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Convex Stability Analysis of Mamdani-Like Fuzzy Systems with Singleton Consequents

Anh-Tu Nguyen*, *Senior Member, IEEE*, Amine Dehak, Thierry-Marie Guerra, Michio Sugeno, *Life Member, IEEE*

This paper is dedicated to the memory of Dr. Amine Dehak, a former PhD student of the first author, who suddenly passed away in November 2022. He will be remembered for his great contributions, his passion for research and his kindness of character. We who knew him will miss him forever.

***Abstract*—We study the stability of a class of discrete-time fuzzy systems with singleton consequents, called Mamdani-like fuzzy systems. The parametric expressions, specific to this class of fuzzy systems, are leveraged to derive stability analysis conditions via Finsler's lemma and Lyapunov stability tools. This allows avoiding the major challenge in dealing with high-dimensional cases, encountered in the related literature when using the classical state-space representation. Moreover, the information of the piecewise region partition can be fully taken into account in the stability analysis with the well-known $S-$procedure to further reduce the stability conservatism. The stability of Mamdani-like fuzzy systems can be checked by solving a set of linear matrix inequalities (LMIs), that is numerically tractable with a suitable semidefinite programming software. Several numerical and physically motivated examples are provided to illustrate the effectiveness of the proposed stability analysis results.**

***Index Terms*—Fuzzy systems, fuzzy rules, stability analysis, singleton consequents, Mamdani fuzzy systems.**

I. Introduction

Fuzzy control has become one of the most prominent approaches to deal with complex industrial plants, essentially nonlinear systems. Since Mamdani's seminal paper published in 1974 on fuzzy modeling and control [1], many fuzzy control systems have been proposed [2]. Depending on the consequents of IF-THEN rules, three classes of fuzzy systems can be distinguished as suggested in [3]: Mamdani fuzzy systems [4], Takagi-Sugeno (TS) fuzzy systems [5] and singleton-type fuzzy systems [3]. Mamdani fuzzy systems are defined by IF-THEN rules with linguistic consequents as [4]

$$\begin{aligned}\text{Rule}\quad R_i:\quad &\text{IF}\quad x_1 \text{ is } \mathcal{G}_1^i,\ x_2 \text{ is } \mathcal{G}_2^i, \ldots, x_n \text{ is } \mathcal{G}_n^i\\ &\text{THEN}\quad y \text{ is } \mathcal{F}^i,\ i=1,2,\ldots,m\end{aligned} \tag{1}$$

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A.-T. Nguyen and T.-M. Guerra are with the LAMIH laboratory, UMR CNRS 8201, Université Polytechnique Hauts-de-France, Valenciennes, France. A.-T. Nguyen is also with the INSA Hauts-de-France, Valenciennes, France. E-mail: name.surname@uphf.fr.

A. Dehak was with the LAMIH laboratory, UMR CNRS 8201, Université Polytechnique Hauts-de-France, Valenciennes, France.

M. Sugeno is with the Tokyo Institute of Technology, Yokohama, Japan. E-mail: michio.sugeno@gmail.com.

*Corresponding author: Anh-Tu Nguyen (nguyen.trananhtu@gmail.com).

where $\mathcal{F}^i$ and $\mathcal{G}_j^i$, $j=1,2,\ldots,n$, are fuzzy sets, m is the number of fuzzy rules. Despite its practical interests in control of industrial processes, Mamdani fuzzy control has received a lot of criticisms from control community due to the lack of a systematic stability analysis framework [2]. Indeed, most of Mamdani fuzzy control schemes are remain model-free and essentially heuristic [6]. TS fuzzy systems are with functional consequents [5], defined by

$$\begin{aligned}\text{Rule}\quad R_i:\quad &\text{IF}\quad x_1 \text{ is } \mathcal{G}_1^i,\ x_2 \text{ is } \mathcal{G}_2^i, \ldots, x_n \text{ is } \mathcal{G}_n^i\\ &\text{THEN}\quad y \text{ is } f_i(x),\ i=1,2,\ldots,m\end{aligned} \tag{2}$$

where $x=\begin{bmatrix}x_1 & x_2 & \ldots & x_n\end{bmatrix}^\top$, and the consequent functions $f_i(x)$, for $i=1,2,\ldots,m$, are usually linear as

$$f_i(x)=b_i+a_{1i}x_1+a_{2i}x_2+\cdots+a_{ni}x_n$$

where the parameters b_i and a_{ji}, for $j=1,\ldots,n$, are constant. TS fuzzy systems with nonlinear consequents or polynomial consequents have been also studied, see for instance [7], [8] and [9], [10], respectively. The functional characteristics of the consequents makes possible to derive a systematic framework for stability analysis of TS fuzzy systems [11]. Despite a tremendous advance, reducing the stability analysis conservatism and dealing with complex systems with a large number of nonlinearities and/or states still remain challenging within TS fuzzy framework [2], [12], [13]. Singleton-type fuzzy systems are defined with fuzzy rules of the form [3]

$$\begin{aligned}\text{Rule}\quad R_i:\quad &\text{IF}\quad x_1 \text{ is } \mathcal{G}_1^i,\ x_2 \text{ is } \mathcal{G}_2^i, \ldots, x_n \text{ is } \mathcal{G}_n^i\\ &\text{THEN}\quad y \text{ is } b_i,\ i=1,2,\ldots,m\end{aligned} \tag{3}$$

where the consequents b_i, for $i=1,2,\ldots,m$, are singletons, *i.e.*, real numbers. Note from (1), (2) and (3) that singleton-type fuzzy systems are a special case of the two other classes of fuzzy systems when the linguistic consequents of Mamdani fuzzy systems and the functional consequents of TS fuzzy systems are simplified to singletons. Moreover, since linguistic labels may be assigned to singletons of singleton-type fuzzy systems, they can be called Mamdani-like fuzzy systems [14]. These fuzzy systems have been also called piecewise multi-affine systems due to their input-output multiaffine relation with respect to an affine-in-control system [15].

Mamdani-like fuzzy modeling has been found advantageous to study nonlinear systems for many reasons [3]. First, it can be used to represent complex systems where only input-output

data are available. This is particularly useful within industrial contexts to study complex plants with no acceptable analytical descriptions. Moreover, a Mamdani-like fuzzy model can be also directly obtained from the mathematical expression of a given nonlinear system. Second, as other classes of fuzzy systems, Mamdani-like fuzzy systems are universal approximators for any smooth nonlinear functions [16]. Third, they can be conveniently implemented using look-up tables, which are widely used in industry for model approximation and control implementation. Finally and specifically, both the state vector and the system dynamics of Mamdani-like fuzzy systems can be represented by *parametric expressions* [3], enabling a systematic framework for stability analysis [15], which remains open for Mamdani fuzzy systems. Despite these advantages, due to theoretical challenges, most of works on Mamdani-like fuzzy modeling have been mainly devoted to application aspects rather than theory [13].

