigates the control synthesis for Takagi-Sugeno (TS) fuzzy systems via a model reconstruction approach. Firstly, based on a nonparallel distributed compensation (NPDC) control scheme and a fuzzy Lyapunov function candidate (FLF), LMI-based synthesis conditions are derived to guarantee the asymptotic stability of the closed-loop TS fuzzy system. Then the operation domain of the premise variables is divided into several subregions to provide more flexibility in control design. And a new local model reconstruction method is proposed to transform the time-derivative of the FLF into new fuzzy forms with transformed membership functions in each subregion. This enables better exploitation of information on membership functions, resulting in relaxed conditions for both stabilization and robust $\mathscr{H}_\infty$ control. Two simulation examples are provided to verify the advantages of the proposed TS fuzzy control approach.

Keywords: Takagi-Sugeno fuzzy systems; Model reconstruction; Conservatism; Fuzzy Lyapunov functions

1. Introduction

Over the last decades, TS fuzzy model-based approaches have become one of the most promising frameworks to deal with stability analysis, controller and observer synthesis for nonlinear systems [1]. This is because TS fuzzy modeling can be used to approximate any smooth nonlinear system with any degree of accuracy. In particular, TS fuzzy models are constructed by blending local linear submodels together with nonlinear membership functions (MFs) [2]. The resulting polytopic structure of TS fuzzy models permits some possible extensions of results from linear control theory to nonlinear systems [3–7].

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For TS fuzzy modeling, two main approaches have been used to derive TS fuzzy models from nonlinear systems: sector nonlinearity approach [8] and local linearization approach [9]. Using Lyapunov stability theory and TS fuzzy modeling, sufficient conditions can be derived for stability analysis and controller/observer synthesis of nonlinear systems in the form of linear matrix inequalities (LMIs) which can be effectively solved with standard numerical solvers [1]. However, one of the major challenges of TS fuzzy model-based approaches consists in reducing the design conservatism, which is unavoidable within TS fuzzy framework [10]. For fuzzy model-based control, it has been demonstrated that the choice of various controller forms and Lyapunov function candidates plays a central role to solve this challenge [11]. The parallel distributed compensation (PDC) scheme, in which the fuzzy controller shares same MFs with the fuzzy system, has been widely adopted [12–15]. To reduce the conservatism of the control results, non-parallel distributed compensation (NPDC) approaches have been investigated [16–18]. For these approaches, controller MFs that differ from the plant MFs can be utilized. This technique has been exploited in various situations, NPDC controller has been developed for fuzzy descriptor systems in [19]; a generalized non-quadratic parameter-dependent controller has been proposed in [20]. However, there is still no systematic way to derive MFs for NPDC controller design. To deal with this control issue, a class of switching controllers, depending on the division of the premise variable operation domain, can be found in [21–23], which yielded more relaxed control results.

Quadratic Lyapunov functions (QLF) have been widely exploited in early works to derive sufficient LMI conditions for stability analysis and control synthesis of TS fuzzy systems [24,25]. It is important to note that it is hard to find a common Lyapunov matrix that satisfies the stability conditions of all local subsystems of the TS fuzzy system. To reduce the conservatism implied by QLF-based approaches, a piecewise Lyapunov function (PLF) approach has been proposed in [26–29]. Switching Lyapunov function (SLF) based approaches have been discussed in [30,31], which allows the local Lyapunov function switches from one subregion to another. Several other approaches have been also proposed to reduce the design conservatism, e.g., fuzzy Lyapunov function based method [32–34], line-integral Lyapunov function based method [35], polynomial fuzzy Lyapunov function based method [36] for polynomial fuzzy systems, see [10] for a recent survey. However, how to involve the information on the membership functions into stability conditions to reduce the conservatism has not been well addressed in the literature. For this aspect, considering the shape of MF, the overlap bounds are used to derive relaxed stability conditions as in [37]. Then one can find polynomial approach as in [38], which also reduces the conservatism by using membership functions. Recently, piecewise membership functions approximation methods have been proposed to incorporate MFs information into stability analysis [39,40]. If the number of the premise variable division is small, the MFs approximation errors may be large which could still introduce the conservatism. A large division number can significantly reduce the conservatism at the expense of increased numerical burdens. Therefore, deriving LMI-based control synthesis conditions with a reduced level of conservatism and a reasonable numerical complexity is still an open problem within TS fuzzy control framework.

Motivated by the above challenge, this paper presents a new control method based on a reconstruction of TS fuzzy systems. The contributions of this article are as follows.

1. A subregion based non-parallel distribution compensation (NPDC) control scheme is proposed based on the premise variable equilibrium enveloping technique, which introduces more flexibility in the control design.
2. A local model reconstruction approach is proposed to transform the double fuzzy summation items of the Lyapunov stability condition into a new fuzzy summation form, aiming to reduce conservatism in stabilization and robust control design conditions by incorporating additional membership functions information.

This paper is organized as follows. Section 2 describes the control synthesis problem and provides a preliminary result, which is useful for the development of the new control method. Section 3 presents the main results on model reconstruction based control synthesis. Two numerical examples are given in Section 4 to point out the interests of the proposed method. Section 5 concludes the paper.

Notation. $\mathbb{N}_l^+$ is referred to a positive number set $\{1, 2, \cdots, l\}$, R^n denotes a n-dimensional Euclidean space. For a matrix X, X^T denotes its transpose matrix, and $X > 0$ means that X is positive definite. I denotes the identity matrix of appropriate dimension. $\frac{\partial *}{\partial *}$ is the partial derivative operator. $|*|$ is absolute value of $*$. card$(*)$ denotes the number of elements in $*$, and $||*||_2 = \{* : \int_o^\infty ||*(t)||^2 dt < \infty\}$ is defined as an $\mathbf{L}_2$ norm. In block matrices, the symbol $\star$ stands for the terms deduced by symmetry. Arguments are omitted when their meaning is clear.

2. Problem statement and preliminaries

Consider a class of TS fuzzy systems of the form

$$\begin{aligned} &\textbf{Rule}\, i: \quad \text{IF } z_1(t) \text{ is } M_{i1} \ldots \text{ and } z_p(t) \text{ is } M_{ip} \\ &\qquad\quad \text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \end{aligned} \tag{1}$$

where M_{ij} represents fuzzy sets, $i \in \mathbb{N}_l^+$ and $j \in \mathbb{N}_p^+$ with l and p denoting the number of fuzzy rules and the number of premise variables, respectively. $z(t) = [z_1(t) \ \ldots \ z_p(t)]^T \in \mathrm{R}^p$ indicates the premise variable vector, $x(t) \in \mathrm{R}^n$ means the system state, $u(t) \in \mathrm{R}^m$ is the control input vector, $A_i \in \mathrm{R}^{n\times n}$ and $B_i \in \mathrm{R}^{n\times m}$ are the state-space matrices corresponding to the ith fuzzy rule. Using the center-of-gravity method for defuzzification [1], the TS fuzzy system (1) can be represented in the compact form

$$\dot{x}(t) = \sum_{i=1}^{l} h_i(z(t))(A_i x(t) + B_i u(t)). \tag{2}$$

The membership functions are defined as

$$h_i(z(t)) = \frac{\prod_{j=1}^{p} \Psi_{ij}(z_j(t))}{\sum_{i=1}^{l}\prod_{j=1}^{p} \Psi_{ij}(z_j(t))}, \quad \forall i \in \mathbb{N}_l^+, \tag{3}$$

where $\Psi_{ij}(z_j(t))$ represents the membership grade of $z_j(t)$ in the respective fuzzy set M_{ij}. Note that the MFs satisfy the following convex sum property:

$$\sum_{i=1}^{l} h_i(z) = 1, \quad 0 \le h_i(z) \le 1, \quad \forall i \in \mathbb{N}_l^+. \tag{4}$$

For brevity, we denote $h_i = h_i(z(t))$ in the subsequent sections. For the control synthesis of system (2), we consider a NPDC fuzzy controller of the form

$$u = \sum_{i=1}^{l} h_i K_i (P_h)^{-1} x, \tag{5}$$

where the matrices $K_i \in \mathrm{R}^{m\times n}$ and $P_h = \sum_{j=1}^{l} h_j P_j \in \mathrm{R}^{n\times n}$, with $P_j > 0$, are to be determined. The closed-loop TS fuzzy system can be defined from (2) and (5) as

$$\dot{x} = \sum_{i=1}^{l}\sum_{j=1}^{l} h_i h_j (A_i + B_i K_j P_h^{-1}) x. \tag{6}$$

For a practical system, system state and control input are usually constrained due to physical limitations, thus one can give the following descriptions:

$$x \in \Omega = \{x \in \mathrm{R}^n \,||c_o^T x| \le 1, c_o \in \mathrm{R}^n, o \in \mathbb{N}_\eta^+\}, \tag{7}$$

$$\|u\|_2 \le \mu. \tag{8}$$

where Ω is a bounded system state set, c_o is a parameter vector, η denotes the number of possible constraints cases on system state, and $\mu > 0$ represents the upper bound on control input.

