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# Unknown input observers for time-varying delay Takagi-Sugeno fuzzy systems with unmeasured nonlinear consequents

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## Abstract

This paper investigates the problem of unknown input observer (UIO) design for Takagi–Sugeno (TS) fuzzy systems with time-varying delay. For observer design, all the unmeasured nonlinearities are isolated in the nonlinear consequent parts of TS fuzzy systems. This allows dealing with the well-known issue related to unmeasured premise variables for TS fuzzy observer design in a more effective fashion via the mean value theorem. Using the unknown input (UI) decoupling technique, no *a priori* information on the UI is needed for the proposed TS fuzzy observer design, which can guarantee the asymptotic estimations of both the system state and the unknown input. Employing a fuzzy Lyapunov-Krasovskii functional with a suitable augmented vector, the effect of time-varying delay on the estimation error dynamics is taken into account in the observer design. Numerical examples are presented to demonstrate the effectiveness of the proposed fuzzy UIO design method for nonlinear time-varying delay systems.

*Keywords:* Takagi-Sugeno fuzzy models, time-varying delay, unknown inputs, unmeasured premise variables.

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## 1. Introduction

State estimation of nonlinear systems in the presence of unknown inputs (UIs) has received increasing research attention. The main reason is that, within different application contexts, unknown inputs can be seen  
5 as unmodeled dynamics, faults in engineering systems, uncertain disturbances, and attack signals in secure communication or cyber-physical systems [1, 2, 3, 4, 5, 6]. Consequently, simultaneous estimation of the state of a dynamical system and its unknown input has become a key aspect in several practical applications [7, 8, 9].

10 Takagi–Sugeno (TS) fuzzy model-based technique [10] has become one of the most effective tools to control and estimation of nonlinear systems. TS fuzzy modeling together with Lyapunov stability theory, conditions for stability analysis, control, and observer design can be described in terms of linear matrix inequality (LMI) constraints, which can be effectively solved via  
15 numerical solvers. **Therefore, TS fuzzy-model-based methods have received considerable attention. Within this context, a great deal of research effort has been devoted to the design of TS fuzzy observers for nonlinear systems [11, 12, 13, 14]. In particular, researchers have increasingly focused on developing TS fuzzy observer design methods in the presence of UIs, as UIs have become** **R1.1**  
20 **a key role in fault diagnosis [15, 16, 17, 8, 18] and robust and fault-tolerant control [19, 20, 21].**

It is noteworthy that when the premise variables can be measured, many results existing for linear observer design can be extended to TS fuzzy systems [22, 23]. However, the corresponding results can only be applied to  
25 a restrictive class of nonlinear systems. Hence, TS fuzzy observer design with unmeasured premise variables must be considered, which leads to a challenging TS fuzzy observer design problem due to mismatching nonlinear terms involved in the estimation error dynamics [11, 24]. One of the most common approaches to deal with the mismatching terms is based on the Lipschitz property of the membership functions (MFs) [22, 25, 26]. However, as  
30 pointed out in [24], this approach generally leads to over-conservative results. To avoid this major drawback, the mean value theorem has been exploited for TS fuzzy observer design with unmeasured nonlinearities [11, 4, 27]. Unknown input observers (UIOs) have been also developed for TS fuzzy systems  
35 with unmeasured premise variables [26, 28, 4, 29, 30, 27]. A proportional-integral observer structure has been proposed in [28] for continuous-time TS fuzzy models with unknown inputs, which are assumed to be of polynomial

forms. The authors in [26] have proposed a method to estimate the uncertainties of TS fuzzy systems using UIO techniques. However, the results in [26] require *a posteriori* check on the Lipschitzian property of the nonlinear mismatching terms, caused by unmeasured premise variables, which can be restrictive for practical uses. Using a UI decoupling method for nonlinear dynamics, the issue of unmeasured premise variables in TS fuzzy observer synthesis can be overcome in [29, 30]. In [4, 27], nonlinear systems have been reformulated in a specific TS fuzzy form with unmeasured nonlinear consequent parts to design UIOs while avoiding unmeasured premise variables. **It is worth mentioning that TS fuzzy models with nonlinear consequent parts have also been employed to deal with the stabilization problem of nonlinear systems, see [31, 32] and related references.**

Time delays are omnipresent in many control engineering applications, for instance, packet-based communication systems, networked control systems, population dynamics, epidemics [33]. Furthermore, the presence of time delay in the system dynamics can lead to critical issues related to poor performance, oscillation, and instability. Hence, due to the theoretical significance and practical relevance, it is essential to take into account the influences of time delays in the study of dynamical systems. Within the context of time-delay TS fuzzy models, Lyapunov–Krasovskii stability theory has been largely exploited to derive LMI-based conditions to design TS fuzzy controllers and observers, see for instance [34, 35, 36, 37, 38, 39, 40, 41, 42, 43] and related references. Following this idea, UIOs have been also developed for time-delay TS fuzzy systems. The robust fault detection problem has been addressed in [44, 45] for TS fuzzy systems subject to unknown input and time-varying delay. Robust fault estimation has been studied for a class of delayed TS fuzzy systems in [46], in which the system state and the actuator fault can be simultaneously estimated using a learning observer scheme. In [47], the estimation of both actuator and sensor faults have been studied for switched discrete-time TS fuzzy systems with constant time delay. The authors in [48] have developed a fault-tolerant control scheme for TS fuzzy systems with multiple time-varying delays as well as sensor and actuator faults. In [49], an observer-based control method has been proposed for fractional-order uncertain TS fuzzy systems with unknown input and time delay. It is worth mentioning that the above-mentioned results on TS fuzzy UIO design are only concerned with the case of measured premise variables, which can be restrictive for engineering applications. Notice also that UIOs have been also developed for Lipschitz nonlinear systems with constant time-

delay in [50, 51, 52]. Despite a great advance, there is a lack of literature on UIO design for delayed TS fuzzy systems with unmeasured premise variables.

Motivated by the aforementioned discussion, this paper presents novel conditions to design UIOs for nonlinear systems with time-varying delays and unmeasured nonlinearities. To this end, nonlinear systems are represented by TS fuzzy model, where all unmeasured nonlinearities are regrouped in the nonlinear consequents. By means of the mean value theorem [53], this enables an effective way to deal with the major issue related to unmeasured premise variables in TS fuzzy observer design. The effect of time-varying delay on the estimation error dynamics is explicitly taken into account in the observer design procedure via a fuzzy Lyapunov-Krasovskii functional (LKF) with a suitable augmented vector to reduce the design conservativeness. The Wirtinger-based summation inequality is used together with Moon's inequality as well as the well-known Finsler's lemma to derive numerically tractable UIO design conditions. Notice that the resulting UIO design for TS fuzzy systems with time-varying delay and unmeasured nonlinearities has not been reported in the open literature. The main contributions of this paper can be summarized as follows.

### R2.3

- Using fuzzy LKF stability tools together with various relaxation techniques, a new set of *delay-dependent* LMI conditions are derived to design UIOs for TS fuzzy systems with time-varying delay and unmeasured nonlinearities.
- The designed fuzzy observer allows to simultaneously and asymptotically estimate both the system state and the UIs without requiring any *a priori* information on UIs.

*Notation.*  $\mathbb{Z}^+$  is the set of positive integers. For a natural number  $m$ ,  $\mathbb{N}_{\leq m} = \{1, 2, \dots, m\}$ .  $\mathbb{S}_+^n$  is the set of symmetric positive definite matrices of order  $n$ .  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space, and  $\mathbb{R}^{m \times n}$  is the set of all  $m \times n$  real matrices. For two vectors  $x, y \in \mathbb{R}^n$ ,  $x_i$  denotes the  $i$ th entry of  $x$ ; the convex hull of vectors  $\gamma$  and  $\beta$  is denoted as  $\text{co}(\beta, \gamma) = \{\lambda\beta + (1-\lambda)\gamma : \lambda \in [0, 1]\}$ . For  $i \in \mathbb{N}_{\leq m}$ , we denote  $\sigma_m(i) = [0, \dots, 0, 1_{i\text{th}}, 0, \dots, 0]^\top \in \mathbb{R}^m$ . For a matrix  $Y$ ,  $Y \succ 0$  means that  $Y$  is positive definite, and  $\text{He}\{Y\} = Y + Y^\top$ . If the existence is ensured,  $Y^\dagger$  denotes the Moore-Penrose pseudo-inverse of matrix  $Y$ , *i.e.*,  $Y^\dagger = (Y^\top Y)^{-1} Y^\top$ . A block diagonal matrix whose elements are matrices  $A$  and  $B$  is denoted by  $\text{diag}(A, B)$ . The identity matrix of order  $n$  is denoted by  $I_n$  and the null matrix of order  $n \times m$  by  $0_{n \times m}$ . If the

dimensions of both identity and null matrices are straightforwardly deduced, they are omitted.

## 2. Problem Statement

115 After a system description, this section formulates the UIO design problem for TS fuzzy systems with time-varying delays and unmeasured nonlinearities.