Based on a quadratic Lyapunov function, Sugeno first set a theoretical foundation for stability analysis of Mamdani-like fuzzy models in the seminal paper [3]. Taking into account the parametric expression of the system state, an equivalent representation of triangular membership functions (MFs), this result has been further improved in [14], [17] to analytically derive *necessary and sufficient* stability conditions. Note that these latter have not been achieved with TS fuzzy-model-based stability approaches, especially for the approaches where the corresponding MFs are considered as "uncertainty" for stability analysis [18]. However, the stability conditions in [3], [14], [17] are reformulated as nonlinear matrix inequalities, which induce numerical difficulties. More importantly, these stability results can be only applied to second-order systems. Due to the specific piecewise polytopic affine form of the state-space representation of Mamdani-like fuzzy models, extensions to high-dimension cases are very challenging [13]. Due to these drawbacks, the research mainstream has been to consider Mamdani-like fuzzy controllers as nonlinear controllers, then the Mamdani-like fuzzy stability analysis is reformulated as a nonlinear stability approach based on absolute stability theory [19], [20], sliding mode control [21]–[23], passivity control approach [24], adaptive fuzzy control [25]–[30], TS fuzzy control [31], [32], etc. Although these results are also based on Lyapunov stability theory, a *fundamental* difference should be made clear with the works developed by Sugeno and coworkers [3], [14], [17]. That is, the above stability and control results have mainly focused on the characteristics of Mamdani-like fuzzy controllers performed on linear/nonlinear/TS fuzzy objective systems, or using Mamdani-like fuzzy modeling to represent their unknown/uncertain dynamics. Hence, the stability analysis of Mamdani-like fuzzy control systems can be treated using conventional nonlinear control [33] or TS fuzzy control [11], which is not the initial motivation of Mamdani-like fuzzy control systems [13]. Indeed, as Mamdani-like fuzzy systems are a special case of Mamdani fuzzy systems and TS fuzzy systems, the primary goal for Mamdani-like fuzzy control is to develop an analysis framework for nonlinear complex plants, embedded in Mamdani-like fuzzy models. This aims at inheriting the advantages while limiting the respective drawbacks of these two classes of fuzzy systems, *i.e.*, a systematic theoretical tool to deal with complex systems with a reduced conservativeness while being linguistically understandable to incorporate the expert's skills and experience [3]. Developing such a theoretical framework for Mamdani-like fuzzy systems is expected to be particularly challenging due to the novel fuzzy mathematics and linguistical features involved in the stability analysis [3], [13], [34].

Recently, using piecewise Lyapunov functions, a systematic stability framework has been established in [15] for continuous-time Mamdani-like fuzzy systems. In contrast to the related works [3], [14], [17], parametric expressions, a specific representation of Mamdani-like fuzzy systems, have been fully exploited instead of state-space representation to derive stability analysis conditions. The basic idea is based on the fact that for each piecewise region, the system state and its dynamics can be determined by convex combinations of the respective vertex values. This has paved the way for numerical stability analysis approaches for Mamdani-like fuzzy systems with any order. This paper can be considered as a discrete-time counterpart of the work in [15], concerned with the continuous-time stability analysis. Dealing with the discrete-time case is revealed to be more challenging since the stability of Mamdani-like fuzzy systems must be guaranteed when the system state can *arbitrarily* jump from one region to another, possibly nonadjacent. This phenomenon does not appear in the continuous-time case. Hence, additional technical treatments are required to ensure that the variation of the Lyapunov function along the system trajectory is decreasing despite these arbitrary jumps between piecewise regions. Specifically, the main contributions can be summarized as follows.

- Using the specific parametric expressions via Finsler's lemma, we establish Lyapunov-based stability conditions for discrete-time Mamdani-like fuzzy systems of any order, which is not the case in [3], [14], [31], [32].
- We propose a novel piecewise MFs-dependent Lyapunov function for stability analysis. Moreover, to further reduce the conservatism, the information of the piecewise region partition can be fully taken into account in the Lyapunov conditions via the $S-$procedure.
- The stability of discrete-time Mamdani-like fuzzy systems can be conveniently checked by solving a set of LMI constraints.

Notation. $\mathcal{I}_N$ denotes the subset of natural numbers $\{1, 2, \ldots, N\}$, $\mathbb{R}$ is the field of real numbers, and $\mathbb{Z}_+$ is the field of nonnegative integer numbers. For a vector $x \in \mathbb{R}^n$ and $i \in \mathcal{I}_n$, x_i denotes the ith entry of x. For a matrix X, $X^\top$ indicates its transpose. For any square matrix X, $X > 0$ indicates a symmetric positive definite matrix, $X \succeq 0$ means that all its elements are nonnegative, and $\mathrm{He}X = X + X^\top$. We denote I_n as the identity matrix of dimension n, and $0_{n\times m}$ as the $n \times m$ zero matrix. The symbol $\star$ stands for matrix blocks that can be deduced by symmetry. The explicit dimensions of both identity and null matrices are omitted if straightforwardly deduced. Moreover, the time dependency of the variables is dropped when convenient.

II. Fuzzy Systems with Singleton Consequents

To construct a fuzzy model with singleton consequents, we consider that the system state $x \in \mathbb{R}^n$ is bounded, *i.e.*, $\underline{x}_i \leq x_i \leq \overline{x}_i$, $i \in \mathcal{I}_n$, where $\underline{x}_i$ and $\overline{x}_i$ denote respectively the upper and lower bounds of the ith entry of x. Hence, x belongs to the set $\mathscr{R} = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$. We also partition the system state-space as

$$\underline{x}_j = \sigma_j^{[1]} < \sigma_j^{[2]} < \ldots < \sigma_j^{[N_j+1]} = \overline{x}_j, \quad j \in \mathcal{I}_n. \tag{4}$$

Let $\mathscr{K}_v = \mathcal{I}_{N_1+1} \times \ldots \times \mathcal{I}_{N_n+1}$ be the set of multi-indices corresponding to all the vertices induced by the partition (4) and $\mathscr{K}_r = \mathcal{I}_{N_1} \times \ldots \times \mathcal{I}_{N_n}$ the set of multi-indices corresponding to the regions. For $i = (i_1, \ldots, i_n) \in \mathscr{K}_r$, the region $\left[\sigma_1^{[i_1]}, \sigma_1^{[i_1+1]}\right] \times \ldots \times \left[\sigma_n^{[i_n]}, \sigma_n^{[i_n+1]}\right]$ is denoted by $\mathscr{R}_i$ and $\mathscr{K}_i = \{i_1, i_1+1\} \times \ldots \times \{i_n, i_n+1\}$ is the set of multi-indices corresponding to all vertices of $\mathscr{R}_i$. For $k \in \mathscr{K}_i$, σ_k is the vertex of $\mathscr{R}_i$ whose jth component is defined as $\sigma_j^{[k_j]}$, for $j \in \mathcal{I}_n$. The partition of the state space into hyper-rectangular piecewise regions for the two-dimension (2D) case is illustrated in Fig. 1.

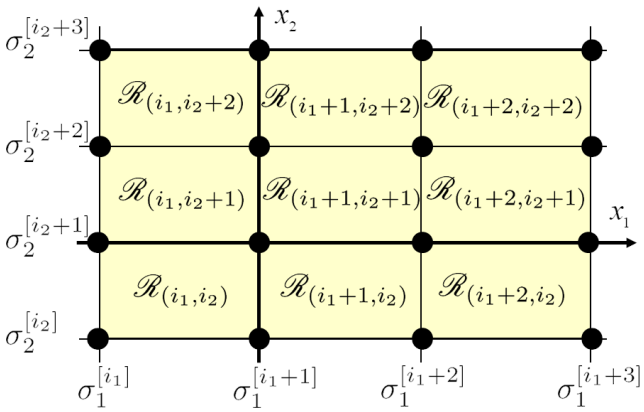

Fig. 1: Partition of the state space into hyper-rectangular piecewise regions for a 2D case.