Problem 1. This paper investigates a model reconstruction approach to design a fuzzy controller (5) for the TS fuzzy system (2) such that the closed-loop system (6) is asymptotically stable while further reducing the design conservatism.

Assumption 1. $\forall r \in \mathbb{N}_p^+$, it is assumed that the premise variable z_r is a linear function of the system state x as $z_r = \mathscr{F}_r(x)$, where $\mathscr{F}_r(x) = a_r x + b_r$, $a_r \in \mathrm{R}^{1\times n}$ and $b_r \in \mathrm{R}$ are known constant coefficients.

Remark 1. It is reasonable to assume the premise variable is a function of the system state, which can be found in many practical systems, as an autonomously parallel parking control system in [41] or a cart inverted pendulum system in [42].

Using fuzzy Lyapunov function based methods [33,39], the following Lemma provides a technical basis for our theoretical developments.

Lemma 1. *$\forall i, j \in \mathbb{N}_l^+$, $r \in \mathbb{N}_p^+$, $o \in \mathbb{N}_\eta^+$, system (2) with fuzzy controller (5) will be asymptotically stable, if there exist positive definite matrices $P_i \in \mathrm{R}^{n\times n}$, a symmetric matrix $Q \in \mathrm{R}^{n\times n}$ and matrices $K_i \in \mathrm{R}^{m\times n}$, such that*

$$P_j + Q > 0, \tag{9}$$

$$\Psi_{ij} + \sum_{r=1}^{p}\sum_{j=1}^{l} \Phi_{rj} < 0, \tag{10}$$

$$\begin{bmatrix} -P_j & P_j c_o \\ \star & -I \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} -P_j & K_j^T \\ \star & -\mu^2 I \end{bmatrix} < 0, \tag{12}$$

where $\Psi_{ij} = P_j A_i^T + K_j^T B_i^T + A_i P_j + B_i K_j$, $\Phi_{rj} = \overline{|\dot{z}_r|}\ \overline{|\chi_{rj}|}(P_j + Q)$, $\overline{|\dot{z}_r|} \geq |\dot{z}_r|$, $\overline{|\chi_{rj}|} \geq |\chi_{rj}|$ are predefined scalars, and the initial system state x_0 should be subject to:

$$\begin{bmatrix} -1 & x_0^T \\ \star & -P_j \end{bmatrix} < 0. \tag{13}$$

Proof. Let us consider the following fuzzy Lyapunov function candidate:

$$V(x) = x^T P_h^{-1} x. \tag{14}$$

The time derivative of $V(x)$ along the solution of system (6) is defined as

$$\begin{aligned} \dot{V}(x) =& \dot{x}^T P_h^{-1} x + x^T \frac{dP_h^{-1}}{dt} x + x^T P_h^{-1} \dot{x} \\ =& \sum_{i=1}^{l}\sum_{i=1}^{l} h_i h_j x^T (A_i^T P_h^{-1} + K_j^T B_i^T P_h^{-1}) x - x^T P_h^{-1} \dot{P}_h P_h^{-1} x + \\ & x^T (P_h^{-1} A_i + P_h^{-1} B_i K_j P_h^{-1}) x \\ =& \sum_{i=1}^{l}\sum_{i=1}^{l} h_i h_j \xi^T (P_j A_i^T + K_j^T B_i^T + A_i P_j + B_i K_j)\xi - \xi^T \dot{P}_h \xi, \end{aligned} \tag{15}$$

where $\xi = P_h^{-1} x$. We denote $f_{ij} = h_i h_j$ and $\chi_{rj} = \frac{\partial h_j}{\partial z_r}$. It follows from conditions (9) and (10) that

$$\sum_{i=1}^{l}\sum_{j=1}^{l} f_{ij}\Psi_{ij} - \sum_{r=1}^{p}\sum_{j=1}^{l} \dot{z}_r \chi_{rj}(P_j + Q) \leq \sum_{i=1}^{l}\sum_{j=1}^{l} f_{ij}\Psi_{ij} + \sum_{r=1}^{p}\sum_{j=1}^{l} \Phi_{rj} < 0. \tag{16}$$

From (16), we show that $\dot{V}(x) < 0$, for $\forall x \neq 0$. Based on $\dot{z}_r = a_r \dot{x}$ and (6)-(8), we can derive a specified upper bound $\overline{|\dot{z}_r|}$ of $\dot{z}_r$. Using the invariant set technique, the constraints on state and control input will be derived into the form of LMIs in the following parts. For the system state constraints, define an initial invariant ellipsoid as

$$\varepsilon = \{x_0 \in R^n | x_0^T P_h^{-1} x_0 \leq 1\}. \tag{17}$$

where x_0 is the initial states. With conditions (9)-(10), $V(x) < x_0^T P_h^{-1} x_0$ can be obtained. Considering the state constraint (7) which equals to $x^T c_o c_o^T x \leq 1$ and the limits (17), it is required that $c_o c_o^T \leq P_h^{-1}$. Then with Schur's complement, (11) can be deduced. For the control input limits, one has:

$$\begin{aligned}\max \|u\|_2^2 =& \max_{x\in\varepsilon} \|K_h P_h^{-1} x\|_2^2 \\ =& x^T P_h^{-1} K_h^T K_h P_h^{-1} x \\ =& x^T P_h^{-\frac{1}{2}} (P_h^{-\frac{1}{2}} K_h^T K_h P_h^{-\frac{1}{2}}) P_h^{-\frac{1}{2}} x. \end{aligned} \tag{18}$$

It can be obtained from (17) and (18) that

$$\max \|u\|_2^2 \le \lambda_{\max}(P_h^{-\frac{1}{2}} K_h^T K_h P_h^{-\frac{1}{2}}). \tag{19}$$

The control input constraints (8) will be satisfied if the following formula holds:

$$\lambda_{\max}(P_h^{-\frac{1}{2}} K_h^T K_h P_h^{-\frac{1}{2}}) \le \mu^2. \tag{20}$$

Pre- and post-multiplying by $diag\{I, P_h^{-\frac{1}{2}}\}$ with (12), and using Schure's complement, (20) can be guaranteed. Thus, control input is effectively constrained.

One should also notice that the initial state should be restricted by $x_0^T P_h^{-1} x_0 \le 1$, which can be guaranteed by (13). This completes the proof. □

Remark 2. The detailed derivation of $\frac{dP_h^{-1}}{dt}$ is listed as follows:

$$P_h P_h^{-1} = I. \tag{21}$$

The derivatives for both sides of the above equation are:

$$\dot{P}_h P_h^{-1} + P_h \frac{dP_h^{-1}}{dt} = 0, \tag{22}$$

$$P_h \frac{dP_h^{-1}}{dt} = -\dot{P}_h P_h^{-1}. \tag{23}$$

For the item P_h is positive definite in fuzzy Lyapunov function, we have:

$$\frac{dP_h^{-1}}{dt} = -P_h^{-1} \dot{P}_h P_h^{-1}. \tag{24}$$

Remark 3. Due to limitations on the control input and state, one must assure the system initial state $x_0 \in \Omega$. To obtain the largest initial feasible state region, the following optimization problem could be utilized:

$$\begin{aligned} &\min \quad \lambda \\ &s.t.\ (11),\ (13)\ and\ P_j \le \lambda I\ hold,\ j \in \mathbb{N}_l^+. \end{aligned} \tag{25}$$

3. Main results

This section presents a model reconstruction control method to further reduce the design conservatism in Lemma 1, and the case that disturbance exists is also considered.

3.1. Model reconstruction method for fuzzy control

At first, to provide more control design flexibility, the operation region Θ of the premise variables is divided into q adjacent subregions $\Theta = \bigcup_{k=1}^{q} \Theta_k$, where Θ_k denotes the k-th subregion and the integer $q > 0$ is the number of subregions. Without loss of generality, we assume that there is a mapping between premise variable and system state as $z = \gamma(x)$. For the region division, note that one of the subregions must contain the premise variable equilibrium $z_0 = \gamma(x)|_{x=0}$ to avoid an infinite switching action when the system state approaches to its equilibrium. Based on this division principle, we consider the following local fuzzy controller for the k-th subregion:

$$u = \sum_{i=1}^{l} h_i(\Theta_k) K_i(\Theta_k) P_h^{-1} x, \tag{26}$$

where the matrices of the kth subregion $K_i(\Theta_k) \in \mathrm{R}^{m\times n}$, for $i \in \mathbb{N}_l^+$, must be determined. From the expressions of system (2) and controller (26), the closed-loop TS fuzzy system is defined as

$$\dot{x} = \sum_{i=1}^{l}\sum_{j=1}^{l} h_i(\Theta_k) h_j(\Theta_k)(A_i + B_i K_j(\Theta_k) P_h^{-1}) x. \tag{27}$$

Remark 4. Subregion-based methods offer additional design flexibility and can improve robustness by involving more controller gain matrices, as shown in Eq. (26). However, determining the optimal number of subregions requires numerous tests, resulting in increased computational complexity.