### 2.1. System Description

Consider a nonlinear system with time-varying delays given by

$$\begin{aligned} x_{k+1} &= A(\mu_k)x_k + A_d(\mu_k)x_{k-\tau_k} + f(\mu_k, u_k) + G(\mu_k)\phi(x_k) + Dd_k, \quad \forall k \in \mathbb{Z}^+ \\ y_k &= Cx_k \\ x_k &= \varphi_k, \quad k \in \{-\bar{\tau}, \dots, 0\}, \end{aligned} \quad (1)$$

120 being  $x_k \in \mathcal{D}_x \subseteq \mathbb{R}^{n_x}$  the state,  $y_k \in \mathbb{R}^{n_y}$  the system output,  $u_k \in \mathbb{R}^{n_u}$  the *known* input,  $d_k \in \mathbb{R}^{n_d}$  is the *unknown* input (UI),  $\mu_k \in \mathbb{R}^{n_\mu}$  the vector of measurable premise variables,  $\varphi_k$  the initial condition, and  $\tau_k$  is a bounded time-varying delay. The elements of function  $f : \mathcal{D}_x \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$  can be measured from the output whereas the nonlinear function  $\phi : \mathcal{D}_x \rightarrow \mathbb{R}^{n_\phi}$  depends on *unmeasured* states.

Using the fuzzy modeling technique [10], the nonlinear system with time-varying delay (1) can be reformulated as a Takagi-Sugeno fuzzy model with nonlinear consequents:

$$\begin{aligned} \text{RULE } \mathcal{R}_i : & \text{ IF } \mu_1 \text{ is } \mathcal{M}_1^i \text{ and } \dots \text{ and } \mu_{n_\mu} \text{ is } \mathcal{M}_{n_\mu}^i. \text{ THEN} \\ & \begin{cases} x_{k+1} = A_i x_k + A_{di} x_{k-\tau_k} + Dd_k + f(\mu_k, u_k) + G_i \phi(x_k) \\ y_k = Cx_k \\ x_k = \varphi_k, \quad k \in \{-\bar{\tau}, \dots, 0\}, \end{cases} \end{aligned} \quad (2)$$

being  $\mathcal{R}_i$  the  $i$ th fuzzy rule, the matrices  $(A_i, A_{di}, G_i)$  are known,  $r$  denotes the number of fuzzy rules, and  $\mathcal{M}_j^i$ , for  $i \in \mathbb{N}_{\leq r}$  and  $j \in \mathbb{N}_{\leq n_\mu}$ , are the fuzzy sets. The fuzzy membership functions are defined as

$$h_i(\mu_k) = \frac{\prod_{j=1}^{n_\mu} \lambda_j^i(\mu_j)}{\sum_{i=1}^r \prod_{j=1}^{n_\mu} \lambda_j^i(\mu_j)}, \quad i \in \mathbb{N}_{\leq r}, \quad (3)$$

where  $\lambda_j^i(\mu_j)$  represents the membership grade of  $\mu_j$  with respect to the fuzzy set  $\mathcal{M}_j^i$ . Notice that the MFs defined in (3) belong to the unit simplex  $\mathcal{H}$  with  $r$  vertices

$$\mathcal{H} = \left\{ h \in \mathbb{R}^r : \sum_{i=1}^r h_i = 1, 0 \leq h_i \leq 1, i \in \mathbb{N}_{\leq r} \right\},$$

where  $h = [h_1(\mu_k), h_2(\mu_k), \dots, h_r(\mu_k)]$ . Using the center-of-gravity method for defuzzification [10], the following compact form can be obtained for the TS fuzzy system (2):

$$\begin{aligned} x_{k+1} &= A(h)x_k + A_d(h)x_{k-\tau_k} + Dd_k + f(\mu_k, u_k) + G(h)\phi(x_k) \\ y_k &= Cx_k \\ x_k &= \varphi_k, \quad k \in \{-\bar{\tau}, \dots, 0\}, \end{aligned} \tag{4}$$

with

$$[A(h) \quad A_d(h) \quad G(h)] = \sum_{i=1}^r h_i(\mu_k) [A_i \quad A_{di} \quad G_i].$$

125 For UIO design, the following assumptions for system (1) equivalently represented in form (4) are considered.

**Assumption 1.** *The time-varying delay  $\tau_k$  is assumed to be known, with  $\underline{\tau} \leq \tau_k \leq \bar{\tau}$ , for  $\forall k \in \mathbb{Z}^+$ , where the lower bound  $\underline{\tau}$  and the upper bound  $\bar{\tau}$  are given positive integers.*

**R3**  
**R4.1**

**Assumption 2.** *The nonlinear function  $\phi(x) : \mathcal{D}_x \rightarrow \mathbb{R}^{n_\phi}$  is differentiable with respect to  $x$  and satisfies*

$$\underline{\phi}_{ij} \leq \frac{\partial \phi_i}{\partial x_j}(x) \leq \bar{\phi}_{ij}, \quad \forall i \in \mathbb{N}_{\leq n_\phi}, \quad \forall j \in \mathbb{N}_{\leq n_x}, \tag{5}$$

130 with  $\underline{\phi}_{ij} = \min_{x \in \mathcal{D}_x} \left( \frac{\partial \phi_i}{\partial x_j}(x) \right)$  and  $\bar{\phi}_{ij} = \max_{x \in \mathcal{D}_x} \left( \frac{\partial \phi_i}{\partial x_j}(x) \right)$ .

**Assumption 3.** *The matrix  $D$  is of full column rank, the output matrix  $C$  is of full row rank, and  $\text{rank}(CD) = \text{rank}(D)$ .*

**R2.1**

**Remark 1.** *The rank condition in Assumption 3 has been widely employed for UIO design to decouple the effect of UIs on the estimation error dynamics [54, 4, 55]. Notice also that a large variety of nonlinearities satisfies*

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condition (5), e.g., the class of differentiable Lipschitzian nonlinear functions [23, 53, 52]. Moreover, for fuzzy observer design, Assumption 2 is not restrictive, since TS fuzzy models are commonly constructed within a state-space compact set  $\mathcal{D}_x$ , especially when the sector nonlinearity approach [10] is used.

## 2.2. Problem Definition

For the estimation of system (4), the following time-delay UIO structure is considered:

$$\begin{aligned} z_{k+1} &= N(h)z_k + MA_d(h)\hat{x}_{k-\tau_k} + L(h)y_k + M\hat{\Lambda}_k \\ \hat{x}_k &= z_k - Fy_k, \quad z_0 = 0 \\ \hat{d}_k &= (CD)^\dagger(y_{k+1} - C\hat{\Lambda}_k - CA(h)\hat{x}_k - CA_d(h)\hat{x}_{k-\tau_k}), \end{aligned} \quad (6)$$

where  $\hat{\Lambda}_k = G(h)\phi(\hat{x}_k) + f(\mu_k, u_k)$ ,  $\hat{x}_k$  is the state estimate,  $\hat{d}_k$  is the UI estimate, and

$$M = I + FC. \quad (7)$$

Notice that Assumption 3 guarantees the existence of  $(CD)^\dagger$ . The MF-dependent matrices  $L(h) \in \mathbb{R}^{n_x \times n_y}$ ,  $N(h) \in \mathbb{R}^{n_x \times n}$ , and the matrix  $F \in \mathbb{R}^{n_x \times n_y}$ , are gains to be determined defined as

$$[N(h) \quad L(h)] = \sum_{i=1}^r h_i(\mu_k) [N_i \quad L_i].$$

**R5.2**

**Remark 2.** Note from (6) that for UI estimation, it is required to have the future information of the measured output  $y_{k+1}$ . It is emphasized that this UI estimation expression is only used in theoretical analysis. In practice, the UI estimation should be implemented as follows:

$$\hat{d}_{k-1} = (CD)^\dagger(y_k - C\hat{\Lambda}_{k-1} - CA(h_-)\hat{x}_{k-1} - CA_d(h_-)\hat{x}_{k-\tau_{k-1}}).$$

where  $h_- = [h_1(\mu_{k-1}), h_2(\mu_{k-1}), \dots, h_r(\mu_{k-1})]$ . Since the past values of all signals involved in the above estimation expression are available, we can always estimate the one-step-back value  $\hat{d}_{k-1}$  of the UI. This practical issue is common for UI estimation with decoupling-based approaches [4, 7, 26, 52, 55].

**Remark 3.** All unmeasured nonlinearities of system (4) are regrouped in the nonlinear consequent  $\phi(x)$ . Then, the MFs  $h_i(\mu_k)$ , for  $i \in \mathbb{N}_{\leq r}$ , only depend on measured premise variables. Therefore, for real-time implementation, these MFs can be directly employed to construct the TS fuzzy observer (6) without any estimated premise variables.