For a hyper-rectangle region $\mathscr{R}_i$, for $i \in \mathscr{K}_r$, we consider the following set of fuzzy rules:

$$\begin{aligned} &\textsc{If} \;\; x_1(t) \text{ is } \eta_1^{[k_1]}(x_1), \ldots, \; x_n(t) \text{ is } \eta_n^{[k_n]}(x_n) \\ &\textsc{Then} \;\; x(t+1) \text{ is } f_k, \quad k \in \mathscr{K}_i \end{aligned} \tag{5}$$

where $f_k \in \mathbb{R}^n$ is the singleton vector, *i.e.*, vector of real scalars. To derive the parametric expressions for system (5), we assume that the normalized membership functions $\eta_j^{[k_j]}(x_j)$, for $j \in \mathcal{I}_n$ and $k_j \in \mathcal{I}_{N_j+1}$, are of a triangular form, defined as

$$\eta_j^{[k_j]}(x_j) = \begin{cases} \dfrac{x_j - \sigma_j^{[k_j-1]}}{\sigma_j^{[k_j]} - \sigma_j^{[k_j-1]}}, & \text{if } x_j \in \left[\sigma_j^{[k_j-1]}, \sigma_j^{[k_j]}\right] \\ & \text{and } j \geq 2 \\ \dfrac{\sigma_j^{[k_j+1]} - x_j}{\sigma_j^{[k_j+1]} - \sigma_j^{[k_j]}}, & \text{if } x_j \in \left[\sigma_j^{[k_j]}, \sigma_j^{[k_j+1]}\right] \\ & \text{and } j \leq N_j \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Note that the triangular MFs (6) satisfy the following convexity property [35]:

$$\begin{aligned} &\eta_j^{[k_j]}(x_j) \geq 0, \quad \sum_{k_j=i_j}^{i_j+1} \eta_j^{[k_j]}(x_j) = 1, \quad j \in \mathcal{I}_n \\ &\sum_{k \in \mathscr{K}_i} \eta_k(x) = \sum_{k \in \mathscr{K}_v} \eta_k(x) = 1. \end{aligned} \tag{7}$$

With these MFs, $x(t+1)$ can be inferred from the fuzzy rules (5) by taking the weighted average of f_k as follows [3]:

$$x(t+1) = \sum_{k \in \mathscr{K}_i} \eta_k(x) f_k, \quad \eta_k(x) = \prod_{j=1}^{n} \eta_j^{[k_j]}(x_j). \tag{8}$$

The following parametric expression of $x(t)$ can be directly derived from those of the triangular MFs defined in (6) as shown in [3, Proposition 1]:

$$x(t) = \sum_{k \in \mathscr{K}_i} \eta_k(x) \sigma_k. \tag{9}$$

Based on (8) and (9), the parametric expressions of the fuzzy model with singleton consequents, defined on $\mathscr{R} = \bigcup_{i \in \mathscr{K}_r} \mathscr{R}_i$, can be expressed as follows [15]:

$$\begin{aligned} x(t+1) &= \sum_{k \in \mathscr{K}_v} \eta_k(x) f_k \\ x(t) &= \sum_{k \in \mathscr{K}_v} \eta_k(x) \sigma_k. \end{aligned} \tag{10}$$

Note from (10) that the weights with respect to x_j, for $j \in \mathcal{I}_n$, in the premises are computed by the multiplication of $\eta_j^{[k_j]}(x_j)$, $k_j \in \mathcal{I}_{N_j+1}$. Hence, the fuzzy reasoning used here is characterized by normalized triangular MFs, multiplicative weights calculation, and weighted average aggregation [3]. For each piecewise region $\mathscr{R}_i$, by setting $\eta_k(x) = 1$ or 0, for $k \in \mathscr{K}_i$, we can recover from the parametric expressions (8) and (9) the fuzzy singleton rules: $x(t) = \sigma_k \mapsto x(t+1) = f_k$ at the vertices of the region $\mathscr{R}_i$. Then, for each piecewise region, the parametric expressions (8) and (9) can be understood as linear interpolation expressions to compute $x(t)$ and $x(t+1)$ from the vertex values. The interpolation procedure for a 2D case is illustrated in Fig. 2.

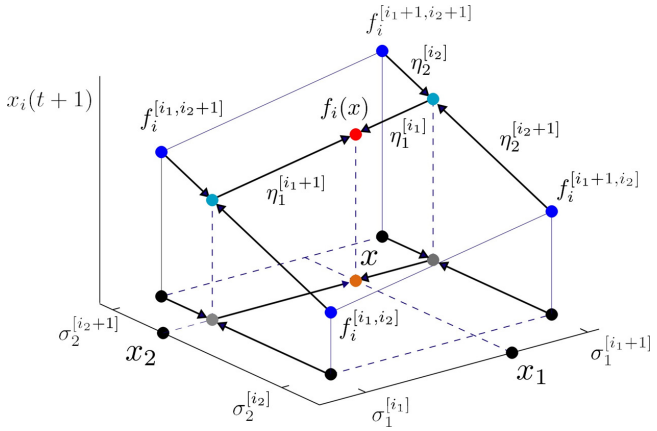

Fig. 2: Interpolation procedure of $x(t)$ and $x_i(t+1)$ for a 2D case using parametric expressions.

Remark 1. The fuzzy model with singleton consequents (5), or its parametric expression form (10), can be used to approximate any nonlinear system [3], [15]

$$x(t+1) = f(x(t)) \tag{11}$$

where $f(x) : \mathscr{R} \to \mathbb{R}^n$ is a smooth nonlinear function, and the singleton vector in (5) is defined as $f_k = f(\sigma_k)$, for $k \in \mathscr{K}_i$. The approximation can be done with any arbitrary accuracy on $\mathscr{R}$ by increasing the number of piecewise regions with (4).

Remark 2. Many types of MFs exist for fuzzy modeling, *e.g.*, Gaussian functions or trapezoidal functions. However, using triangular MFs allows to directly derive the parametric expression (9) of $x(t)$ as shown in [3]. This particular representation of the fuzzy model (10) is crucial to study its stability analysis. Moreover, triangular MFs have been the most employed in fuzzy control and applications [35].

For stability analysis, we assume that the equilibrium $x \equiv 0$ of system (10) corresponds to the vertex σ_{k_0} of the state-space partition, for a given $k_0 \in \mathscr{K}_v$. Let $\mathscr{K}_Z$ be the set of multi-indices for regions containing the origin which is called *zero-regions*, and $\mathscr{K}_{NZ} = \mathscr{K}_r \backslash \mathscr{K}_Z$ is the set of multi-indices for *non-zero regions*. We also denote $\mathscr{K}_i^* = \mathscr{K}_i \backslash \{k_0\}$, for $i \in \mathscr{K}_Z$, and $\mathscr{K}_v^* = \mathscr{K}_v \backslash \{k_0\}$. Then, each point in zero-regions can be computed by interpolating $2^n - 1$ vertices, except for the origin, as expressed by

$$\begin{aligned} x(t+1) &= \sum_{k \in \mathscr{K}_i^*} \eta_k(x) f_k \\ x(t) &= \sum_{k \in \mathscr{K}_i^*} \eta_k(x) \sigma_k. \end{aligned} \tag{12}$$

The following assumption is considered for the stability analysis of discrete-time fuzzy systems with singleton consequents.

Assumption 1. When the system state transits from the region $\mathscr{R}_i$ to the region $\mathscr{R}_j$ at the time sample t, the dynamics of the fuzzy system (10) is governed by the dynamics within the region $\mathscr{R}_i$ at that time sample.

A similar assumption can be found in [36] for stability analysis of piecewise discrete-time linear systems. For future uses, we define a set $\mathscr{T}$ that represents all possible transitions from one region to another for any sample $t \in \mathbb{Z}_+$, that is

$$\mathscr{T} = \{i, j \in \mathscr{K}_r : \ x(t) \in \mathscr{R}_i, \ x(t+1) \in \mathscr{R}_j, \ i \neq j\}.$$

By construction, the state space of the fuzzy system (10) is partitioned into hyper-rectangle regions. Then, exploiting the information of the region structure in the stability analysis can help to reduce the conservatism of the results [37]. To this end, we construct the continuity matrices $\bar{L}_i$, for $i \in \mathscr{K}_r$, such that

$$\bar{L}_i \bar{x} \succeq 0, \quad x \in \mathscr{R}_i, \quad i \in \mathscr{K}_r \tag{13}$$

where $\bar{x} = \begin{bmatrix} x^\top & 1 \end{bmatrix}^\top$, $\bar{L}_i = \begin{bmatrix} L_i & l_i \end{bmatrix}$, with $l_i = 0$, $i \in \mathscr{K}_Z$.