To clearly describe the proposed control method, a sequence representation is used to convert multi-dimensional subscripts into single-dimensional subscripts. To this end, we denote

$$\mathscr{F}(\Im_1, \Im_2, \cdots, \Im_\varpi) = \sum_{i=1}^{\varpi-1}\left[(\Im_i - 1)\prod_{j=i+1}^{\varpi} l_j\right] + \Im_\varpi, \tag{28}$$

$\forall \Im_i \in \mathbb{N}_{l_i}^+$ and $\forall i \in \mathbb{N}_\varpi^+$. For example, if $\varpi = 2$ and $l_1 = l_2 = 2$, then we have $\mathscr{F}(\Im_1 = 1, \Im_2 = 1) = 1$, $\mathscr{F}(\Im_1 = 1, \Im_2 = 2) = 2$, $\mathscr{F}(\Im_1 = 2, \Im_2 = 1) = 3$, and $\mathscr{F}(\Im_1 = 2, \Im_2 = 2) = 4$.

Remark 5. Using the notation (28), we can rewrite

$$\sum_{\Im_1=1}^{l_1}\sum_{\Im_2=1}^{l_2}\cdots\sum_{\Im_\varpi=1}^{l_\varpi} f_{\Im_1\ldots\Im_\varpi} = \sum_{\ell=1}^{\prod_{i=1}^{\varpi} l_i} f_{\ell=\mathscr{F}(\Im_1,\Im_2,\ldots,\Im_\varpi)}, \tag{29}$$

where $f_{\Im_1\ldots\Im_\varpi} = h_{\Im_1}\ldots h_{\Im_\varpi}$.

Let us define the number $\bar{\tau}_{1\ell}^k$ of the minimal values of $f_\ell(\Theta_k) = h_i(\Theta_k)h_j(\Theta_k)$, with $i, j \in \mathbb{N}_l^+$ and $\ell = \mathscr{F}(i, j)$, as

$$\bar{\tau}_{1\ell}^k = \mathrm{card}(z^{k_{\min}}), \quad z^{k_{\min}} = \arg\min_{z\in\Theta_k}[f_\ell]. \tag{30}$$

Similarly, we define $\bar{\tau}_{2\ell}$ as the number of the maximal values of f_ℓ

$$\bar{\tau}_{2\ell}^k = \mathrm{card}(z^{k_{\max}}), \quad z^{k_{\max}} = \arg\max_{z\in\Theta_k}[f_\ell]. \tag{31}$$

The following theorem provides a model reconstruction method for fuzzy control synthesis to reduce the conservatism with respect to the result in Lemma 1.

Theorem 1. *$\forall i, j \in \mathbb{N}_l^+$ $o \in \mathbb{N}_\eta^+$, $r \in \mathbb{N}_p^+$, $k \in \mathbb{N}_q^+$ where q is the number of subregions divided on z, $\jmath = \mathscr{F}(r, j)$, $\ell = \mathscr{F}(i, j)$, $\tau_{1\ell}^k \in \mathbb{N}_{\bar{\tau}_{1\ell}^k}^+$, $\tau_{2\ell}^k \in \mathbb{N}_{\bar{\tau}_{2\ell}^k}^+$, $s^k \in \mathbb{N}_{\delta^k}^+$, $\delta^k = \sum_{\ell=1}^{l^2}(\bar{\tau}_{1\ell}^k + \bar{\tau}_{2\ell}^k)$. System (2) is asymptotically stable with local state feedback law (26), if there exist matrices $K_j(\Theta_k) \in \mathrm{R}^{m\times n}$, $P_j = P_j^T > 0 \in \mathrm{R}^{n\times n}$ and $Q(\Theta_k) = Q^T(\Theta_k) \in \mathrm{R}^{n\times n}$ such that*

$$P_j + Q(\Theta_k) > 0, \tag{32}$$

$$\Xi_{s^k}(\Theta_k) + \sum_{\jmath=1}^{pl}\Phi_\jmath(\Theta_k) < 0, \tag{33}$$

$$\begin{bmatrix} -P_j & P_j c_o \\ \star & -I \end{bmatrix} < 0, \tag{34}$$

$$\begin{bmatrix} -P_j & K_j^T(\Theta_k) \\ \star & -\mu^2 I \end{bmatrix} < 0, \tag{35}$$

where

$$\Xi_{s^k}(\Theta_k) = \begin{cases} \kappa_{1\ell}^k \Psi_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho_{\tau_{1\ell}^k \ell'}^k \Psi_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{1\ell}^k\} \\ \kappa_{2\ell}^k \Psi_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho'^k_{\tau_{2\ell}^k \ell'} \Psi_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{2\ell}^k\} \end{cases}$$

$$\Psi_\ell(\Theta_k) = P_j A_i^T + K_j^T(\Theta_k) B_i^T + A_i P_j + B_i K_j(\Theta_k)$$

$$\Phi_J(\Theta_k) = \overline{|\dot{z}_r^k|}\ \overline{|\chi_j^k|}(P_j + Q(\Theta_k))$$

with $\overline{|\dot{z}_r^k|} \geq |\dot{z}_r^k|$ *and* $\overline{|\chi_j^k|} \geq |\chi_j^k|$ *are predefined scalars,* $\kappa_{1\ell}^k$ *and* $\kappa_{2\ell}^k$ *are the minimal and maximal values of* f_ℓ*, for* $z \in \Theta_k$*, and their respective numbers* $\bar{\tau}_{1\ell}^k$ *and* $\bar{\tau}_{2\ell}^k$*,* $\rho_{\tau_{1\ell}\ell'}^k = f_{\ell'}(z_{\tau_{1\ell\neq\ell'}}^{k\min})$*,* $\rho'^k_{\tau_{2\ell}\ell'} = f_{\ell'}(z_{\tau_{2\ell\neq\ell'}}^{k\max})$*,* $z_{\tau_{1\ell}}^{k\min}$ *and* $z_{\tau_{1\ell}^k}^{k\min}$ *denote the* $\tau_{1\ell}^k$*-th and* $\tau_{2\ell}^k$*-th vector of* $z^{k\min}$ *and* $z^{k\max}$*, respectively. And the initial system state* x_0 *should be subject to:*

$$\begin{bmatrix} -1 & x_0^T \\ \star & -P_j \end{bmatrix} < 0. \tag{36}$$

Proof. Based on Lemma 1, $\dot{V}(x)$ defined in (15) and (29), a sufficient condition to make $\dot{V}(x) < 0$, $\forall x \neq 0$ can be given as:

$$\sum_{\ell=1}^{l^2} f_\ell(\Theta_k)\Psi_\ell(\Theta_k) + \sum_{j=1}^{pl} \Phi_J(\Theta_k) < 0. \tag{37}$$

In each subregion, the model reconstruction process is carried out to involve information from $f_\ell(\Theta_k)$ into condition in (37). To this end, we define a vector $f(\Theta_k) = [f_1(\Theta_k), ..., f_\ell(\Theta_k), ..., f_{l^2}(\Theta_k)]$, where $f_\ell(\Theta_k) \in [\kappa_{1\ell}^k, \kappa_{2\ell}^k]$ satisfying $0 \leq \kappa_{1\ell}^k < \kappa_{2\ell}^k \leq 1$. And, given a class of fuzzy sets as $\Upsilon_{s^k} = [\varsigma_1^k, ..., \varsigma_\ell^k, ..., \varsigma_{l^2}^k]$, where $s^k \in \mathbb{N}_{\delta^k}^+$, $\delta^k = \sum_{\ell=1}^{l^2}(\bar{\tau}_{1\ell}^k + \bar{\tau}_{2\ell}^k)$. Since $\sum_{\ell=1}^{l^2} f_\ell(\Theta_k) = 1$, if $\varsigma_\ell^k = \kappa_{1\ell}^k$ *or* $\kappa_{2\ell}^k$, other elements in Υ_{s^k} can be given as $\varsigma_{\ell'\neq\ell}^k = \rho_{\tau_{1\ell}^k\ell'}^k$ or $\varsigma_{\ell'\neq\ell}^k = \rho'^k_{\tau_{2\ell}^k\ell'}$, where $\kappa_{1\ell}^k + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho_{\tau_{1\ell}^k\ell'}^k = 1$ and $\kappa_{2\ell}^k + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho'^k_{\tau_{2\ell}^k\ell'} = 1$. Then, a reconstruction matrix for $\sum_{\ell=1}^{l^2} f_\ell(\Theta_k)\Psi_\ell(\Theta_k)$ in (37) is derived as

$$\begin{aligned} \Xi_{s^k}(\Theta_k) &= \Upsilon_{s^k}\left[(\Psi_1(\Theta_k))^T \quad ... \quad (\Psi_{l^2}(\Theta_k))^T \right]^T \\ &= \begin{cases} \kappa_{1\ell}^k \Psi_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho_{\tau_{1\ell}^k\ell'}^k \Psi_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{1\ell}\} \\ \kappa_{2\ell}^k \Psi_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho'^k_{\tau_{2\ell}^k\ell'} \Psi_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{2\ell}\} \end{cases}. \end{aligned} \tag{38}$$