The state estimation error is defined as  $e_k = \hat{x}_k - x_k$ . Then, from (6) and (7), one has  $e_k = z_k - Fy_k - x_k$ , or

$$e_k = z_k - Mx_k. \quad (8)$$

Hence, from (4), (6) and (8), the estimation error dynamics can be defined as

$$\begin{aligned} e_{k+1} &= z_{k+1} - Mx_{k+1} \\ &= N(h)e_k + MA_d(h)e_{k-\tau_k} - MDd_k + (L(h)C + N(h)M - MA(h))x_k + \\ &\quad MG(h)\delta_\phi, \end{aligned} \quad (9)$$

where  $\delta_\phi = \phi(\hat{x}_k) - \phi(x_k)$ . To convert this nonlinear mismatch term  $\delta_\phi$  into a function of  $e_k$  for UIO design, the following mean value theorem is used.

**Lemma 1** ([53, Proposition 2]). Let  $\psi(x) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\psi}$  and  $\beta, \gamma \in \mathbb{R}^{n_x}$ . If  $\psi(x)$  is differentiable with respect to  $x$  on  $\text{co}(\beta, \gamma)$ , then there exist constant vectors  $c_i \in \text{co}(\beta, \gamma)$ ,  $\forall i \in \mathbb{N}_{\leq n_\psi}$ , such that

$$\psi(\beta) - \psi(\gamma) = \left( \sum_{i=1}^{n_\psi} \sum_{j=1}^{n_x} \sigma_{n_\psi}(i) \sigma_{n_x}^\top(j) \frac{\partial \psi_i}{\partial x_j}(c_i) \right) (\beta - \gamma).$$

The nonlinear term  $\delta_\phi$  in (9) can be rewritten using Assumption 2 and Lemma 1 as

$$\delta_\phi = \phi(\hat{x}_k) - \phi(x_k) = \Psi e_k, \quad (10)$$

where

$$\Psi = \left[ \sum_{i=1}^{n_\phi} \sum_{j=1}^{n_x} \sigma_{n_\phi}(i) \sigma_{n_x}^\top(j) \frac{\partial \phi_i}{\partial x_j}(\chi_i) \right], \quad (11)$$

with  $\chi_i \in \text{co}(x, \hat{x})$ . Defining  $\rho_{ij} = \frac{\partial \phi_i}{\partial x_j}(\chi_i)$ ,  $\forall i \in \mathbb{N}_{\leq n_\phi}$  and  $\forall j \in \mathbb{N}_{\leq n_x}$ , note from (5) that the matrix  $\Psi$  is with elements varying within a bounded convex set, whose set of  $2^{n_x n_\phi}$  vertices is given by

$$\mathcal{V}_\phi = \left\{ \rho = [\rho_{11}, \dots, \rho_{1n_x}, \dots, \rho_{n_\phi n_x}] : \rho_{ij} \in \left[ \underline{\phi}_{ij}, \bar{\phi}_{ij} \right] \right\}, \quad (12)$$

with  $\underline{\phi}_{ij}$  and  $\bar{\phi}_{ij}$  defined as in (5). From (9) and (10), the error dynamics can be rewritten as R4.5

$$e_{k+1} = \mathcal{N}(h, \Psi)e_k + MA_d(h)e_{k-\tau_k} - MDd_k + (N(h)M + L(h)C - MA(h))x_k, \quad (13)$$

where  $\mathcal{N}(h, \Psi) = \sum_{i=1}^r h_i(\mu_k)\mathcal{N}_i(\Psi)$ , and

$$\mathcal{N}_i(\Psi) = N_i + MG_i\Psi. \quad (14)$$

Remark that  $\mathcal{N}(h, \Psi)$  can be represented in a polytopic form as  $\mathcal{N}(h, \rho_{lj}) = \sum_{i=1}^r h_i(\mu_k)\mathcal{N}_i(\rho_{lj})$  with R4.3

$$\mathcal{N}_i(\rho_{lj}) = N_i + MG_i \sum_{l=1}^{n_\phi} \sum_{j=1}^{n_x} \sigma_{n_\phi}(l)\sigma_{n_x}^\top(j)\rho_{lj}.$$

The following UIO design problem for TS with time-varying delay fuzzy systems can be now formulated.

155 **Problem 1.** Consider the nonlinear system with time-varying delay (1), represented by the Takagi-Sugeno fuzzy model (4). Design an UIO (6) such that  $\hat{x}_k$  and  $\hat{d}_k$  asymptotically converge to  $x_k$  and  $d_k$ , respectively, for  $\forall \tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$ ,  $k \in \mathbb{Z}$ .

The following technical lemmas are useful to develop time-varying delay UIOs as stated in Problem 1. 160

**Lemma 2** (Wirtinger's summation inequality [56]). Consider a given matrix  $R \in \mathbb{S}_+^n$ , integers  $\gamma > \beta$ . Then, for any sequence of discrete-time variable  $x : [\beta, \gamma] \rightarrow \mathbb{R}^n$ , the following inequalities hold:

$$\sum_{i=\beta}^{\gamma-1} \eta_i^\top R \eta_i \geq \frac{1}{\gamma - \beta} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & \left(\frac{\gamma - \beta + 1}{\gamma - \beta - 1}\right) 3R \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}, \quad (15)$$

$$\sum_{i=\beta}^{\gamma-1} \eta_i^\top R \eta_i \geq \frac{1}{\gamma - \beta} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}, \quad (16)$$

with  $\eta_i = x_i - x_{i-1}$ , and

$$\Theta_1 = x_\gamma - x_\beta, \quad \Theta_2 = x_\gamma + x_\beta - \frac{2}{\gamma - \beta + 1} \sum_{i=\beta}^{\gamma} x_i.$$

**Lemma 3** (Moon's inequality [57]). *For any matrices  $R_1, R_2 \in \mathbb{S}_+^n$  and  $Y_1, Y_2 \in \mathbb{R}^{2n \times n}$ , and any scalar  $\varrho \in (0, 1)$ , the following inequality holds:*

$$\begin{bmatrix} \frac{1}{\varrho} R_1 & 0 \\ \varrho Y_1 & \frac{1}{1-\varrho} R_2 \end{bmatrix} \succeq \Theta_R(\varrho), \quad (17)$$

with

$$\Theta_R(\varrho) = \text{He} \{ Y_1 [I_n \ 0_n] + Y_2 [0_n \ I_n] \} - \varrho Y_1 R_1^{-1} Y_1^\top - (1 - \varrho) Y_2 R_2^{-1} Y_2^\top.$$

**Remark 4.** *The use of Wirtinger's inequality together with Moon's inequality allows to deal with multiple summations, involved in LKF-based stability analysis conditions, in a less conservative fashion compared to the standard reciprocally convex combination lemma [58]. Hence, these technical tools are employed here to derive LKF-based conditions to design UIOs for TS fuzzy systems with time-varying delays.*

### 3. UIO Design for Time-Varying Delay TS Fuzzy Systems

This section presents conditions to design UIOs for TS fuzzy systems with time-varying delays.

#### 3.1. Existence of Time-Varying Delay Unknown Input Observers

The following theorem provides the existence conditions of time-varying delay fuzzy UIOs.

**Theorem 1.** *Given positive integers  $\underline{\tau}$  and  $\bar{\tau}$ , with  $\underline{\tau} < \bar{\tau}$ , and a positive scalar  $\epsilon$ . There is an asymptotic UIO of the form (6), if there exist MF-dependent matrices  $L(h) \in \mathbb{R}^{n_x \times n_y}$ ,  $N(h) \in \mathbb{R}^{n_x \times n_x}$ , positive definite matrices  $P(h) \in \mathbb{S}_+^{3n_x}$ ,  $R_1, R_2, Z_1, Z_2 \in \mathbb{S}_+^{n_x}$ , and matrices  $F \in \mathbb{R}^{n_x \times n_y}$ ,  $W \in \mathbb{R}^{n_x \times n_x}$  and  $Y_1, Y_2 \in \mathbb{R}^{8n_x \times 2n_x}$ , such that*

$$L(h)C + N(h)M - MA(h) = 0, \quad (18)$$

$$MD = 0, \quad (19)$$

$$\begin{bmatrix} \Phi(h, h_+, \underline{\tau}) + \text{He} \{ \mathcal{W}^\top \mathcal{B}(h, \Psi) \} & Y_2 \\ Y_2^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (20)$$

$$\begin{bmatrix} \Phi(h, h_+, \bar{\tau}) + \text{He} \{ \mathcal{W}^\top \mathcal{B}(h, \Psi) \} & Y_1 \\ Y_1^\top & -\tilde{Z}_1 \end{bmatrix} \prec 0, \quad (21)$$

for  $h, h_+ \in \mathcal{H}$  and  $\Psi \in \mathcal{V}_\phi$ . The matrices  $\Phi(h, h_+, \underline{\tau})$ ,  $\Phi(h, h_+, \bar{\tau})$ ,  $\mathcal{W}$  and  $\mathcal{B}(h, \Psi)$  in (20)–(21) are defined as