Remark 3. A systematic method to compute $\bar{L}_i$, for $i \in \mathscr{K}_r$, is given in [38]. However, since the hyper-rectangular state-space partition of system (10) is structurally simple, we can exactly determine the dimension of the continuity matrix in (13), namely $\bar{L}_i \in \mathbb{R}^{2n \times (n+1)}$ and $L_i \in \mathbb{R}^{2n \times n}$, which is generally not the case for piecewise control systems [36], [38].

The following Finsler's lemma is useful to convert checking the sign of a quadratic form over a subspace into solving an LMI problem for stability analysis.

Lemma 1 ([39])**.** Consider a vector $\pi \in \mathbb{R}^n$ and two matrices $\mathbf{Q} = \mathbf{Q}^\top \in \mathbb{R}^{n \times n}$ and $\mathbf{R} \in \mathbb{R}^{m \times n}$ such that $\mathrm{rank}(\mathbf{R}) < n$, the following statements are equivalent:

i) $\pi^\top \mathbf{Q} \pi < 0, \ \forall \pi \in \{\pi \in \mathbb{R}^n : \ \pi \neq 0, \ \mathbf{R}\pi = 0\}$
ii) $\exists \mathbf{M} \in \mathbb{R}^{n \times m}$ such that $\mathbf{Q} + \mathbf{M}\mathbf{R} + \mathbf{R}^\top \mathbf{M}^\top < 0$.

Based on Lemma 1, the parametric expressions of $x(t+1)$ and $x(t)$ in (10) can be fully exploited to derive the stability analysis conditions for the Mamdani-like fuzzy system (5).

III. Stability Analysis for Fuzzy Systems with Singleton Consequents

Using a common quadratic Lyapunov function is unnecessarily restrictive for analysis of the fuzzy system (10), since the dynamics given by the parametric expressions in (8) and (9) is only valid within the region $\mathscr{R}_i$. Consequently, the following theorem presents a new method for stability analysis of system (10), which relies on a fuzzy Lyapunov function, expressed in a parametric expression form.

Theorem 1. Consider the fuzzy system with singleton consequents in (10). If there exist a positive definite matrix $P_{k_0} \in \mathbb{R}^{n \times n}$, matrices $P_k \in \mathbb{R}^{n \times n}$, for $k \in \mathscr{K}_v^*$, symmetric matrices with nonnegative entries $U_q \in \mathbb{R}^{2n \times 2n}$ and $W_q \in \mathbb{R}^{2n \times 2n}$, for $q \in \mathscr{K}_r$, $Q_{ij} \in \mathbb{R}^{2n \times 2n}$, for $i, j \in \mathscr{T}$, matrices $Y_{1i} \in \mathbb{R}^{n \times n}$, $Y_{2i} \in \mathbb{R}^{n \times n}$, $Y_{3i} \in \mathbb{R}^{1 \times n}$, $Z_{1i} \in \mathbb{R}^{n \times n}$, $Z_{2i} \in \mathbb{R}^{n \times n}$, $Z_{3i} \in \mathbb{R}^{1 \times n}$, for $i \in \mathscr{K}_Z$, matrices $\bar{Y}_{1i} \in \mathbb{R}^{n \times n}$, $\bar{Y}_{2i} \in \mathbb{R}^{(n+1) \times n}$, $\bar{Y}_{3i} \in \mathbb{R}^{1 \times n}$, $\bar{Z}_{1i} \in \mathbb{R}^{n \times (n+1)}$, $\bar{Z}_{2i} \in \mathbb{R}^{(n+1) \times (n+1)}$, $\bar{Z}_{3i} \in \mathbb{R}^{1 \times (n+1)}$, for $i \in \mathscr{K}_{NZ}$, matrices $R_{1ij} \in \mathbb{R}^{n \times n}$, $R_{2ij} \in \mathbb{R}^{n \times n}$, $R_{3ij} \in \mathbb{R}^{1 \times n}$, $S_{1ij} \in \mathbb{R}^{n \times n}$, $S_{2ij} \in \mathbb{R}^{n \times n}$, $S_{3ij} \in \mathbb{R}^{1 \times n}$, for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_Z$, $j \in \mathscr{K}_r$, matrices $\bar{R}_{1ij} \in \mathbb{R}^{n \times n}$, $\bar{R}_{2ij} \in \mathbb{R}^{(n+1) \times n}$, $\bar{R}_{3ij} \in \mathbb{R}^{1 \times n}$, $\bar{S}_{1ij} \in \mathbb{R}^{n \times (n+1)}$, $\bar{S}_{2ij} \in \mathbb{R}^{(n+1) \times (n+1)}$, $\bar{S}_{3ij} \in \mathbb{R}^{1 \times (n+1)}$, for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_{NZ}$, $j \in \mathscr{K}_r$, satisfying the following linear matrix inequalities (14)–(19), shown at the top of the next page, with

$$\bar{P}_k = \begin{bmatrix} I_n & 0_{n \times 1} \end{bmatrix}^\top P_k \begin{bmatrix} I_n & 0_{n \times 1} \end{bmatrix}, \quad \hat{\sigma}_k = \begin{bmatrix} \sigma_k \\ 1 \end{bmatrix}$$

for $k \in \mathscr{K}_i$ and $i \in \mathscr{K}_{NZ}$. Then, the origin of the fuzzy system (10) is asymptotically stable, *i.e.*, $x(t)$ exponentially tends to the origin for every trajectory in $\mathscr{R}$.

Proof. Let us define

$$\mathcal{V}(x) = x^\top \sum_{k \in \mathscr{K}_v} \eta_k(x) P_k x \tag{20}$$

with $P_{k_0} > 0$ as assumed by Theorem 1. Note that $\mathcal{V}(x)$ shares the same triangular MFs (6) as the fuzzy system (10), and $\mathcal{V}(0) = 0$. By the convexity property in (7) and constraint (13), if conditions (14) and (15) are verified, it follows that

$$\mathcal{V}(x) = x^\top \sum_{k \in \mathscr{K}_i} \eta_k(x) P_k x > 0, \quad x \neq 0, \quad i \in \mathscr{K}_r.$$

$$\sigma_l^\top P_k \sigma_m - \sigma_l^\top L_i U_i L_i \sigma_m > 0, \quad i \in \mathscr{K}_Z,\ k \in \mathscr{K}_i,\ l,m \in \mathscr{K}_i^* \tag{14}$$

$$\sigma_l^\top P_k \sigma_m - \bar{\sigma}_l^\top \bar{L}_i U_i \bar{L}_i \bar{\sigma}_m > 0, \quad i \in \mathscr{K}_{NZ},\ k,l,m \in \mathscr{K}_i \tag{15}$$

$$\text{He}\begin{bmatrix} \bar{Y}_{1i} + \frac{1}{2}P_p & \bar{Z}_{1i} & -\bar{Y}_{1i} f_k - \bar{Z}_{1i}\sigma_k \\ \bar{Y}_{2i} & \bar{Z}_{2i} + \frac{1}{2}(\bar{L}_i^\top W_i \bar{L}_i - \bar{P}_k) & -\bar{Y}_{2i} f_k - \bar{Z}_{2i}\sigma_k \\ \bar{Y}_{3i} & \bar{Z}_{3i} & -\bar{Y}_{3i} f_k - \bar{Z}_{3i}\sigma_k \end{bmatrix} < 0, \quad i \in \mathscr{K}_{NZ},\ k,p \in \mathscr{K}_i \tag{16}$$