The corresponding membership functions for $\Xi_{s^k}(\Theta_k)$ are defined as $\phi_{s^k}(f(\Theta_k))$, thus condition (37) is equivalent to

$$\sum_{s^k=1}^{\delta^k} \phi_{s^k}(f(\Theta_k))\Xi_{s^k}(\Theta_k) + \sum_{J=1}^{pl} \Phi_J(\Theta_k) < 0. \tag{39}$$

This completes the proof. □

Remark 6. In kth subregion, based on the classical sector nonlinearity approach for fuzzy modeling [9], the s^k-th rule for the newly derived fuzzy summation $\sum_{s^k=1}^{\delta^k} \phi_{s^k}(f(\Theta_k))\Xi_{s^k}(\Theta_k)$ can be described as '$f(\Theta_k)$ is Υ_{s^k}'. When $f_\ell(\Theta_k) = \kappa_{1\ell}^k$, the other elements $f_{\ell'}(\Theta_k) = \rho_{\tau_{1\ell}^k\ell',\ell'\neq\ell}^k$, and the weighting function $\beta_0(\Theta_k)$ can be characterized as

$$\beta_0^\ell(\Theta_k) = \frac{\kappa_{2\ell}^k - f_\ell(\Theta_k)}{\kappa_{2\ell}^k - \kappa_{1\ell}^k}, \quad \beta_{0,\ell'\neq\ell}^{\vartheta_{\ell\ell'}}(\Theta_k) = \frac{\rho_{\tau_{1\ell}^k\ell'}^k - f_{\ell'}(\Theta_k)}{\rho_{\tau_{1\ell}^k\ell'}^k - \rho'^k_{\tau_{2\ell}^k\ell'}}. \tag{40}$$

Table 1
Computational complexity for different methods.

Methods	Number of LMIs	Number of decision variables
Theorem 1 in [39]	$2l^2+4l+2$	$mnl+\frac{1}{2}n(n+1)(l^2+2l+4)$
Theorem 3 in [43]	$2^p q+2ql$	$n(n+1)q+mnq+\frac{1}{2}n(n+1)ql$
Lemma 1	$l^2+(\eta+3)l$	$mnl+\frac{1}{2}n(n+1)(l+1)$
Theorem 1	$\sum_{k=1}^{q}\sum_{\ell=1}^{l^2}\bar{\tau}_\ell^k+(2q+\eta+1)l$	$qmnl+\frac{1}{2}n(n+1)(l+q)$

For $f_\ell(\Theta_k)=\kappa_{2\ell}^k$ and $f_{\ell'}(\Theta_k)=\rho^k_{\tau^k_{2\ell}\ell',\ell'\neq\ell}$, the weighting function β_1 can be defined as

$$\beta_1^\ell(\Theta_k)=\frac{f_\ell(\Theta_k)-\kappa_{1\ell}^k}{\kappa_{2\ell}^k-\kappa_{1\ell}^k},\quad \beta_{1,\ell'\neq\ell}^{\vartheta_\ell\ell'}=\frac{f_{\ell'}(\Theta_k)-\rho'^k_{\tau^k_{2\ell}\ell'}}{\rho^k_{\tau^k_{1\ell}\ell'}-\rho'^k_{\tau^k_{2\ell}\ell'}}. \tag{41}$$

Then, the new membership functions can be directly deduced as

$$\phi_{s^k}(f(\Theta_k))=\frac{\wp_\mu^\ell(\Theta_k)}{\sum_{\mu=0}^{1}\sum_{\ell=1}^{l^2}\sum_{\tau_\ell=1}^{\bar{\tau}_\ell}\wp_\mu^\ell(\Theta_k)}, \tag{42}$$

where $\wp_\mu^\ell(\Theta_k)=\beta_\mu^\ell(\Theta_k)\prod_{\ell'=1,\ell'\neq\ell}^{l}\beta_\mu^{\tau_\ell^k\ell'}(\Theta_k)$.

Remark 7. For the derivation of the new membership functions in Remark 6, it is required that $f_{\ell'}$ satisfies

$$\min\{\rho^k_{\tau^k_{1\ell}\ell'},\rho'^k_{\tau^k_{2\ell}\ell'}\}\le f_{\ell'}^k\le\max\{\rho^k_{\tau^k_{1\ell}\ell'},\rho'^k_{\tau^k_{2\ell}\ell'}\}, \tag{43}$$

with $\ell,\ell'=1,2,...,l^2$ and $\ell\neq\ell'$. If condition (43) does not hold for $f_{\ell'}$, then the corresponding weighting functions $\beta_0(\Theta_k)$ and $\beta_1(\Theta_k)$ will be set as zero. Note that it follows directly from (40) and (41) that $\beta_0^\ell(\Theta_k)+\beta_1^\ell(\Theta_k)=1$ and $\beta_{0,\ell'\neq\ell}^{\tau_\ell^k\ell'}(\Theta_k)+\beta_{1,\ell'\neq\ell}^{\tau_\ell^k\ell'}(\Theta_k)=1$.

Remark 8. The computational complexity comparison among several control methods is summarized in Table 1. We can see the number of LMI conditions in Theorem 1 is dependent on $\bar{\tau}_\ell^k$ and q. Compared with traditional control results, the division number q in Theorem 1 may introduce many extra decision variables when q becomes large. If $q=1$, a global control result will be generated by Theorem 1. Therefore, one can choose the number of subregions based on specific task requirements to balance the control performance and the computational burdens.

3.2. H_∞ robust control based on model reconstruction

Based on the above control results, a robust synthesis method can be directly derived in the following. To this end, we consider a disturbed TS fuzzy system of the following form:

$$\dot{x}=\sum_{i=1}^{l}h_i(\Lambda_i x+B_i u+E_i\omega), \tag{44}$$

$$y=\sum_{i=1}^{l}h_i C_i x, \tag{45}$$

where $E_i\in\mathrm{R}^{n\times d}$ and $C_i\in\mathrm{R}^{o\times n}$ are parameter matrices, $\omega\in\mathrm{R}^d$ is the unknown external disturbance satisfying $\omega\in\mathbf{L}_2(0,\infty)$, and $y\in\mathrm{R}^o$ is the controlled output.

Corollary 1. *$\forall i,j\in\mathbb{N}_l^+$ $o\in\mathbb{N}_\eta^+$, $r\in\mathbb{N}_p^+$, $k\in\mathbb{N}_q^+$ where q is the number of subregions divided on z, $\jmath=\mathscr{F}(r,j)$, $\ell=\mathscr{F}(i,j)$, $\tau_{1\ell}^k\in\mathbb{N}_{\bar{\tau}_{1\ell}^k}^+$, $\tau_{2\ell}^k\in\mathbb{N}_{\bar{\tau}_{2\ell}^k}^+$, $s^k\in\mathbb{N}_{\delta^k}^+$, $\delta^k=\sum_{\ell=1}^{l^2}(\bar{\tau}_{1\ell}^k+\bar{\tau}_{2\ell}^k)$, the fuzzy controller (26) robustly stabilizes*

system (44) with the $\mathscr{H}_\infty$ performance index $\|y\|_2^2 \le \gamma^2\|\omega\|_2^2$ under zero initial condition, if there exist matrices $K_j(\Theta_k) \in \mathrm{R}^{m\times n}$, $P_j = P_j^T > 0 \in \mathrm{R}^{n\times n}$ and $Q(\Theta_k) = Q^T(\Theta_k) \in \mathrm{R}^{n\times n}$ and a positive scalar γ such that

$$P_j + Q(\Theta_k) > 0, \tag{46}$$

$$\Gamma_{s^k}(\Theta_k) < 0, \tag{47}$$

$$\begin{bmatrix} -P_j & P_j c_o \\ \star & -I \end{bmatrix} < 0, \tag{48}$$

$$\begin{bmatrix} -P_j & K_j^T(\Theta_k) \\ \star & -\mu^2 I \end{bmatrix} < 0, \tag{49}$$