$$\begin{aligned}
\Phi(h, h_+, \underline{\tau}) &= \vartheta(h, h_+) - \text{He} \{ \Gamma^\top(\underline{\tau}) P(h) F_2 \} + \text{He} \{ \Gamma^\top(\underline{\tau}) P(h_+) F_1 \} \\
&\quad + \Gamma^\top(\underline{\tau}) P(h_+) \Gamma(\underline{\tau}) - \Gamma^\top(\underline{\tau}) P(h) \Gamma(\underline{\tau}), \\
\Phi(h, h_+, \bar{\tau}) &= \vartheta(h, h_+) - \text{He} \{ \Gamma^\top(\bar{\tau}) P(h) F_2 \} + \text{He} \{ \Gamma^\top(\bar{\tau}) P(h_+) F_1 \} \\
&\quad + \Gamma^\top(\bar{\tau}) P(h_+) \Gamma(\bar{\tau}) - \Gamma^\top(\bar{\tau}) P(h) \Gamma(\bar{\tau}), \\
\vartheta(h, h_+) &= F_1^\top (P(h_+) + \Pi_1) F_1 - F_2^\top P(h) F_2 + \tilde{R} - G_1^\top \tilde{Z}_1 G_1 \\
&\quad - \text{He} \{ Y_1 G_2 + Y_2 G_3 \}, \tag{22}
\end{aligned}$$

$$P(h) = \sum_{i=1}^r h_i(\mu_k) P_i, \quad P(h_+) = \sum_{q=1}^r h_q(\mu_{k+1}) P_q,$$

$$\tilde{Z}_1 = \text{diag}(Z_1, 3\delta(\underline{\tau}) Z_1), \quad \tilde{Z}_2 = \text{diag}(Z_2, 3Z_2),$$

$$\Pi_1 = \text{diag}(\underline{\tau}^2 Z_1 + \tilde{\tau}^2 Z_2, 0_{2n_x}),$$

$$\tilde{R} = \text{diag}(0, R_1, R_2 - R_1, 0_{n_x}, -R_2, 0_{3n_x}),$$

$$\mathcal{W} = \epsilon W^\top v_1 + W^\top v_2 + W^\top v_4,$$

$$\mathcal{B}(h, \Psi) = -v_1 + \mathcal{N}(h, \Psi) v_2 + M A_d(h) v_4,$$

$$F_1 = \begin{bmatrix} v_1 - v_2 \\ v_6 - v_3 \\ v_7 + v_8 - v_4 - v_5 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0_{n_x \times 8n_x} \\ v_6 - v_2 \\ v_7 + v_8 - v_3 - v_4 \end{bmatrix},$$

$$\Gamma(\underline{\tau}) = \begin{bmatrix} v_2 \\ \underline{\tau} v_6 \\ \tilde{\tau} v_8 \end{bmatrix}, \quad \Gamma(\bar{\tau}) = \begin{bmatrix} v_2 \\ \bar{\tau} v_6 \\ \tilde{\tau} v_7 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} v_2 - v_3 \\ v_2 + v_3 - 2v_6 \end{bmatrix}, \quad G_2 = \begin{bmatrix} v_3 - v_4 \\ v_3 + v_4 - 2v_7 \end{bmatrix}, \quad G_3 = \begin{bmatrix} v_4 - v_5 \\ v_4 + v_5 - 2v_8 \end{bmatrix},$$

with  $v_i = [0_{n_x \times (i-1)n_x} \quad I_{n_x} \quad 0_{n_x \times (8-i)n_x}]$ , for  $i \in \mathbb{N}_{\leq 8}$ ,  $\tilde{\tau} = \bar{\tau} - \underline{\tau}$ , and

$$\delta(\underline{\tau}) = \begin{cases} 1, & \text{if } \underline{\tau} = 1, \\ \frac{\underline{\tau} + 1}{\underline{\tau} - 1}, & \text{if } \underline{\tau} > 1. \end{cases}$$

*Proof.* Using the algebraic constraints (18) and (19), it is possible to rewrite the error dynamics (13) as

$$e_{k+1} = \mathcal{N}(h, \Psi)e_k + MA_d(h)e_{k-\tau_k}. \quad (23)$$

The following fuzzy Lyapunov-Krasovskii functional is employed to ensure the stability analysis of the time-varying delay system (23),

$$V(e_k) = V_1(e_k) + V_2(e_k) + V_3(e_k), \quad (24)$$

where

$$\begin{aligned} V_1(e_k) &= \varpi_k^\top P(h) \varpi_k, \\ V_2(e_k) &= \sum_{i=k-\underline{\tau}}^{k-1} e_i^\top R_1 e_i + \sum_{i=k-\bar{\tau}}^{k-\underline{\tau}-1} e_i^\top R_2 e_i, \\ V_3(e_k) &= \underline{\tau} \sum_{i=-\underline{\tau}+1}^0 \sum_{j=k+i}^k \eta_j^\top Z_1 \eta_j + \bar{\tau} \sum_{i=-\bar{\tau}+1}^{-\underline{\tau}} \sum_{j=k+i}^k \eta_j^\top Z_2 \eta_j, \\ \varpi_k &= \left[ e_k^\top \quad \sum_{i=k-\underline{\tau}}^{k-1} e_i^\top \quad \sum_{i=k-\bar{\tau}}^{k-\underline{\tau}-1} e_i^\top \right]^\top, \\ \eta_i &= e_i - e_{i-1}. \end{aligned} \quad (25)$$

Since the matrices  $P(h)$ ,  $R_1$ ,  $R_2$ ,  $Z_1$  and  $Z_2$  are positive definite, the LKF defined in (24) is also positive definite. Let us define the following augmented vector:

$$\zeta_k = \left[ e_{k+1}^\top \quad e_k^\top \quad e_{k-\underline{\tau}}^\top \quad e_{k-\tau_k}^\top \quad e_{k-\bar{\tau}}^\top \quad \nu_{1k}^\top \quad \nu_{2k}^\top \quad \nu_{3k}^\top \right]^\top \quad (26)$$

with

$$\nu_{1k} = \frac{1}{\underline{\tau} + 1} \sum_{i=k-\underline{\tau}}^k e_i, \quad \nu_{2k} = \frac{1}{\tau_k - \underline{\tau} + 1} \sum_{i=k-\tau_k}^{k-\underline{\tau}} e_i, \quad \nu_{3k} = \frac{1}{\bar{\tau} - \tau_k + 1} \sum_{i=k-\bar{\tau}}^{k-\tau_k} e_i.$$

Now, it is possible to compute an upper bound of the difference  $\Delta V(e_k) = V(e_{k+1}) - V(e_k)$ , where  $V(e_k)$  is defined in (24). To this end, we express  $\Delta V(e_k)$  in the function of  $\zeta_k$ . We have

$$\begin{aligned} \Delta V_1(e_k) &= V_1(e_{k+1}) - V_1(e_k) \\ &= \varpi_{k+1}^\top P(h_+) \varpi_{k+1} - \varpi_k^\top P(h) \varpi_k. \end{aligned} \quad (27)$$

From the definitions of  $\varpi_k$  in (25), and  $F_1, F_2$  in (22), it follows that

$$\varpi_k = (F_2 + \Gamma(\tau_k))\zeta_k \quad \text{and} \quad \varpi_{k+1} = (F_1 + \Gamma(\tau_k))\zeta_k, \quad (28)$$

with

$$\Gamma(\tau_k) = \begin{bmatrix} v_2 \\ \underline{\tau}v_6 \\ (\tau_k - \underline{\tau})v_7 + (\bar{\tau} - \tau_k)v_8 \end{bmatrix}.$$

From (27) and (28),  $\Delta V_1(e_k)$  can be rewritten as

$$\Delta V_1(e_k) = \zeta_k^\top \left[ (F_1 + \Gamma(\tau_k))^\top P(h_+) (F_1 + \Gamma(\tau_k)) - (F_2 + \Gamma(\tau_k))^\top P(h) (F_2 + \Gamma(\tau_k)) \right] \zeta_k. \quad (29)$$

Moreover, the difference  $\Delta V_2(e_k) = V_2(e_{k+1}) - V_2(e_k)$  can be computed as

$$\Delta V_2(e_k) = e_k^\top R_1 e_k + e_{k-\underline{\tau}}^\top (R_2 - R_1) e_{k-\underline{\tau}} - e_{k-\bar{\tau}}^\top R_2 e_{k-\bar{\tau}},$$

which can be described in the compact form

$$\Delta V_2(e_k) = \zeta_k^\top \tilde{R} \zeta_k, \quad (30)$$

where  $\tilde{R}$  is defined in (22). Furthermore, the computation of the difference  $\Delta V_3(e_k) = V_3(e_{k+1}) - V_3(e_k)$  results in

$$\Delta V_3(e_k) = \eta_{k+1}^\top (\underline{\tau}^2 Z_1 + \tilde{\tau}^2 Z_2) \eta_{k+1} - \underline{\tau} \sum_{i=k-\underline{\tau}+1}^k \eta_i^\top Z_1 \eta_i - \tilde{\tau} \sum_{i=k-\bar{\tau}+1}^{k-\underline{\tau}} \eta_i^\top Z_2 \eta_i,$$

which can be rewritten as

$$\begin{aligned} \Delta V_3(e_k) &= \eta_{k+1}^\top (\underline{\tau}^2 Z_1 + \tilde{\tau}^2 Z_2) \eta_{k+1} - \underline{\tau} \sum_{i=k-\underline{\tau}+1}^k \eta_i^\top Z_1 \eta_i \\ &\quad - \tilde{\tau} \sum_{i=k-\bar{\tau}+1}^{k-\tau_k} \eta_i^\top Z_2 \eta_i - \tilde{\tau} \sum_{i=k-\tau_k+1}^{k-\underline{\tau}} \eta_i^\top Z_2 \eta_i. \end{aligned} \quad (31)$$