$$\text{He}\begin{bmatrix} Y_{1i} + \frac{1}{2}P_p & Z_{1i} & -Y_{1i} f_k - Z_{1i}\sigma_k \\ Y_{2i} & Z_{2i} + \frac{1}{2}(L_i^\top W_i L_i - P_k) & -Y_{2i} f_k - Z_{2i}\sigma_k \\ Y_{3i} & Z_{3i} & -Y_{3i} f_k - Z_{3i}\sigma_k \end{bmatrix} < 0, \quad i \in \mathscr{K}_Z,\ k,p \in \mathscr{K}_i,\ m \in \mathscr{K}_i^* \tag{17}$$

$$\text{He}\begin{bmatrix} R_{1ij} + \frac{1}{2}P_p & S_{1ij} & -R_{1ij} f_k - S_{1ij}\sigma_k \\ R_{2ij} & S_{2ij} + \frac{1}{2}(L_i^\top Q_{ij} L_i - P_k) & -R_{2ij} f_k - S_{2ij}\sigma_k \\ R_{3ij} & S_{3ij} & -R_{3ij} f_k - S_{3ij}\sigma_k \end{bmatrix} < 0, \quad \begin{array}{l} i,j \in \mathscr{T},\ i \in \mathscr{K}_Z,\ j \in \mathscr{K}_r \\ p \in \mathscr{K}_j,\ k \in \mathscr{K}_i,\ m \in \mathscr{K}_i^* \end{array} \tag{18}$$

$$\text{He}\begin{bmatrix} \bar{R}_{1ij} + \frac{1}{2}P_p & \bar{S}_{1ij} & -\bar{R}_{1ij} f_k - \bar{S}_{1ij}\bar{\sigma}_k \\ \bar{R}_{2ij} & \bar{S}_{2ij} + \frac{1}{2}(\bar{L}_i^\top Q_{ij} \bar{L}_i - \bar{P}_k) & -\bar{R}_{2ij} f_k - \bar{S}_{2ij}\bar{\sigma}_k \\ \bar{R}_{3ij} & \bar{S}_{3ij} & -\bar{R}_{3ij} f_k - \bar{S}_{3ij}\bar{\sigma}_k \end{bmatrix} < 0, \quad \begin{array}{l} i,j \in \mathscr{T},\ i \in \mathscr{K}_Z,\ j \in \mathscr{K}_r \\ p \in \mathscr{K}_j,\ k \in \mathscr{K}_i \end{array} \tag{19}$$

Then, by the interpolation procedure with the positive membership functions $\eta_k(x)$, for $k \in \mathscr{K}_v$, the function $\mathcal{V}(x)$ is positive definite. Therefore, the function $\mathcal{V}(x)$ defined in (20) is a proper Lyapunov function candidate.

For stability analysis, we examine the difference $\Delta\mathcal{V}_x = \mathcal{V}(x(t+1)) - \mathcal{V}(x(t))$ along the trajectories of system (10). Note that there are only four possible cases that must be distinguished. To study the stability, we also consider Assumption 1, which is useful for the proofs of Cases 3 and 4.

a) Case 1: $x(t) \in \mathscr{R}_i$ and $x(t+1) \in \mathscr{R}_i$, for $i \in \mathscr{K}_{NZ}$. To derive the stability conditions, the variation of $\mathcal{V}(x)$ along the solution of the fuzzy system (10) is required to be negative definite, that is

$$\Delta\mathcal{V}_x = x(t+1)^\top \mathcal{P}_{i_+} x(t+1) - x(t)^\top \mathcal{P}_i x(t) < 0 \tag{21}$$

for $i \in \mathscr{K}_{NZ}$, with

$$\mathcal{P}_{i_+} = \sum_{p \in \mathscr{K}_i} \eta_p(x(t+1)) P_p, \quad \mathcal{P}_i = \sum_{k \in \mathscr{K}_i} \eta_k(x(t)) P_k. \tag{22}$$

Since $W_i \succeq 0$ and by the construction of $\bar{L}_i$ in (13), using the well-known $S-$procedure [39], it is clear that condition (21) holds if

$$\begin{aligned} x(t+1)^\top \mathcal{P}_{i_+} x(t+1) - \bar{x}(t)^\top \bar{\mathcal{P}}_i \bar{x}(t) \\ + \bar{x}(t)^\top \bar{L}_i^\top W_i \bar{L}_i \bar{x}(t) < 0 \end{aligned} \tag{23}$$

with $\bar{\mathcal{P}}_i = \begin{bmatrix} I_n & 0_{n\times 1} \end{bmatrix}^\top \mathcal{P}_i \begin{bmatrix} I_n & 0_{n\times 1} \end{bmatrix}$, for $i \in \mathscr{K}_{NZ}$. Note that inequality (23) can be rewritten in the form

$$\xi(t)^\top \bar{\mathcal{W}}_i \xi(t) < 0, \quad i \in \mathscr{K}_{NZ} \tag{24}$$

where

$$\xi(t) = \begin{bmatrix} x(t+1) \\ \bar{x}(t) \\ 1 \end{bmatrix}, \quad \bar{\mathcal{W}}_i = \begin{bmatrix} \mathcal{P}_{i_+} & 0 & 0 \\ \star & \bar{L}_i^\top W_i \bar{L}_i - \bar{\mathcal{P}}_i & 0 \\ \star & \star & 0 \end{bmatrix}.$$

Moreover, for conciseness the expressions in (8) and (9) can be rewritten in the following respective compact forms:

$$\begin{aligned} x(t+1) &= \mathcal{F}_i \\ \bar{x}(t) &= \bar{\Sigma}_i \end{aligned} \tag{25}$$

for $i \in \mathscr{K}_{NZ}$, where

$$\mathcal{F}_i = \sum_{k \in \mathscr{K}_i} \eta_k(x) f_k, \quad \bar{\Sigma}_i = \sum_{k \in \mathscr{K}_i} \eta_k(x) \bar{\sigma}_k, \quad \bar{\sigma}_k = \begin{bmatrix} \sigma_k \\ 1 \end{bmatrix}.$$

It follows from (25) that

$$\bar{\mathcal{N}}_i \xi(t) = 0, \quad i \in \mathscr{K}_{NZ} \tag{26}$$

with

$$\bar{\mathcal{N}}_i = \begin{bmatrix} I_n & 0_{n\times(n+1)} & -\mathcal{F}_i \\ 0_{(n+1)\times n} & I_{n+1} & -\bar{\Sigma}_i \end{bmatrix}, \quad \xi(t) = \begin{bmatrix} x(t+1) \\ \bar{x}(t) \\ 1 \end{bmatrix}.$$

Note that the algebraic expression (26) is simply obtained from (25) by matrix manipulation. By Finsler's lemma, condition (24) holds under the equality constraint (26), if there exist matrices $\bar{\mathcal{M}}_i$ such that

$$\bar{\mathcal{W}}_i + \bar{\mathcal{M}}_i \bar{\mathcal{N}}_i + \bar{\mathcal{N}}_i^\top \bar{\mathcal{M}}_i^\top < 0, \quad i \in \mathscr{K}_{NZ}. \tag{27}$$

Let us partition

$$\bar{\mathcal{M}}_i = \begin{bmatrix} \bar{Y}_{1i} & \bar{Z}_{1i} \\ \bar{Y}_{2i} & \bar{Z}_{2i} \\ \bar{Y}_{3i} & \bar{Z}_{3i} \end{bmatrix}.$$