where

$$\Gamma_{s^k}(\Theta_k) = \begin{cases} \kappa_{1\ell}^k \Delta_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho_{\tau_{1\ell}^k \ell'}^k \Delta_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{1\ell}^k\} \\ \kappa_{2\ell}^k \Delta_\ell(\Theta_k) + \sum_{\ell'=1,\ell'\neq\ell}^{l^2} \rho'^k_{\tau_{2\ell}^k \ell'} \Delta_{\ell'}(\Theta_k), & \ell \in \{\ell | f_\ell = \kappa_{2\ell}^k\} \end{cases}$$

$$\Delta_\ell(\Theta_k) = \begin{bmatrix} \Psi_\ell(\Theta_k) + \sum_{J=1}^{pl} \Phi_J(\Theta_k) & E_i & P_j C_i^T \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix}$$

$$\Psi_\ell(\Theta_k) = P_j A_i^T + K_j^T(\Theta_k) B_i^T + A_i P_j + B_i K_j(\Theta_k)$$

$$\Phi_j(\Theta_k) = \overline{|\dot{z}_r^k|}\ \overline{|\chi_j^k|}(P_j + Q(\Theta_k))$$

with $\overline{|\dot{z}_r^k|} \ge |\dot{z}_r^k|$ and $\overline{|\chi_j^k|} \ge |\chi_j^k|$ are predefined scalars, $\chi_j^k = h_j^{z_r}(\Theta_k)$; $\kappa_{1\ell}^k$ and $\kappa_{2\ell}^k$ are the minimal and maximal values of f_ℓ for $z \in \Theta_k$ and their respective numbers $\bar{\tau}_{1\ell}^k$ and $\bar{\tau}_{2\ell}^k$, $\rho_{\tau_{1\ell}\ell'}^k = f_{\ell'}(z_{\tau_{1\ell\neq\ell'}}^{k_{\min}})$, $\rho'^k_{\tau_{2\ell}\ell'} = f_{\ell'}(z_{\tau_{2\ell\neq\ell'}}^{k_{\max}})$, $z_{\tau_{1\ell}}^{k_{\min}}$ and $z_{\tau_{1\ell}}^{k_{\min}}$ denote the $\tau_{1\ell}^k$-th and $\tau_{2\ell}^k$-th vector of $z^{k_{\min}}$ and $z^{k_{\max}}$, respectively. And the initial system state x_0 should be subject to:

$$\begin{bmatrix} -1 & x_0^T \\ \star & -P_j \end{bmatrix} < 0. \tag{50}$$

Proof. Consider the fuzzy Lyapunov function $V(x)$ defined in (14). The time derivative of $V(x)$ along the solution of system (44) is defined as

$$\begin{aligned} \dot{V}(x) = &\sum_{i=1}^{l}\sum_{j=1}^{l} h_i(\Theta_k) h_j(\Theta_k) \xi^T (P_h A_i^T + K_j^T(\Theta_k) B_i^T + A_i P_h + B_i K_j(\Theta_k))\xi \\ &+ \omega^T \sum_{i=1}^{l} h_i(\Theta_k) E_i^T \xi + \xi^T \sum_{i=1}^{l} h_i(\Theta_k) E_i \omega - \xi^T \dot{P}_h \xi, \end{aligned} \tag{51}$$

with $\xi = P_h^{-1} x$. Based on Lemma 1 and the notation (29), it follows that

$$\begin{aligned} \dot{V}(x) \le &\xi^T \sum_{\ell=1}^{l^2} f_\ell(\Theta_k) \Psi_{ij}(\Theta_k) \xi + \omega^T \sum_{i=1}^{l} h_i(\Theta_k) E_i^T \xi \\ &+ \xi^T \sum_{i=1}^{l} h_i(\Theta_k) E_i \omega + \xi^T \sum_{J=1}^{rp} \Phi_J(\Theta_k) \xi. \end{aligned} \tag{52}$$

Under zero-initial condition $x(0) = 0$, we define

$$\mathscr{J} = \int_0^T (y^T y - \gamma^2 \omega^T \omega + \dot{V}(x)) dt - V_T, \tag{53}$$

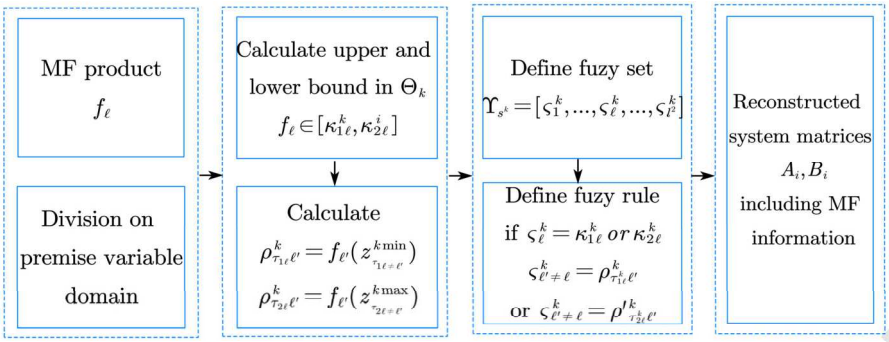

Fig. 1. Diagram of model reconstruction method.

with $V_T = x^T(T)P_h^{-1}x(T) > 0$. It follows from (53) that

$$\mathscr{J} \leq \int_0^T \begin{bmatrix} \xi \\ \omega \end{bmatrix}^T \begin{bmatrix} \left\{ \begin{array}{c} \Psi_f(\Theta_k) + \sum_{J=1}^{pl} \Phi_J(\Theta_k) \\ +P_h C_h^T(\Theta_k) C_h(\Theta_k) P_h \end{array} \right\} & E_h(\Theta_k) \\ \star & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \xi \\ \omega \end{bmatrix} dt - V_T, \tag{54}$$

where $\Psi_f(\Theta_k) = \sum_{\ell=1}^{l^2} f_\ell(\Theta_k)\Psi_{ij}(\Theta_k)$, $C_h(\Theta_k) = \sum_{i=1}^{l} h_i(\Theta_k)C_i$, and $E_h(\Theta_k) = \sum_{i=1}^{l} h_i(\Theta_k)E_i$. Using Schur complement lemma, condition $\mathscr{J} < 0$ is guaranteed if

$$\begin{bmatrix} \Psi_f(\Theta_k) + \sum_{J=1}^{pl} \Phi_J(\Theta_k) & E_h(\Theta_k) & P_h C_h^T(\Theta_k) \\ \star & -\gamma^2 I & 0 \\ \star & \star & -I \end{bmatrix} < 0. \tag{55}$$

The rest of the proof follows the same arguments as those of Theorem 1. □

Remark 9. To better describe the model reconstruction method with Theorem 1 and Corollary 1, a flowchart illustrating the reconstruction process is shown in Fig. 1. It should be noted that the numerical calculated parameters $\kappa^k_{1\ell}$, $\kappa^k_{2\ell}$ $\rho^k_{\tau^k_{2\ell}\ell'}$ and $\rho^k_{\tau^k_{2\ell}\ell'}$ will introduce MF information into system matrices A_i and B_i, thus the stabilization and robust control conditions can be relaxed. Moreover, as the number of premise variable subregions increases, more MF information will be introduced and more control gain matrices could be utilized, which could further provide flexibility and reduce the conservatism in LMI conditions.

4. Numerical examples

This section presents two numerical examples to demonstrate the effectiveness of the proposed control method. All LMI constraints are solved using LMI Toolbox in Matlab R2020b.