From (29), (30) and (31), the difference  $\Delta V(e_k) = V(e_{k+1}) - V(e_k)$  can be expressed by

$$\begin{aligned} \Delta V(e_k) &= \zeta_k^\top \left[ (F_1 + \Gamma(\tau_k))^\top P(h_+) (F_1 + \Gamma(\tau_k)) - (F_2 + \Gamma(\tau_k))^\top P(h) (F_2 + \Gamma(\tau_k)) \right] \zeta_k \\ &\quad + \zeta_k^\top \tilde{R} \zeta_k + \eta_{k+1}^\top (\underline{\tau}^2 Z_1 + \tilde{\tau}^2 Z_2) \eta_{k+1} - \Sigma_1 - \Sigma_2 - \Sigma_3, \end{aligned} \quad (32)$$

where

$$\begin{aligned}
\Sigma_1 &= \tau \sum_{i=k-\tau+1}^k \eta_i^\top Z_1 \eta_i, \\
\Sigma_2 &= \tilde{\tau} \sum_{i=k-\tilde{\tau}+1}^{k-\tau_k} \eta_i^\top Z_2 \eta_i, \\
\Sigma_3 &= \tilde{\tau} \sum_{i=k-\tau_k+1}^{k-\tau} \eta_i^\top Z_2 \eta_i.
\end{aligned} \tag{33}$$

Applying the Wirtinger's inequality (15) to the summation  $\Sigma_1$ , and (16) to the summations  $\Sigma_2$  and  $\Sigma_3$  defined in (33), with matrices  $F_1$ ,  $F_2$ ,  $G_1$ ,  $G_2$  and  $G_3$  defined in (22), the following upper bound can be computed for  $\Delta V(e_k)$  in (32):

**R5.3**

$$\begin{aligned}
\Delta V(e_k) &\leq \zeta_k^\top \left[ (F_1 + \Gamma(\tau_k))^\top P(h_+) (F_1 + \Gamma(\tau_k)) \right] \zeta_k \\
&\quad + \zeta_k^\top \left[ -(F_2 + \Gamma(\tau_k))^\top P(h) (F_2 + \Gamma(\tau_k)) + \tilde{R} + F_1^\top \Pi_1 F_1 \right] \zeta_k \\
&\quad + \zeta_k^\top \left[ -G_1^\top \tilde{Z}_1 G_1 - \begin{bmatrix} G_2 \\ G_3 \end{bmatrix}^\top \begin{bmatrix} \frac{\tilde{\tau}}{\tau_k - \tau} \tilde{Z}_2 & 0_{2n_x} \\ 0_{2n_x} & \frac{\tilde{\tau}}{\tilde{\tau} - \tau_k} \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} G_2 \\ G_3 \end{bmatrix} \right] \zeta_k.
\end{aligned} \tag{34}$$

Then, applying the Moon's inequality (17) to the right-hand side expression in (34) with  $\varrho = \frac{\tau_k - \tau}{\tilde{\tau}}$ , it follows that

$$\Delta V(e_k) \leq \zeta_k^\top \Xi(h, h_+, \tau_k) \zeta_k, \tag{35}$$

with

$$\begin{aligned}
\Xi(h, h_+, \tau_k) &= F_1^\top (P(h_+) + \Pi_1) F_1 + \tilde{R} + \text{He} \{ \Gamma^\top(\tau_k) P(h_+) F_1 \} \\
&\quad + \Gamma^\top(\tau_k) P(h_+) \Gamma(\tau_k) - \Gamma^\top(\tau_k) P(h) \Gamma(\tau_k) - F_2^\top P(h) F_2 \\
&\quad - \text{He} \{ \Gamma^\top(\tau_k) P(h) F_2 \} - G_1^\top \tilde{Z}_1 G_1 - \text{He} \{ Y_1 G_2 + Y_2 G_3 \} \\
&\quad + \varrho Y_1 \tilde{Z}_2^{-1} Y_1^\top + (1 - \varrho) Y_2 \tilde{Z}_2^{-1} Y_2^\top.
\end{aligned}$$

It follows from (35) that the error dynamics (23) is asymptotically stable if

$$\zeta_k^\top \Xi(h, h_+, \tau_k) \zeta_k < 0. \tag{36}$$



Moreover, from the definitions of  $\mathcal{B}(h, \Psi)$  in (22) and  $\zeta_k$  in (26), we have

$$\mathcal{B}(h, \Psi)\zeta_k = 0. \quad (37)$$

Using the well-known Finsler's lemma, one has that condition (36) is verified under constraint (37) if

$$\Xi(h, h_+, \tau_k) + \mathcal{W}^\top \mathcal{B}(h, \Psi) + \mathcal{B}(h, \Psi)^\top \mathcal{W} \prec 0, \quad (38)$$

with  $\mathcal{W}$  specifically chosen as in (22) to obtain a tractable problem with LMI constraints.

Since the condition (38) is affine with respect to  $\tau_k$ , it can be satisfied for all  $\tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$  if it is satisfied at the vertices of the interval  $\tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$ . Therefore, by applying the Schur complement lemma, it is possible to notice that condition (38) is equivalent to

$$\begin{bmatrix} \Phi(h, h_+, \underline{\tau}) + \text{He}\{\mathcal{W}^\top \mathcal{B}(h, \Psi)\} & Y_2 \\ Y_2^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (39)$$

$$\begin{bmatrix} \Phi(h, h_+, \bar{\tau}) + \text{He}\{\mathcal{W}^\top \mathcal{B}(h, \Psi)\} & Y_2 \\ Y_2^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (40)$$

with  $\Phi(h, h_+, \underline{\tau})$  and  $\Phi(h, h_+, \bar{\tau})$  defined as in (22).

Notice that inequalities (39) and (40) are exactly equal to inequalities (20) and (21), respectively, which are affine with respect to  $\tau_k$  implying that (20) and (21) are negative definite for all  $\tau_k \in \{\underline{\tau}, \dots, \bar{\tau}\}$ . Hence, if LMIs (20) and (21) are feasible the asymptotic stability of system (23) is ensured for any integer delay  $\tau_k \in \{\underline{\tau}, \underline{\tau} + 1, \dots, \bar{\tau}\}$ . Finally, it is proved that the UI estimate  $\hat{d}_k$  defined in (6) asymptotically converges to  $d_k$ . To this end, the expression of the unknown input  $d_k$  can be retrieved from (4) as **R4.2**

$$d_k = (CD)^\dagger [y_{k+1} - C(\Lambda_k + A(h)x_k + A_d(h)x_{k-\tau_k})]. \quad (41)$$

Then, it is possible to compute the UI estimation error  $\varepsilon_k = d_k - \hat{d}_k$  from (6) and (41) as

$$\varepsilon_k = (CD)^\dagger C (A(h)e_k + A_d(h)e_{k-\tau_k} + G(h)\delta_\phi).$$

Using (10), it is possible to rewrite the UI estimation error  $\varepsilon_k$  as

$$\varepsilon_k = (CD)^\dagger C (\mathcal{A}(h, \Psi)e_k + A_d(h)e_{k-\tau_k}), \quad (42)$$

with  $\mathcal{A}(h, \Psi) = \sum_{i=1}^r h_i(\mu_k) \mathcal{A}_i(\Psi)$ , and

$$\mathcal{A}_i(\Psi) = A_i + G_i \Psi, \quad (43)$$

where  $\Psi$  is defined in (11). We can see from (42) that if  $e_k \rightarrow 0$ , then  
 180  $e_{k-\tau_k} \rightarrow 0$  and  $\varepsilon_k \rightarrow 0$ , when  $k \rightarrow \infty$ . This concludes the proof.  $\square$

Notice that Theorem 1 cannot be straightforwardly solved for UIO design **R1.3**  
 because of the nonlinear matrix equalities (18)–(19), the nonlinear matrix  
 inequalities (20)–(21), and their dependency on the membership functions.  
 To deal with these issues, the following lemma is employed and numerically  
 185 tractable conditions to design the TS fuzzy UIO (6) for system (4) are derived.

**Lemma 4** ([59]). *Consider a matrix  $W \in \mathbb{R}^{m \times n}$ , with  $m \geq n$ , and a matrix  
 $M \in \mathbb{R}^{k \times n}$ . Matrix  $X$  of the form  $X = ZW^\dagger + Y(I - WW^\dagger)$ , with an  
 arbitrary matrix  $Y \in \mathbb{R}^{k \times m}$ , is a solution of  $XW = M$  when the condition  
 $MW^\dagger W = M$  holds.*

190 Based on the results in Theorem 1 and Lemma 4, the following theorem  
 presents LMI-based conditions to design TS fuzzy UIOs for system (4).