Then, condition (27) is rewritten as

$$\text{He}\begin{bmatrix} \bar{Y}_{1i} + \frac{1}{2}\mathcal{P}_{i_+} & \bar{Z}_{1i} & -\bar{Y}_{1i}\mathcal{F}_i - \bar{Z}_{1i}\bar{\Sigma}_i \\ \bar{Y}_{2i} & \bar{\mathscr{Z}}_{2i} & -\bar{Y}_{2i}\mathcal{F}_i - \bar{Z}_{2i}\bar{\Sigma}_i \\ \bar{Y}_{3i} & \bar{Z}_{3i} & -\bar{Y}_{3i}\mathcal{F}_i - \bar{Z}_{3i}\bar{\Sigma}_i \end{bmatrix} < 0 \tag{28}$$

with $\bar{\mathscr{Z}}_{2i} = \bar{Z}_{2i} + \frac{1}{2}(\bar{L}_i^\top W_i \bar{L}_i - \bar{\mathcal{P}}_i)$, for $i \in \mathscr{K}_{NZ}$. By the convexity property (7), we deduce that condition (16) implies inequality (28), which guarantees that $\Delta\mathcal{V}_x < 0$, for all $x(t) \in \mathscr{R}_i$, $x(t+1) \in \mathscr{R}_i$ and $i \in \mathscr{K}_{NZ}$.

b) Case 2: $x(t) \in \mathscr{R}_i$ and $x(t+1) \in \mathscr{R}_i$, for $i \in \mathscr{K}_Z$. It is important to note that the vertices corresponding to the origin are excluded in the parametric expressions of $x(t)$ and $x(t+1)$ in (12) for zero-regions. Hence, these parametric expressions do not correspond to convex combinations of the vertex values.

A convexification procedure via some changes of variables is required to deal with this issue. To this end, we define

$$\alpha(x(t)) = \sum_{m\in\mathscr{K}_{i^*}} \eta_m(x(t)) = 1 - \eta_{k_0}(x(t))$$

for $x(t) \in \mathscr{R}_i\backslash\{0\}$, $i \in \mathscr{K}_Z$. For simplicity, we denote $\alpha(x) = \alpha(x(t))$. Note that $\alpha(x) > 0$, for $x \neq 0$. Then, we perform the changes of variables $x^*(t) = \frac{1}{\alpha(x)}x(t)$ and

$$\begin{aligned} x^*(t+1) &= \frac{1}{\alpha(x)}x(t+1), \quad \eta_m^*(x) = \frac{1}{\alpha(x)}\eta_m(x) \\ \mathcal{F}_i^* &= \sum_{m\in\mathscr{K}_{i^*}} \eta_m^*(x) f_m, \qquad \Sigma_i^* = \sum_{m\in\mathscr{K}_{i^*}} \eta_m^*(x)\sigma_m. \end{aligned}$$

Then, the parametric expressions of $x(t)$ and $x(t+1)$ in (12) can be equivalently represented in the following convex combination form, for $x \in \mathscr{R}_i\backslash\{0\}$, $i \in \mathscr{K}_Z$:

$$\begin{aligned} x^*(t+1) &= \sum_{m\in\mathscr{K}_{i^*}} \eta_m^*(x) f_k = \mathcal{F}_i^* \\ x^*(t) &= \sum_{m\in\mathscr{K}_{i^*}} \eta_m^*(x)\sigma_k = \Sigma_i^*. \end{aligned} \tag{29}$$

Following similar steps as in the proof of Case 1 for system (29), we can prove that condition (17) implies that

$$\text{He}\begin{bmatrix} Y_{1i} + \frac{1}{2}\mathcal{P}_{i_+} & Z_{1i} & -Y_{1i}\mathcal{F}_i^* - Z_{1i}\Sigma_i^* \\ Y_{2i} & \mathscr{Z}_{2i} & -Y_{2i}\mathcal{F}_i^* - Z_{2i}\Sigma_i^* \\ Y_{3i} & Z_{3i} & -Y_{3i}\mathcal{F}_i^* - Z_{3i}\Sigma_i^* \end{bmatrix} < 0 \tag{30}$$

with $\mathscr{Z}_{2i} = Z_{2i} + \frac{1}{2}(L_i^\top W_i L_i - \mathcal{P}_i)$, for $i \in \mathscr{K}_Z$. Condition (30), in turn, equivalently implies that

$$\alpha(x)^2 x^*(t)^\top \mathcal{W}_i x^*(t) = \Delta\mathcal{V}_x + x(t)^\top L_i^\top W_i L_i x(t) < 0 \tag{31}$$

for $x \in \mathscr{R}_i\backslash\{0\}$ and $i \in \mathscr{K}_Z$, with

$$\mathcal{W}_i = \begin{bmatrix} \mathcal{P}_{i_+} & 0 & 0 \\ \star & L_i^\top W_i L_i - \mathcal{P}_i & 0 \\ \star & \star & 0 \end{bmatrix}.$$

Since $W_i \succeq 0$ and by (13), via the $S-$procedure we can deduce from (31) that $\Delta\mathcal{V}_x < 0$, for $x \in \mathscr{R}_i\backslash\{0\}$ and $i \in \mathscr{K}_Z$.

c) Case 3: $x(t) \in \mathscr{R}_i$ and $x(t+1) \in \mathscr{R}_j$, for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_Z$ and $j \in \mathscr{K}_r$. In this case, the variation of $\Delta\mathcal{V}_x$ along the trajectory of system (10) is given by

$$\begin{aligned} \Delta\mathcal{V}_x &= \mathcal{V}(x(t+1)) - \mathcal{V}(x(t)) \\ &= x(t+1)^\top \mathcal{P}_{j_+} x(t+1) - x(t)^\top \mathcal{P}_i x(t) \end{aligned} \tag{32}$$

where $\mathcal{P}_{j_+}$ and $\mathcal{P}_i$ are defined similarly as in (22) with $i \in \mathscr{K}_Z$ and $j \in \mathscr{K}_r$. Dividing both sides of inequality (32) by $\alpha(x)^2 > 0$, defined in Case 2, it follows that

$$\Delta\mathcal{V}_x^* = x^*(t+1)^\top \mathcal{P}_{j_+} x^*(t+1) - x^*(t)^\top \mathcal{P}_i x^*(t). \tag{33}$$

By the $S-$procedure with $Q_{ij} \succeq 0$ and constraint (13), it follows from (32) and (33) that $\Delta\mathcal{V}_x < 0$ if

$$\Delta\mathcal{V}_x^* + x^*(t)^\top L_i^\top Q_{ij} L_i x^*(t) < 0 \tag{34}$$

for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_Z$ and $j \in \mathscr{K}_r$. Condition (34) can be rewritten in the form

$$\xi^*(t)^\top \mathcal{Q}_{ij} \xi^*(t) < 0, \quad i, j \in \mathscr{T},\ i \in \mathscr{K}_Z,\ j \in \mathscr{K}_r \tag{35}$$

where

$$\xi^*(t) = \begin{bmatrix} x^*(t+1) \\ x^*(t) \\ 1 \end{bmatrix}, \quad \mathcal{Q}_{ij} = \begin{bmatrix} \mathcal{P}_{j_+} & 0 & 0 \\ \star & L_i^\top Q_{ij} L_i - \mathcal{P}_i & 0 \\ \star & \star & 0 \end{bmatrix}.$$

Moreover, it can be obtained from (29) that

$$\mathcal{N}_i^* \xi^*(t) = 0, \quad i \in \mathscr{K}_Z \tag{36}$$

with

$$\mathcal{N}_i^* = \begin{bmatrix} I_n & 0_{n\times n} & -\mathcal{F}_i^* \\ 0_{n\times n} & I_n & -\Sigma_i^* \end{bmatrix}.$$