Example 1. To study the design conservatism between different control methods, we consider the benchmark TS fuzzy system in [39], whose local state-space matrices are given as

$$A_1 = \begin{bmatrix} 1.59 & -7.29 & 0 \\ 0.01 & 0 & 0 \\ 0 & -0.17 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 10.02 & -40.64 & 0 \\ 0.35 & 0.21 & 0 \\ 0 & -0.08 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a & -4.33 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & -0.21 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -b+6 \\ -1 \\ 0 \end{bmatrix},$$

where a and b are parameters, and $-10 \leq x_3 \leq 10$. The corresponding membership functions are defined as

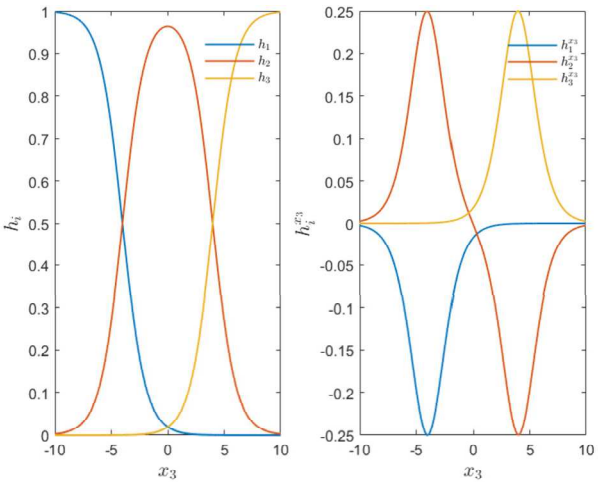

Fig. 2. Membership functions (left) and their partial derivatives (right).

$$\begin{cases} h_1(x_3) = 1 - 1/(1+e^{-(x_3+4)}), \\ h_2(x_3) = 1 - h_1(x_3) - h_3(x_3), \\ h_3(x_3) = 1/(1+e^{-(x_3-4)}). \end{cases}$$

The membership functions and their partial derivative with respect to $z_1 = x_3$ are depicted in Fig. 2. When $q = 1$, one can get $\overline{|\chi_1|} = \overline{|\chi_3|} = 0.2500$, and $\overline{|\chi_2|} = 0.2497$. It is assumed that $|\dot{h}_i| \le 0.1$ for $i \in \mathbb{N}_3^+$. Then considering $|\dot{h}_i| \le \overline{|\chi_i|}\,\overline{|\dot{x}_3|}$, we see the assumption $|\dot{h}_i| \le 0.1$ can be satisfied by

$$\begin{aligned} |\dot{z}_1| &= |\dot{x}_3| \\ &= |-0.17x_2h_1(x_3) - 0.08x_2h_2(x_3) - 0.21x_3h_3(x_3)| \\ &\le 0.4. \end{aligned} \tag{56}$$

To calculate the range of x_2, the following optimization question should be solved:

$$\begin{aligned} &\min \quad f \\ &s.t. \quad (56)\, and \ -10 \le x_3 \le 10. \end{aligned}$$

Thus the upper and lower bounds of x_2 can be obtained by solving such optimization problems by setting $f = x_2$ and $f = -x_2$ via the solver fmincon in Matlab R2022b as $x_2 \in [-10, 5.0798]$. Based on $x_3 \in [-10, 10]$, the following 4 cases could make constraints (7) hold:

$$c_1 = \begin{bmatrix} 0 & -0.1 & 0 \end{bmatrix}, c_2 = \begin{bmatrix} 0 & 0.1 & 0 \end{bmatrix}, c_3 = \begin{bmatrix} 0 & 0 & -0.1 \end{bmatrix}, c_4 = \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix}.$$

Setting the parameters ranges as $a \in [2, 12]$ and $b \in [6, 20]$, the feasible regions obtained from the conditions in Lemma 1 and Theorem 1 ($q = 1$) are shown in Fig. 3. We see that the proposed model reconstruction control method can effectively enlarge the feasibility region compared with the classical control result in Lemma 1.

When $q \ge 2$, consider the following subdomain division method: $\Lambda_k = \{x_3 :\ 20(k-q/2-1)/q \le x_3 \le 20(k-q/2)/q\}$ where q is an odd number. For $q = 2$, $\Lambda_1 = \{x_3 :\ -10 \le x_3 \le \delta\}$ and $\Lambda_2 = \{x_3 :\ \delta \le x_3 \le 10\}$. If q is an even number and $q \ge 4$, $\Lambda_k = \{x_3 | -10 + 2(\delta+10)(k-1)/q \le x_3 \le -10 + 2(\delta+10)k/q, k = 1, 2, \cdots, q/2\}$ and $\Lambda_{q/2+k} = \{x_3 | \delta + 2(10-\delta)(k-1)/q \le x_3 \le \delta + 2(10-\delta)(k-1)/q, k = 1, 2, \cdots, q/2\}$ where $0 < \delta < 10$. The predefined variables are set as $\overline{|\dot{z}_1^k|} = 0.4$ where $k = 1, 2, \cdots, q$. With $q = 2$, the values of $\overline{|\chi_j^k|}$ are given in Table 2. One can find that no feasible solutions for Theorem 1 in [40] and Theorem 1 in [39] are derived for $q = 2$. Since larger subregions division number can help reduce the conservatism and in order to give a more illustrative comparison, $q = 2$ is set for Theorem 1 in this paper, $q = 10$ for Theorem 1 in [40] and Theorem 1 in [39]. Then, the feasibility regions of the above three methods are shown in Fig. 4. From Figs. 3 and 4, we can see that the feasibility regions are enlarged when the number of the subregions increases.

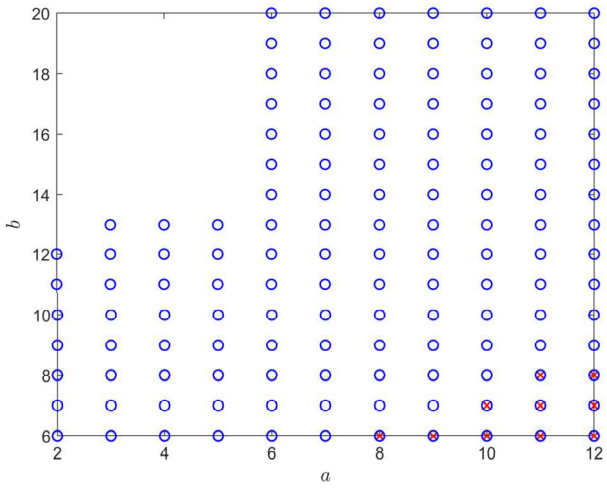

Fig. 3. Feasibility region obtained from Lemma 1 (x) and Theorem 1 (o).

Table 2
Different values of $\overline{|\chi_j^k|}$ with $q = 2$.

$k = 1$	$k = 2$
$\overline{\lvert\chi_1^1\rvert} = 0.2500$	$\overline{\lvert\chi_3^2\rvert} = 0.0177$
$\overline{\lvert\chi_2^1\rvert} = 0.2497$	$\overline{\lvert\chi_3^2\rvert} = 0.2497$
$\overline{\lvert\chi_3^1\rvert} = 0.0177$	$\overline{\lvert\chi_3^2\rvert} = 0.2500$

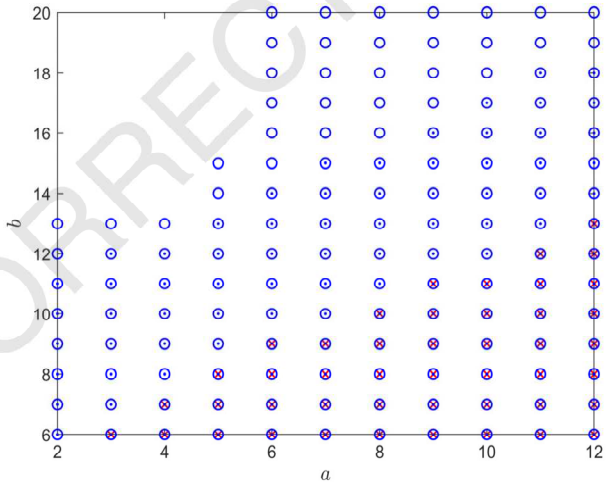

Fig. 4. Feasibility regions obtained from Theorem 1 (o) with $q = 2$, Theorem 1 in [39] (x), and Theorem 1 in [40] (.) with $q = 10$.

For illustrations, we consider the case with $a = 2$, $b = 13$. Considering $\mu = 30$ and solving the LMI conditions in Theorem 1, the following control results can be obtained:

$$K_1(\Theta_1) = \begin{bmatrix} -3.8972 \times 10^4 & 1.1237 \times 10^3 & -137.7643 \end{bmatrix},$$
$$K_2(\Theta_1) = \begin{bmatrix} 9.7642 \times 10^3 & -307.9120 & 130.1952 \end{bmatrix},$$
$$K_3(\Theta_1) = \begin{bmatrix} -5.6504 \times 10^5 & 4.3057 \times 10^4 & -4.8076 \times 10^3 \end{bmatrix},$$
$$K_1(\Theta_2) = \begin{bmatrix} 5.4374 \times 10^5 & -3.2691 \times 10^5 & 427.7246 \end{bmatrix},$$