**Theorem 2.** *Consider the TS fuzzy system with time-varying delay (4), given  
 positive integers  $\underline{\tau}$  and  $\bar{\tau}$ , with  $\underline{\tau} < \bar{\tau}$ , and a positive scalar  $\epsilon$ . There is  
 an asymptotic UIO in the form (6), if there exist positive definite matrices  
 $P_i \in \mathbb{S}_+^{3n_x}$ ,  $R_1, R_2, Z_1, Z_2 \in \mathbb{S}_+^{n_x}$ , and matrices  $X_i \in \mathbb{R}^{n_x \times n_y}$ ,  $Q \in \mathbb{R}^{n_x \times n_y}$ ,  
 $W \in \mathbb{R}^{n_x \times n_x}$ ,  $Y_1, Y_2 \in \mathbb{R}^{8n_x \times 2n_x}$ , for  $i, \in \mathbb{N}_{\leq r}$  such that*

$$\begin{bmatrix} \Phi_{iq}(\underline{\tau}) + \text{He}\{\Upsilon_i(\Psi)\} & Y_2 \\ Y_2^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (44)$$

$$\begin{bmatrix} \Phi_{iq}(\bar{\tau}) + \text{He}\{\Upsilon_i(\Psi)\} & Y_1 \\ Y_1^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (45)$$

for all  $i, q \in \mathbb{N}_{\leq r}$ , and  $\Psi \in \mathcal{V}_\phi$ . The matrices  $\Phi_{iq}(\underline{\tau})$ ,  $\Phi_{iq}(\bar{\tau})$  and  $\Upsilon_i(\Psi)$  are  
 defined as

$$\begin{aligned} \Phi_{iq}(\underline{\tau}) &= \vartheta_{iq} - \text{He}\{\Gamma^\top(\underline{\tau})P_iF_2\} + \Gamma^\top(\underline{\tau})P_q\Gamma(\underline{\tau}) + \text{He}\{\Gamma^\top(\underline{\tau})P_qF_1\} - \Gamma^\top(\underline{\tau})P_i\Gamma(\tau), \\ \Phi_{iq}(\bar{\tau}) &= \vartheta_{iq} - \text{He}\{\Gamma^\top(\bar{\tau})P_iF_2\} + \Gamma^\top(\bar{\tau})P_q\Gamma(\bar{\tau}) + \text{He}\{\Gamma^\top(\bar{\tau})P_qF_1\} - \Gamma^\top(\bar{\tau})P_i\Gamma(\bar{\tau}), \\ \vartheta_{iq} &= F_1^\top(P_q + \Pi_1)F_1 - F_2^\top P_i F_2 + \tilde{R} - G_1^\top \tilde{Z}_1 G_1 - \text{He}\{Y_1 G_2 + Y_2 G_3\}, \\ \Upsilon_i(\Psi) &= \mathcal{I}^\top(\mathcal{Q}\mathcal{A}_i(\Psi)v_2 - Wv_1 - X_i C v_2 + \mathcal{Q}A_{di}v_4), \end{aligned}$$

with  $\mathcal{A}_i(\Psi) = A_i + G_i\Psi$ ,  $\mathcal{S} = (\epsilon v_1 + v_2 + v_4)$ ,  $\mathcal{Q} = W + (WU + QT)C$ , and

$$U = -D(CD)^\dagger \quad (46)$$

$$T = I - (CD)(CD)^\dagger \quad (47)$$

$$Q = WS \quad (48)$$

$$X_i = WH_i, \quad (49)$$

and  $F_1, F_2, G_1, G_2, G_3, \tilde{Z}_1, \tilde{Z}_2, \Pi_1, \tilde{R}, \Gamma(\underline{\tau}), \Gamma(\bar{\tau})$  defined as in (22). Moreover, the matrix gains  $F \in \mathbb{R}^{n_x \times n_y}$ ,  $M \in \mathbb{R}^{n_x \times n_x}$ ,  $N_i \in \mathbb{R}^{n_x \times n_x}$ , and  $L_i \in \mathbb{R}^{n_x \times n_y}$ , for  $\forall i \in \mathbb{N}_{\leq r}$ , of the TS fuzzy UIO (6) can be computed as

$$F = U + ST \quad (50)$$

$$N_i = MA_i - H_iC \quad (51)$$

$$L_i = H_i(I + CF) - MA_iF. \quad (52)$$

*Proof.* It follows from (7) and (19) that

$$F(CD) = -D. \quad (53)$$

Using Lemma 4 and the rank condition in Assumption 3, the solution  $F$  of the matrix equation (53) can be directly obtained from (50), where  $U$  and  $V$  are given in (46) and (47), respectively. Let us denote

$$H_i = L_i + N_iF, \quad \forall i \in \mathbb{N}_{\leq r}. \quad (54)$$

From (7), (18) and (54), we can obtain (51). Moreover, expression (52) can be directly obtained from (51) and (54). Notice also from (7) and (51) that

$$\begin{aligned} N_i &= (I + FC)A_i - H_iC \\ &= [I + (U + QT)C]A_i - H_iC, \quad \forall i \in \mathbb{N}_{\leq r}. \end{aligned} \quad (55)$$

Using expressions (7), (14), (43), (50) and (55), followed by the variable changes (48) and (49), inequalities (44) and (45) can be respectively rewritten as

$$\begin{bmatrix} \Phi_{iq}(\underline{\tau}) + \text{He}\{\mathcal{W}^\top \mathcal{B}_i(\Psi)\} & Y_2 \\ Y_2^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (56)$$

$$\begin{bmatrix} \Phi_{iq}(\bar{\tau}) + \text{He}\{\mathcal{W}^\top \mathcal{B}_i(\Psi)\} & Y_1 \\ Y_1^\top & -\tilde{Z}_2 \end{bmatrix} \prec 0, \quad (57)$$

with  $\mathcal{B}_i(\Psi) = -v_1 + \mathcal{N}_i(\Psi)v_2 + MA_{di}v_4$ . Since  $h, h_+ \in \mathcal{H}$  and  $\Psi \in \mathcal{V}_\phi$ , by convexity property, we prove that conditions (56) and (57) respectively imply (20) and (21). Then, using the result in Theorem 1, the proof can be  
 195 concluded.  $\square$

#### R4.4

**Remark 5.** *In this paper, we consider TS fuzzy systems with constant matrices  $C$  and  $D$  mainly because of the algebraic constraints induced by the proposed UI decoupling method. Indeed, if  $C$  and/or  $D$  are state-dependent, then it would be hard to get a solution from the algebraic equation (53). Moreover, even when the solution of this latter exists, the state-dependent forms of the solution matrices  $U$  and  $T$  in (46) and (47) become too complex to solve the corresponding state-dependent LMI conditions (44) and (45) in that case. This restriction is also observed in recent works on UIO design for TS fuzzy systems without time-varying delay [4, 26]. The design of TS fuzzy UIO for time-delayed nonlinear systems, whose output and UI matrices are also affected by unmeasured nonlinearities, is left for future works.*  
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The procedure to design UIOs for time-varying TS fuzzy systems with unmeasured nonlinear consequents is summarized in Algorithm 1. This design  
 210 procedure also enables determining the maximum allowable upper bound  $\bar{\tau}$  of the time-delay function  $\tau_k$  for a given lower bound  $\underline{\tau}$ .

## 4. Illustrative Examples

This section provides two numerical examples to illustrate the effectiveness of the proposed TS fuzzy unknown input observer design. All LMI-based design conditions are implemented in MATLAB using YALMIP parser and MOSEK solver. In both examples, a line search for  $\epsilon$  with 10 points gridded linearly on a logarithmic scale in  $[1, 10^3]$  has been performed.  
 215 R2.4

**Example 1.** *Consider the following nonlinear time-varying delay system adapted from [60]:*

$$x_{1,k+1} = (1 + T_s)x_{1,k} + cT_s x_{2,k} + T_s \sin(x_{3,k}) - 0.1T_s x_{4,k} + (1 - c)T_s x_{2,k-\tau_k} + T_s(1 + x_{1,k}^2)u_k \quad (58)$$

$$x_{2,k+1} = T_s x_{1,k} + (1 - 2cT_s)x_{2,k} - (1 - c)2T_s x_{2,k-\tau_k}$$

$$x_{3,k+1} = T_s x_{1,k} + (1 - 0.3T_s)x_{3,k} + cT_s x_{1,k}^2 x_{2,k} + (1 - c)T_s x_{1,k}^2 x_{2,k-\tau_k}$$

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**Algorithm 1:** Time-Varying Delay UIO Design

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**Input** : TS fuzzy system (4),  $\epsilon$ ,  $\underline{\tau}$

**Output:** TS fuzzy observer (6) such that

$$\hat{x}_k \rightarrow x_k \text{ and } \hat{d}_k \rightarrow d_k, \text{ when } k \rightarrow \infty$$

**Step 1:** Compute  $U$  with (46) and  $T$  with (47).

**Step 2:** Solve LMI conditions (44) and (45) for  $\bar{\tau} = \underline{\tau}$

**while true do**

$$\bar{\tau} \leftarrow \bar{\tau} + 1$$

Solve conditions (44) and (45) in Theorem 2.