Using Finsler's lemma, condition (35) holds under the equality constraint (36) if there exist matrices $\mathcal{M}_{ij}^*$ such that

$$\mathcal{Q}_{ij} + \mathcal{M}_{ij}^* \mathcal{N}_i^* + \mathcal{N}_i^{*\top} \mathcal{M}_i^{*\top} < 0 \tag{37}$$

for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_Z$ and $j \in \mathscr{K}_r$. We partition

$$\mathcal{M}_{ij} = \begin{bmatrix} R_{1ij} & S_{1ij} \\ R_{2ij} & S_{2ij} \\ R_{3ij} & S_{3ij} \end{bmatrix}.$$

Then, condition (37) can be rewritten as

$$\text{He}\begin{bmatrix} R_{1ij} + \frac{1}{2}\mathcal{P}_{j_+} & S_{1ij} & -R_{1ij}\mathcal{F}_i^* - S_{1ij}\Sigma_i^* \\ R_{2ij} & \mathscr{S}_{2ij} & -R_{2ij}\mathcal{F}_i^* - S_{2ij}\Sigma_i^* \\ R_{3ij} & S_{3ij} & -R_{3ij}\mathcal{F}_i^* - S_{3ij}\Sigma_i^* \end{bmatrix} < 0 \tag{38}$$

with $\mathscr{S}_{2ij} = S_{2ij} + \frac{1}{2}(L_i^\top Q_{ij} L_i - \mathcal{P}_i)$, for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_Z$ and $j \in \mathscr{K}_r$. By the convexity property of the parametric expressions of $\mathcal{F}_i^*$ and Σ_i^* in (29), we can see that condition (18) implies (38), which, in turn, guarantees that $\Delta\mathcal{V}_x < 0$, for $x \neq 0$, in this case.

d) Case 4: $x(t) \in \mathscr{R}_i$ and $x(t+1) \in \mathscr{R}_j$, for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_{NZ}$ and $j \in \mathscr{K}_r$. As for the previous cases, it is required that $\Delta\mathcal{V}_x < 0$ along the trajectory of system (10) in this case, that is

$$x(t+1)^\top \mathcal{P}_{j_+} x(t+1) - x(t)^\top \mathcal{P}_i x(t) < 0 \tag{39}$$

where $\mathcal{P}_{j_+}$ and $\mathcal{P}_i$ are defined similarly as in (22) with $i \in \mathscr{K}_{NZ}$ and $j \in \mathscr{K}_r$. Using the $S-$procedure with $Q_{ij} \succeq 0$ and constraint (13), condition (39) is verified if

$$\begin{aligned} x(t+1)^\top \mathcal{P}_{j_+} x(t+1) - \bar{x}(t)^\top \bar{\mathcal{P}}_i \bar{x}(t) \\ + \bar{x}(t)^\top \bar{L}_i^\top Q_{ij} \bar{L}_i \bar{x}(t) < 0 \end{aligned} \tag{40}$$

for $i, j \in \mathscr{T}$, $i \in \mathscr{K}_{NZ}$ and $j \in \mathscr{K}_r$. We rewrite inequality (40) in the form

$$\xi(t)^\top \bar{\mathcal{Q}}_{ij} \xi(t) < 0, \quad i, j \in \mathscr{T},\ i \in \mathscr{K}_{NZ},\ j \in \mathscr{K}_r \tag{41}$$

where

$$\bar{\mathcal{Q}}_{ij} = \begin{bmatrix} \mathcal{P}_{j_+} & 0 & 0 \\ \star & \bar{L}_i^\top Q_{ij} \bar{L}_i - \bar{\mathcal{P}}_i & 0 \\ \star & \star & 0 \end{bmatrix}.$$

Similar to (26), we have

$$\bar{\mathcal{N}}_i \xi(t) = 0, \quad i \in \mathscr{T} \cap \mathscr{K}_{NZ}. \tag{42}$$

It follows from Finsler's lemma that condition (41) holds under constraint (42) if there exist matrices $\bar{\mathcal{M}}_{ij}$ satisfying

$$\bar{\mathcal{Q}}_{ij} + \bar{\mathcal{M}}_{ij} \bar{\mathcal{N}}_i + \bar{\mathcal{N}}_i^\top \bar{\mathcal{M}}_{ij}^\top < 0 \tag{43}$$

for $i,j \in \mathscr{T}$, $i \in \mathscr{K}_{NZ}$ and $j \in \mathscr{K}_r$. We partition

$$\bar{\mathcal{M}}_{ij} = \begin{bmatrix} \bar{R}_{1ij} & \bar{S}_{1ij} \\ \bar{R}_{2ij} & \bar{S}_{2ij} \\ \bar{R}_{3ij} & \bar{S}_{3ij} \end{bmatrix}.$$

Then, condition (43) can be explicitly rewritten as

$$\text{He}\begin{bmatrix} \bar{R}_{1ij} + \frac{1}{2}\mathcal{P}_{j+} & \bar{S}_{1ij} & -\bar{R}_{1ij}\mathcal{F}_i - \bar{S}_{1ij}\bar{\Sigma}_i \\ \bar{R}_{2ij} & \bar{\mathscr{S}}_{2ij} & -\bar{R}_{2ij}\mathcal{F}_i - \bar{S}_{2ij}\bar{\Sigma}_i \\ \bar{R}_{3ij} & \bar{S}_{3ij} & -\bar{R}_{3ij}\mathcal{F}_i - \bar{S}_{3ij}\bar{\Sigma}_i \end{bmatrix} < 0 \quad (44)$$

with $\bar{\mathscr{S}}_{2ij} = \bar{S}_{2ij} + \frac{1}{2}(\bar{L}_i^\top Q_{ij}\bar{L}_i - \bar{\mathcal{P}}_i)$, for $i,j \in \mathscr{T}$, $i \in \mathscr{K}_{NZ}$ and $j \in \mathscr{K}_r$. Using again the convexity property in (7), we can conclude that condition (19) implies (44), which guarantees that $\Delta\mathcal{V}_x < 0$, for $x \in \mathscr{R}_i$ in this case.

The results of all the above cases lead to $\Delta\mathcal{V}_x < 0$, for $\forall x \in \mathscr{R}\backslash\{0\}$, which concludes the proof. □

The stability analysis conditions in Theorem 1 are strictly expressed in terms of linear matrix inequalities, whose feasibility can be checked using a suitable semidefinite programming software. Conditions (14)–(15) guarantee that the Lyapunov function candidate $\mathcal{V}(x)$, defined in (20), is positive definite. LMI conditions (16)–(17) ensure that the Lyapunov function is decreasing along the trajectory of system (10) in each region, whereas the LMI conditions (18)–(19) ensure that this function decreases when the system state transits from one region to another on the state space $\mathscr{R}$.

Remark 4. The membership-functions-dependent Lyapunov function $\mathcal{V}(x)$ in (20) is expected to yield less conservative stability results than those obtained with a common quadratic Lyapunov function $V(x) = x^\top Px$, with $P > 0$. Indeed, the latter can be regarded as a special case of (20) by imposing $P_k = P$, for $k \in \mathscr{K}_v$. Moreover, we leverage the parametric expression of $x(t)$ in (14) and (15) so that the Lyapunov matrices P_k, for $k \in \mathscr{K}_v^*$, are not explicitly required to be positive definite.

Remark 5. The slack matrices U_q, W_q, for $q \in \mathscr{K}_r$, and Q_{ij}, for $i,j \in \mathscr{T}$, are introduced into the stability analysis via the $S-$procedure [39]. This contributes to reduce the stability conservatism since the hyper-rectangular piecewise-region information of the fuzzy model (10) can be fully exploited via constraint (13).