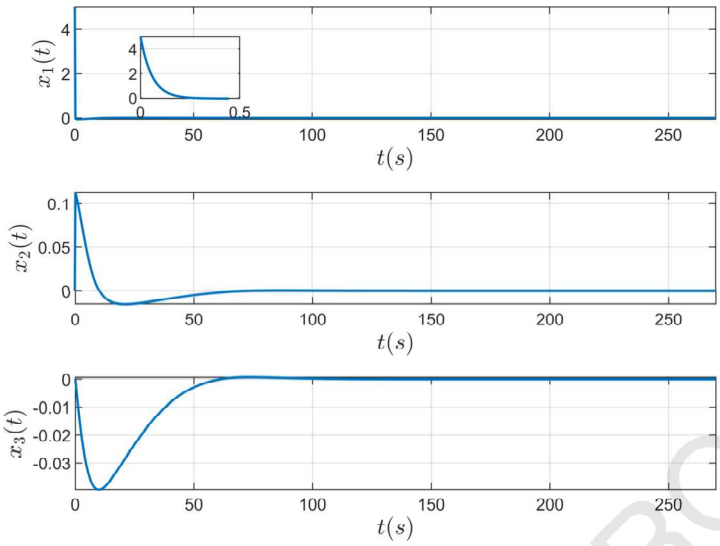

Fig. 5. State trajectories of the closed-loop system obtained from Theorem 1.

$$K_2(\Theta_2) = \begin{bmatrix} -1.1334 \times 10^4 & 6.6084 \times 10^3 & 28.6537 \end{bmatrix},$$
$$K_3(\Theta_2) = \begin{bmatrix} 431.2564 & 53.2637 & -0.8426 \end{bmatrix},$$
$$P_1 = \begin{bmatrix} 1.8079 \times 10^4 & -664.8963 & -66.5934 \\ -664.8963 & 29.0006 & 9.3591 \\ -66.5934 & 9.3591 & 85.4380 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 45.5104 & -36.3666 & 10.3530 \\ -36.3666 & 89.1589 & 9.9405 \\ 10.3530 & 9.9405 & 79.4944 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 187.5800 & -5.4081 & 11.8938 \\ -5.4081 & 1.3677 & -4.9685 \\ 11.8938 & -4.9685 & 79.4226 \end{bmatrix}.$$

Fig. 5 shows the system state trajectories, Fig. 6 depicts the control input and Lyapunov function with initial conditions $x_0 = [5\ 0\ 0]^T$ satisfying $x_0 \in \varepsilon$. We can see that the designed stabilization method can give satisfied control effects. The constraints $x_3 \in [-10, 10]$, $x_2 \in [-10, 5.0798]$, and $V \leq 1$ are all satisfied, which validate Theorem 1.

Example 2. We consider the cart inverted pendulum system presented in [42], whose dynamics can be described by a two-rule TS fuzzy model (44) with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)mgl}{a_1} & -\frac{f_1(M+m)}{a_1} & 0 & \frac{f_0ml}{a_1} \\ 0 & 0 & 0 & 1 \\ -\frac{m^2gl^2}{a_1} & \frac{f_1ml}{a_1} & 0 & \frac{f_0(J+ml^2)}{a_1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\frac{ml}{a_1} \\ 0 \\ \frac{J+ml^2}{a_1} \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}\frac{(M+m)mgl}{a_2} & -\frac{f_1(M+m)}{a_2} & 0 & \frac{f_0ml\cos(\pi/3)}{a_2} \\ 0 & 0 & 0 & 1 \\ -\frac{3\sqrt{3}}{2\pi}\frac{m^2gl^2}{a_2} & \frac{f_1ml\cos(\pi/3)}{a_2} & 0 & \frac{f_0(J+ml^2)}{a_2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -\frac{ml\cos(\pi/3)}{a_2} \\ 0 \\ \frac{J+ml^2}{a_2} \end{bmatrix},$$
$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 & 0 & 0.5 & 0 \end{bmatrix}^T,$$

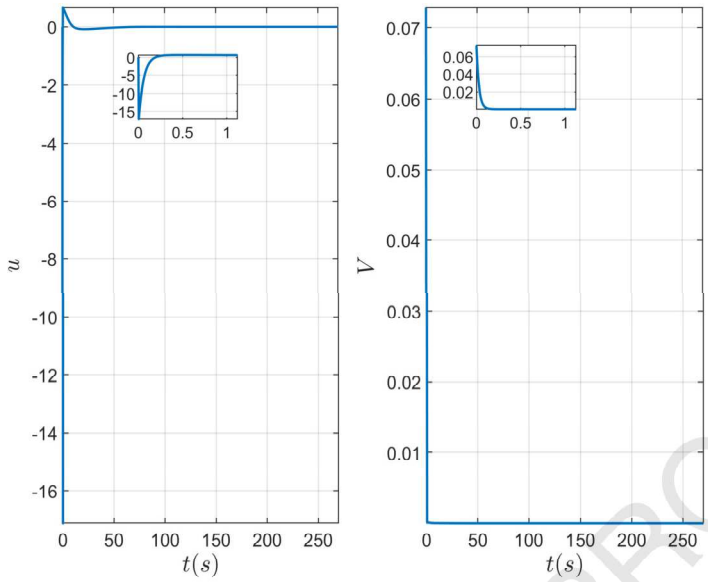

Fig. 6. Control input and Lyapunov function of the closed-loop system obtained from Theorem 1.

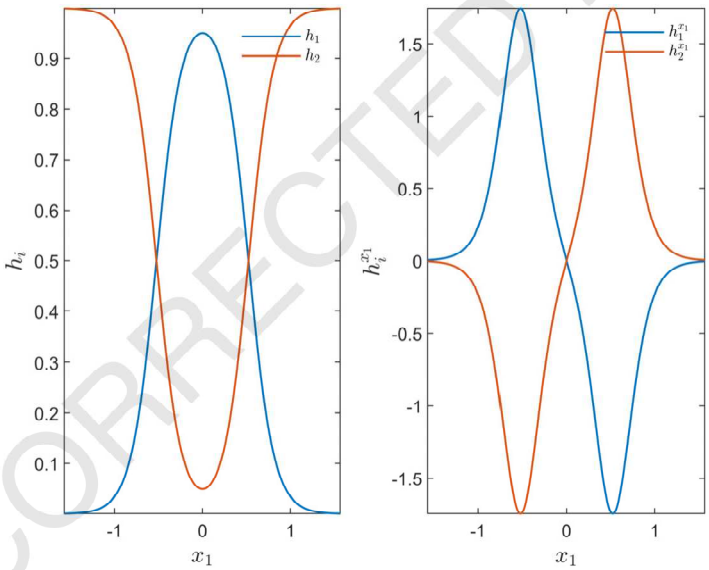

Fig. 7. Membership functions (left) and their partial derivatives (right).

where $a_1 = (M+m)(J+ml^2) - m^2l^2$ and $a_2 = (M+m)(J+ml^2) - m^2l^2\cos(\pi/3)^2$. The parameters of the systems are given by $g = 9.8$ m/s^2, $M = 1.3282$ kg, $m = 0.22$ kg, $f_0 = 22.915$ N/m/s, $f_1 = 0.007056$ N/rad/s, $l = 0.304$ m and $J = 0.004963$ kg $\cdot$ m^2. The corresponding membership functions are defined as

$$\begin{cases} h_1(x_1) = (1 - 1/(1+e^{-7.0(x_1-\pi/6)}))(1/(1+e^{-7.0(x_1+\pi/6)})), \\ h_2(x_1) = 1 - h_1(x_1). \end{cases}$$

The graph of membership functions and their partial derivatives are shown in Fig. 7. And the operation region is partitioned as

$$\Lambda_k = \{x_1 | \pi(k - q/2 - 1)/q \le x_1 \le \pi(k - q/2)/q\},$$

Table 3
Comparison of computational complexity and γ_{min} for different methods.

Method	Number of LMIs	Number of decision variables	γ_{min}
Theorem 6 in [45]	7	54	1.7122
Theorem 3 in [46]	19	46	1.7098
Theorem 2 in [44]	28	54	1.7089
Theorem 1 in [40]	16($q=1$)	284($q=1$)	7.4974($q=1$)
	24($q=3$)	284($q=3$)	5.9684($q=3$)
Corollary 1	20($q=1$)	38($q=1$)	1.7009($q=1$)
	42($q=3$)	74($q=3$)	1.6985($q=3$)

where $k=1,2,\cdots,q$. To study the design conservatism between different control methods, the minimal value of the $\mathscr{H}_\infty$ performance index γ, denoted by γ_{min}, is determined from LMI-based conditions in Corollary 1 (using Matlab function `mincx`). The comparative result of γ_{min} is summarized in Table 3. To solve the stabilization conditions in [44, Theorem 2], the numbers of checking points are set as $d=12$ and $\hat{d}=2$ and its pre-set parameter $\mu=5\times10^{-7}$. The LMI-based conditions in [45, Theorem 6] and Corollary 1 are solved with the same predefined scalars $\phi_{1\sim2}=(\dot{h}_{1\sim2})_{\max}=[1.7466\ 2.7466]$ and $\mu=1\times10^{-6}$. Note that for a fair comparison, the control results in [45, Theorem 6] and [40, Theorem 1] have been adapted for an $\mathscr{H}_\infty$ state-feedback control design. We can see in Table 1 that the proposed design conditions in Corollary 1 provide the best disturbance attenuation performance $\gamma_{\min}$. The corresponding design parameters are selected as $q=3$ and $|\dot{z}_1^i|=4$, $i=1,...,5$. Since $\dot{z}_1=\dot{x}_1=x_2h_1(x_1)+x_2h_2(x_1)+\omega$, one can get $x_2\in[-|\dot{z}_1^i|-|\omega|,|\dot{z}_1^i|+|\omega|]$. Assuming $|\omega|\le0.5$, the coefficient c_o can be calculated as follows:

$$c_1=\begin{bmatrix}2/\pi & 0 & 0 & 0\end{bmatrix}^T, c_2=\begin{bmatrix}-2/\pi & 0 & 0 & 0\end{bmatrix}^T,$$
$$c_3=\begin{bmatrix}0 & 0.2222 & 0 & 0\end{bmatrix}^T, c_4=\begin{bmatrix}0 & -0.2222 & 0 & 0\end{bmatrix}^T.$$

To facilitate the readers, complexity comparison is illustrated in Table 3. Based on the Table, one could choose a suitable robust synthesis method for specific problem by balancing the complexity and conservatism. Also, the computational burden is reduced when compared with Theorem 1 in [40] for the same sub-regions division number $q=15$.