**if unfeasible then**

| Stop while loop.

**end**

**end**

**Step 3:** Get  $W$ ,  $Q$  and  $X_i$ , for  $i \in \mathbb{N}_{\leq r}$ , with the maximum  $\bar{\tau}$  obtained in Step 2.

**Step 4:** Compute  $S$  with (48) and  $H_i$  with (49).

**Step 5:** Compute the observer gains  $F$ ,  $M$ ,  $L_i$  and  $N_i$  with (7), (50), (51) and (52), respectively.

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$$x_{4,k+1} = (1 - T_s)x_{4,k} + T_s \sin(x_{3,k}) + T_s d_k$$

$$y_{1,k} = x_{1,k}$$

$$y_{2,k} = x_{2,k} + x_{4,k}$$

with  $x_{1,k} \in [-a, a]$ ,  $x_{3,k} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  and  $c \in [0, 1]$ . For illustrations, we consider system (58) with  $c = 0.9$ ,  $T_s = 0.5$ , and the parameter  $a$  is used to study the design conservatism. The premise variable is defined as  $\mu_k = x_{1,k}^2$ . The unmeasured nonlinearity is given by  $\phi(x) = \sin(x_{3,k})$ . Then, the set of vertices  $\mathcal{V}_\phi$  in (12) is defined for system (58) as

$$\mathcal{V}_\phi = \{[0 \ 0 \ \rho \ 0] : \rho \in \{-1, 1\}\}.$$

Moreover, system (58) can be rewritten in the form (1) with

$$A(\mu_k) = \begin{bmatrix} 1 + T_s & cT_s & 0 & -0.1T_s \\ T_s & 1 - 2cT_s & 0 & 0 \\ T_s & cT_s\mu_k & 1 - 0.3T_s & 0 \\ 0 & 0 & 0 & 1 - T_s \end{bmatrix},$$

$$A_d(\mu_k) = \begin{bmatrix} 0 & (1 - c)T_s & 0 & 0 \\ 0 & -(1 - c)2T_s & 0 & 0 \\ 0 & (1 - c)T_s\mu_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}^\top, \quad f(\mu_k, u_k) = \begin{bmatrix} T_s(1 + \mu_k)u_k \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} T_s \\ 0 \\ 0 \\ T_s \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

Applying the sector nonlinearity approach [10, Chapter 2] with the measured premise variable  $\mu_k = x_{1,k}^2$ , system (58) can be represented by a two-rule TS fuzzy model (4) with

$$A_1 = \begin{bmatrix} 1.5 & 0.5c & 0 & -0.05 \\ 0.5 & 1 - c & 0 & 0 \\ 0.5 & 0.5ca^2 & 0.85 & 0 \\ 0 & 0 & 0 & 0.50 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 0.5c & 0 & -0.05 \\ 0.5 & 1 - c & 0 & 0 \\ 0.5 & 0 & 0.85 & 0 \\ 0 & 0 & 0 & 0.50 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0 & (1 - c)0.5 & 0 & 0 \\ 0 & -1 + c & 0 & 0 \\ 0 & (1 - c)0.5a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & (1 - c)0.5 & 0 & 0 \\ 0 & c - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

The corresponding membership functions are given by

$$h_1(\mu_k) = \frac{x_{1,k}^2}{a^2}, \quad h_2(\mu_k) = \frac{a^2 - x_{1,k}^2}{a^2}.$$

For this example, we compute the largest value of parameter  $a$ , denoted by  $a^*$ , for which a TS fuzzy UIO can be designed with different values of the

220 upper bound  $\bar{\tau}$  for system (58). To this end, Algorithm 1 is employed with  $\epsilon = 10^3$  and  $\underline{\tau} = 1$ . The corresponding design result is shown in Table 1. As expected, we can see that a direct relation between  $a^*$  and  $\bar{\tau}$ , i.e., the larger value of  $a^*$  leads to a smaller value of  $\bar{\tau}$ .

Table 1: Maximum delay upper bound  $\bar{\tau}$  in function of  $a^*$  in Example 1.

$\bar{\tau}$	2	5	8	11	14	17	20
$a^*$	1153	888	832	720	272	1.5	1

For illustrations, we consider the case with  $a = 832$  and  $\bar{\tau} = 8$ . Applying the design procedure given Algorithm 1, the following observer gains can be obtained:

$$\begin{aligned}
 M &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.2176 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -0.2176 & 0 \\ 0 & -1 \end{bmatrix}, \\
 L_1 &= \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0.3527 & 0.0109 \\ -0.5 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0.3586 & 0.0109 \\ -0.5 & 0 \end{bmatrix}, \\
 N_1 &= 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1.6004 & 0 & -1.5146 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 N_2 &= 10^3 \begin{bmatrix} 0.0009 & 0 & 0 & 0 \\ 0 & 0.0005 & 0 & 0.0004 \\ 0 & 7.8585 & 0.0008 & 7.8586 \\ 0 & 0.0004 & 0 & 0.0005 \end{bmatrix}.
 \end{aligned}$$

225 Fig. 1 depicts the estimation performance of the proposed UIO with respect to the initial condition  $x_0 = [0.25 \ -0.1 \ 0.5 \ -1]^\top$ , the known input  $u_k = -2.5x_{1,k} - 0.7x_{2,k} - x_{3,k}$ , and the unknown input  $d_k = 2 \cos(0.25\pi k) \sin(0.1\pi k)$ . The time-varying delay signal  $\tau_k$  is generated randomly while respecting the minimum bound  $\underline{\tau} = 1$  and the maximum bound  $\bar{\tau} = 8$ . We can see that both the estimates of the state  $\hat{x}_k$  and the unknown input  $\hat{d}_k$  asymptotically  
 230 convergent to  $x_k$  and  $d_k$ , respectively, despite the effect of the time-varying delay  $\tau_k$ .

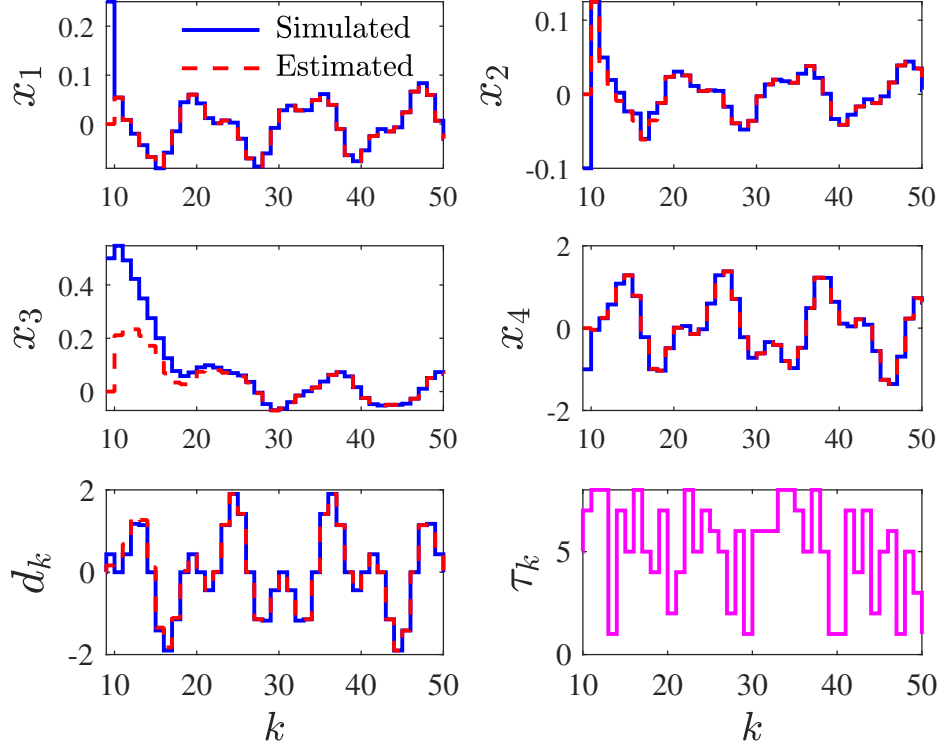


Figure 1: Estimation performance obtained for system (58) with  $a = 832$  and  $\bar{\tau} = 8$  under the effect of time-varying bounded delay.

**Example 2.** We consider the electronic circuit system depicted in Fig. 2, whose dynamics can be described as [61]

$$\begin{aligned}
 x_{1,k+1} &= x_{1,k} - c_{11}x_{1,k-\tau_k} + \frac{c_{12}}{10}x_{1,k}x_{2,k} - c_{13}(u_k + d_k) \\
 x_{2,k+1} &= (1 - c_{25})x_{2,k} + \frac{c_{24}}{10}x_{1,k-\tau_k} \\
 x_{3,k+1} &= (1 - c_{36})x_{3,k} - c_{37}x_{1,k} + \frac{c_{38}}{10}x_{2,k}^2 \\
 y_k &= x_{1,k}
 \end{aligned} \tag{59}$$

being  $x_{1,k}$ ,  $x_{2,k}$  and  $x_{3,k}$  the voltages on the capacitors with  $x_{1,k} \in [-1, 1]$ ,  $x_{2,k} \in [-1, 1]$ , the known input  $u_k$  is the voltage source, and the unknown input  $d_k$  can be viewed as a voltage fault signal. The system parameters are defined as

$$c_{1l} = \frac{T_s}{R_l C_1}, \quad c_{2m} = \frac{T_s}{R_m C_2}, \quad c_{3n} = \frac{T_s}{R_n C_3},$$



for  $l \in \{1, 2, 3\}$ ,  $m \in \{4, 5\}$  and  $n \in \{6, 7, 8\}$ . The parameter values are given by  $T_s = 0.01s$ ,  $R_1 = 0.6k\Omega$ ,  $R_2 = 66.667\Omega$ ,  $R_3 = 0.4k\Omega$ ,  $R_4 = 16.667\Omega$ ,  $R_5 = 0.2k\Omega$ ,  $R_6 = 0.5\Omega$ ,  $R_7 = R_8 = 0.1k\Omega$ ,  $\bar{C}_1 = \bar{C}_2 = \bar{C}_3 = 1mF$ .

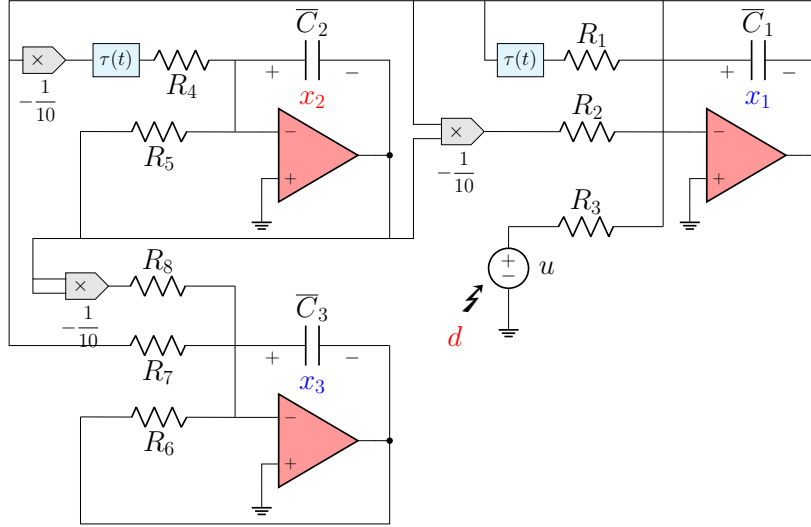


Figure 2: Diagram of the electronic circuit system.

The measured premise variable is defined as  $\mu_k = x_{1,k}$ . With  $x_{2,k} \in [-1, 1]$ , the set of vertices  $\mathcal{V}_\phi$  is defined for the unmeasured nonlinearity  $\phi(x) = x_{2,k}^2$  of system (58) as

$$\mathcal{V}_\phi = \{[0 \ \rho \ 0] : \rho \in \{-2, 2\}\}.$$

Notice that system (59) can be rewritten in the form

$$\begin{aligned} x_{k+1} &= A(\mu_k)x_k + A_d x_{k-\tau_k} + Dd_k + f(u_k) + G\phi(x_k), \\ y_k &= Cx_k \end{aligned} \quad (60)$$

with

$$\begin{aligned} A(\mu_k) &= \begin{bmatrix} 1 & \frac{c_{12}}{10}\mu_k & 0 \\ 0 & 1 - c_{25} & 0 \\ -c_{37} & 0 & 1 - c_{36} \end{bmatrix}, \quad D = \begin{bmatrix} -c_{13} \\ 0 \\ 0 \end{bmatrix}, \\ A_d &= \begin{bmatrix} -c_{11} & 0 & 0 \\ \frac{c_{24}}{10} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \frac{c_{38}}{10} \end{bmatrix}, \\ C &= [1 \ 0 \ 0], \quad f(u_k) = [-c_{13}u_k \ 0 \ 0]^\top, \quad \phi(x_k) = x_{2,k}^2. \end{aligned}$$

Applying the sector nonlinearity approach [10, Chapter 2] with the measured premise variable  $\mu_k = x_{1,k}$ , the electronic circuit system (60) can be represented by a two-rule TS fuzzy model (4) with

$$A_1 = \begin{bmatrix} 1.0 & -0.015 & 0 \\ 0 & 0.95 & 0 \\ -0.1 & 0 & 0.95 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.0167 & 0 & 0 \\ 0.0600 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.0 & 0.015 & 0 \\ 0 & 0.95 & 0 \\ -0.1 & 0 & 0.95 \end{bmatrix}, \quad D = \begin{bmatrix} -0.025 \\ 0 \\ 0 \end{bmatrix}.$$

The corresponding membership functions are given by

$$h_1(\mu_k) = \frac{1 - x_{1,k}}{2}, \quad h_2(\mu_k) = 1 - h_1(\mu_k).$$

235 For this example, the goal is to design a TS fuzzy observer (6) for system (59) with a maximum possible upper bound  $\bar{\tau}$  of the time-varying delay  $\tau_k$  such that  $\hat{x}_k \rightarrow x_k$  and  $\hat{d}_k \rightarrow d_k$ , when  $k \rightarrow \infty$ . Following the design procedure in Algorithm 1 with  $\epsilon = 100$  and  $\underline{\tau} = 1$ , the maximum delay bound  $\bar{\tau} = 160$  can be reached. For illustrative purposes, Fig. 3 depicts the estimation performance of the proposed TS fuzzy UIO with respect to the initial condition  $x_0 = [-0.5 \ 0.5 \ -0.5]^\top$ , the known input  $u_k = 1.5 \cos(0.1\pi k)$ , the unknown input  $d_k = \cos(0.02\pi k) \sin(0.1\pi k)$ , and the time-delay function is selected as  $\tau_k = \text{round}(80.5 + 79.5 \cos(0.02k\pi))$ <sup>1</sup>. As can be seen, the states  $x_k$  of the electronic system (59) together with its unknown input  $d_k$  are accurately estimated under the effect of  $\tau_k$ . Notably, the states  $x_2$  and  $x_3$  are not measured, and even so, the proposed UIO estimates these states quite efficiently. This result illustrates the effectiveness and the practical interest of the proposed TS fuzzy UIO design for nonlinear systems with time-varying delays and unmeasured nonlinearities. **R1.4**

## 250 5. Conclusions

A method has been presented to design unknown input observers for nonlinear systems with time-varying delays and unmeasured nonlinearities. The **R1.6**  
**R2.5**

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<sup>1</sup>round( $x$ ) denotes the function that rounds the entry  $x \in \mathbb{R}$  to the nearest integer using the round half up tie-breaking rule.

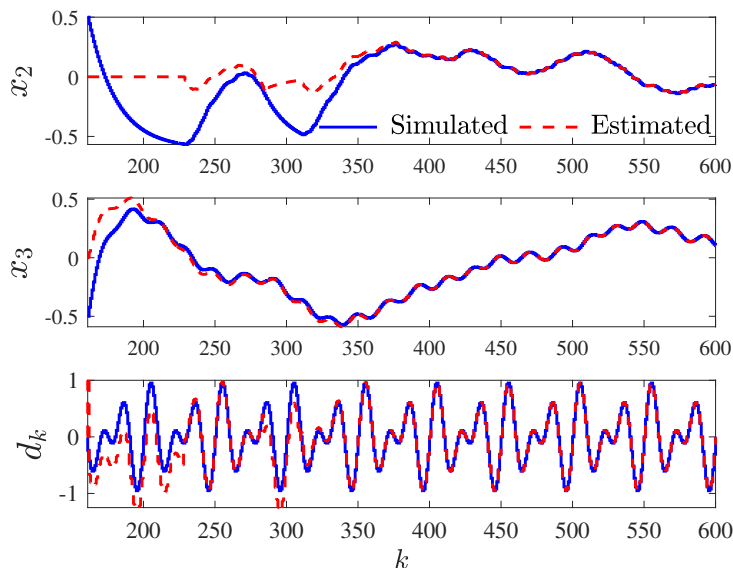


Figure 3: Estimation performance obtained for the electronic circuit system (59) under the effect of  $\tau_k = \text{round}(80.5 + 79.5 \cos(0.02k\pi))$ .

nonlinear system has been reformulated for observer design using TS fuzzy modeling with nonlinear consequents. All unmeasured nonlinearities, isolated in the nonlinear consequents, are assumed to be Lipschitzian such that the mean value theorem can be applied to deal with the well-known mismatch issue in TS observer design caused by unmeasured premise variables. LMI-based conditions have been presented for UIO design, which can guarantee an asymptotic estimation of both the state and the unknown input. The effectiveness and the practical interest of the proposed design procedure are illustrated with two examples. For future works, we focus on adapting the proposed UIO design results for fault-tolerant control of nonlinear systems with unknown time-varying delays.

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