Setting $x_0=[\frac{\pi}{6}\ \ -1\ \ 0\ \ 0]\in\varepsilon$, $\mu=40$ and solving Corollary 1, we can obtain $\gamma_{\min}=1.6985$ and

$$K_1(\Theta_1)=\begin{bmatrix}30.3842 & -82.2830 & 1.1248 & -22.4327\end{bmatrix},$$
$$K_2(\Theta_1)=\begin{bmatrix}22.5284 & -16.1523 & -0.3455 & -40.7876\end{bmatrix},$$
$$K_1(\Theta_2)=\begin{bmatrix}24.9202 & -35.7302 & -1.1369 & -26.4470\end{bmatrix},$$
$$K_2(\Theta_2)=\begin{bmatrix}21.2624 & -11.5503 & -0.1934 & -40.8517\end{bmatrix},$$
$$K_1(\Theta_3)=\begin{bmatrix}30.3842 & -82.2830 & 1.1248 & -22.4327\end{bmatrix},$$
$$K_2(\Theta_3)=\begin{bmatrix}22.5284 & -16.1523 & -0.3455 & -40.7876\end{bmatrix},$$
$$P_1=\begin{bmatrix}2.1051 & -4.7754 & 0.1721 & -0.7123\\ -4.7754 & 27.4518 & 0.1578 & -0.6256\\ 0.1721 & 0.1578 & 0.1147 & -0.0613\\ -0.7123 & -0.6256 & -0.0613 & 0.7579\end{bmatrix},$$
$$P_2=\begin{bmatrix}1.9772 & -3.6448 & 0.1751 & -0.8339\\ -3.6448 & 20.2327 & 0.1465 & -0.9927\\ 0.1751 & 0.1465 & 0.1147 & -0.0667\\ -0.8339 & -0.9927 & -0.0667 & 1.3828\end{bmatrix}.$$

To provide more illustrative results, we consider 3 kinds of disturbance as follows:

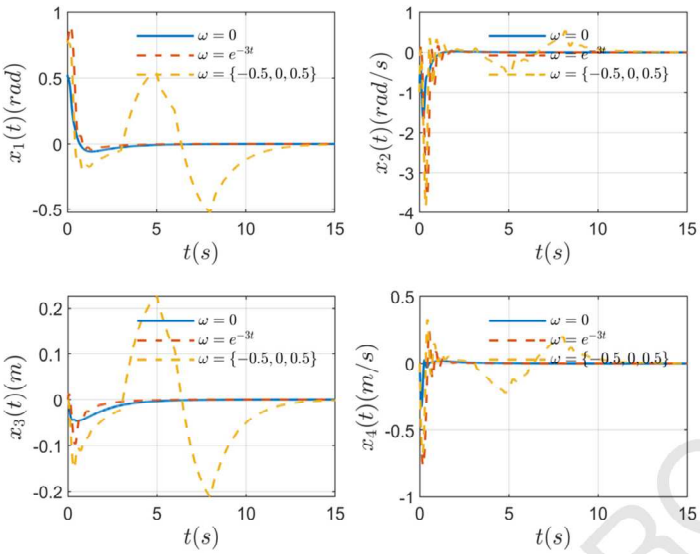

Fig. 8. State trajectories of the closed-loop system obtained from Corollary 1.

Case 1:

$$\omega = 0.$$

Case 2:

$$\omega = e^{-3t}.$$

Case 3:

$$\omega = \begin{cases} 0.5, & 3 \le t \le 5, \\ -0.5, & 6 < t \le 8, \\ 0, & else. \end{cases}$$

The state trajectories under the above 3 types of disturbances are shown in Figs. 8, demonstrating that all constraints are satisfied. It should be noted that satisfied performance is still guaranteed when dealing with decaying disturbance even if its upper bound of amplitude exceeds 0.5. The control inputs and external disturbances are depicted in Figs. 9. The Lyapunov function and $\mathscr{H}_\infty$ performance index under zero-initial condition are shown in Fig. 10, which indicates that its steady value is less than 1.6985.

5. Conclusions

A fuzzy Lyapunov based control method has been investigated for TS fuzzy systems. For the control synthesis, a model reconstruction method by dividing subregions on premise variable was proposed to transform the multiple multiplication terms in fuzzy Lyapunov functions into new fuzzy forms. This allows exploiting better the information of the membership functions for the control design to further reduce the conservatism. And the proposed method was expanded to $\mathscr{H}_\infty$ robust control case. Numerical are given to demonstrate the effectiveness of the proposed control approach in reducing the conservatism with respect to various existing control results. For future research, the approach for effectively restrict z composed by a nonlinear function with x will be investigated.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

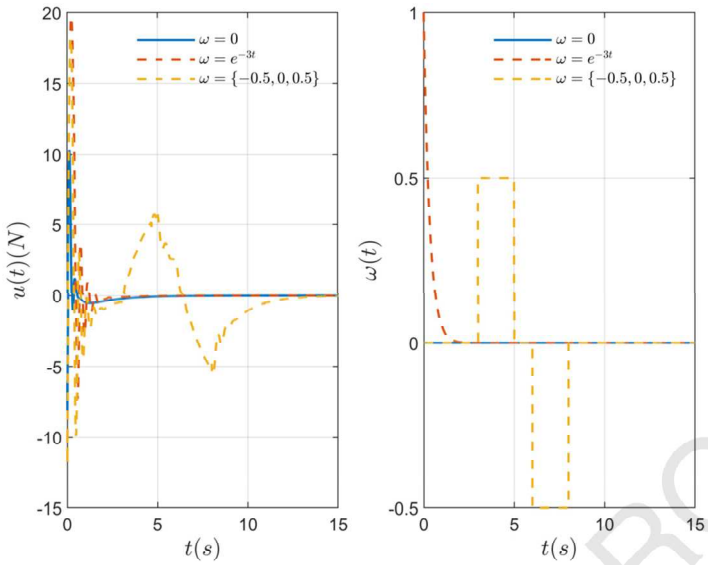

Fig. 9. Control inputs and disturbance of the closed-loop system obtained from Corollary 1.

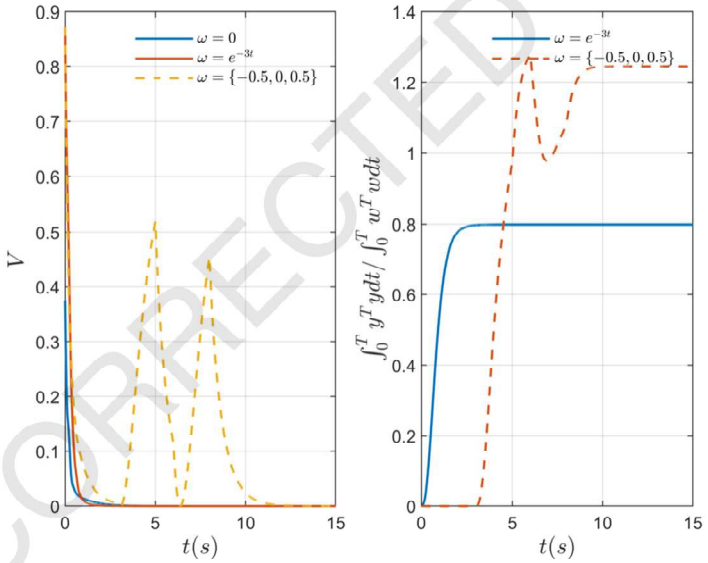

Fig. 10. Lyapunov function and $\mathscr{H}_\infty$ performance index of the closed-loop system obtained from Corollary 1.

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