Remark 6. For the stability results in Theorem 1, the parametric expressions of both $x(t)$ and $x(t+1)$ in (10) can be leveraged using Finsler's lemma [39]. As discussed in [3], expression (9) is a reformulation of the triangular MFs (6), whose information can be considered for stability analysis.

Remark 7. The stability conditions in Theorem 1 do not necessarily imply the stability of system (11) due to the approximation error between this system and its fuzzy model (10). The error characterization and its impacts on stability analysis of fuzzy models with singleton consequents are left for future works. At this stage, we emphasize the tradeoff between the computational burden and the approximation precision in the stability analysis using the fuzzy modeling (5). Specifically, a fine modeling repartition (4) decreases the approximation error, but leads to a more expensive computational cost.

IV. Illustrative Examples

This section provides four numerical examples to illustrate the effectiveness and the conservativeness of the proposed stability results. All the involved LMI-based conditions are solved using YALMIP toolbox with SDPT3 solver [40].

Example 1. This example is used to study the stability of a Mamdani-like fuzzy system, described in a parametric expression form. To this end, we consider the second-order fuzzy model taken from [14], whose the state-space partition and the values of the singleton consequents are given in Table I. Fig. 3 shows the characteristics of this fuzzy model, which is nonlinear. Solving the stability conditions in Theorem 1, we obtain a feasible solution, whose details are omitted for conciseness. The Lyapunov function proving the stability of the considered fuzzy system is depicted in Fig. 4, whereas Fig. 5 shows the corresponding Lyapunov level sets and several trajectories. In contrast to the stability results in [3], [14], we can see in Fig. 5 that the obtained Lyapunov level sets have a nonquadratic form. Moreover, the convergence of the trajectories to the origin in the phase plane confirms that the studied Mamdani-like fuzzy system is stable.

TABLE I: State-Space Partition and Singleton Consequents of Fuzzy Model in Example 1.

$f_1(\sigma_1,\sigma_2)$ / $f_2(\sigma_1,\sigma_2)$		$\sigma_2^{[1]}$	$\sigma_2^{[2]}$	$\sigma_2^{[3]}$	$\sigma_2^{[4]}$	$\sigma_2^{[5]}$
		$-\pi/2$	$-\pi/6$	0	$\pi/6$	$\pi/2$
$\sigma_1^{[1]}$	-2	-1.000 11.47	0.500 9.047	2.000 8.000	3.500 6.952	5.000 4.858
$\sigma_1^{[2]}$	-1	-2.000 4.142	-0.500 2.047	1.000 1.000	2.500 -0.047	4.000 -2.142
$\sigma_1^{[3]}$	0	-4.000 -3.000	-2.500 -1.500	0 0	0.500 1.500	2.000 3.000
$\sigma_1^{[4]}$	1	3.142 2.142	1.047 0.047	-1.000 -1.000	-1.047 -2.047	-3.142 -4.142
$\sigma_1^{[5]}$	2	-5.000 -4.858	-3.500 -6.953	-2.000 -8.000	-0.500 -9.047	1.000 -11.14

Example 2. To evaluate the conservatism of the stability analysis results, we consider the nonlinear system (11), where

$$f(x) = \begin{bmatrix} (1-T_e)x_1 + T_e(\cos(x_1) + 5\sin(x_2))x_2 \\ aT_e\sin(x_2)x_1 + T_e(b+1)x_2 \end{bmatrix} \quad (45)$$

where the parameters are given by $T_e = 0.001$, $a \in [-5,5]$ and $b \in [-5,5]$. The state vector $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ belongs to the set $\mathscr{R} = [-\pi,\pi] \times [-\pi,\pi]$. The fuzzy model (10) is constructed from 25 points of a 5×5 rectangular partition on the state space resulting in 16 piecewise rectangular regions, which can approximate very accurately the nonlinear function (45). To evaluate the conservatism, we compare the proposed result in Theorem 1 and the stability analysis results obtained with TS fuzzy modeling. To this end, using the sector nonlinearity approach [11], the nonlinear system (11) with $f(x)$ given in (45) can be represented by the following TS fuzzy model:

$$x(t+1) = \sum_{i=1}^{4} h_i(x)A_i x(t) \quad (46)$$

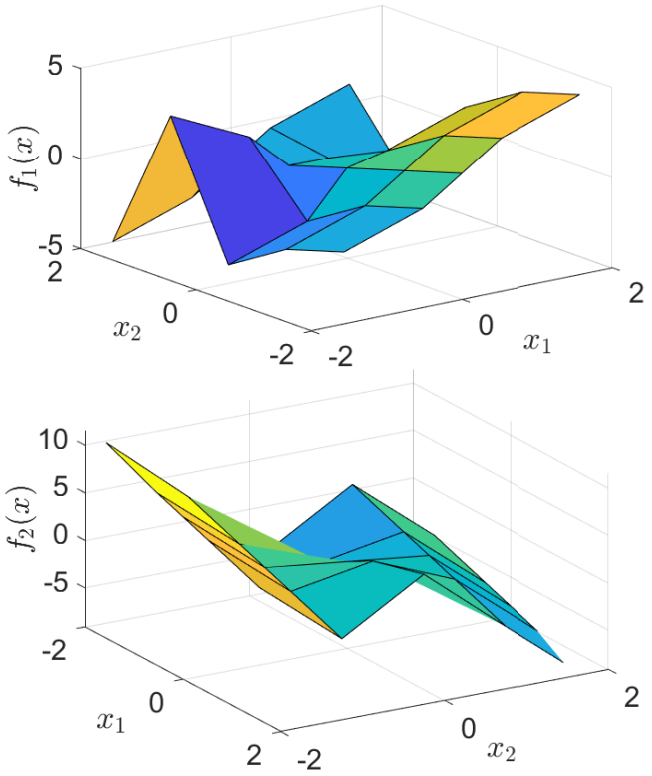

Fig. 3: Nonlinear characteristics of the Mamdani-like fuzzy model with singleton consequents in Example 1.

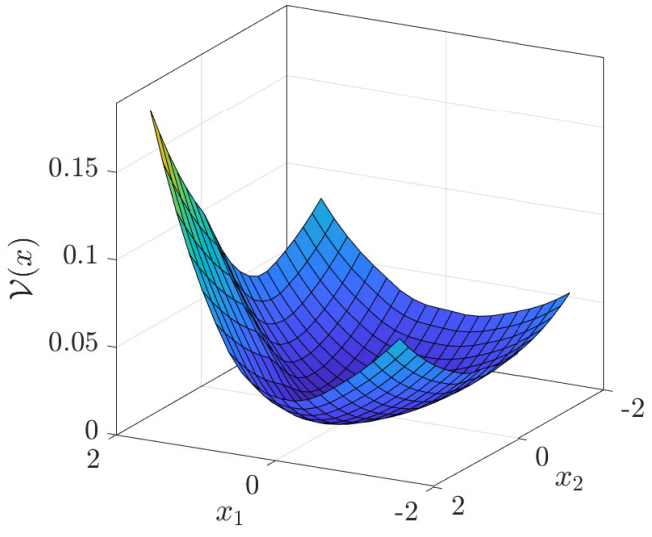

Fig. 4: Nonquadratic Lyapunov function $\mathcal{V}(x)$ obtained with Example 1.

where the local linear submodels A_i, for $i \in \mathcal{I}_4$, are given by

$$A_1 = \begin{bmatrix} 1 - T_e & -6T_e \\ -aT_e & bT_e + 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 - T_e & 4T_e \\ aT_e & bT_e + 1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 - T_e & -4T_e \\ -aT_e & bT_e + 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 - T_e & 6T_e \\ aT_e & bT_e + 1 \end{bmatrix}.$$

The corresponding membership functions $h_i(x)$, for $i \in \mathcal{I}_4$,

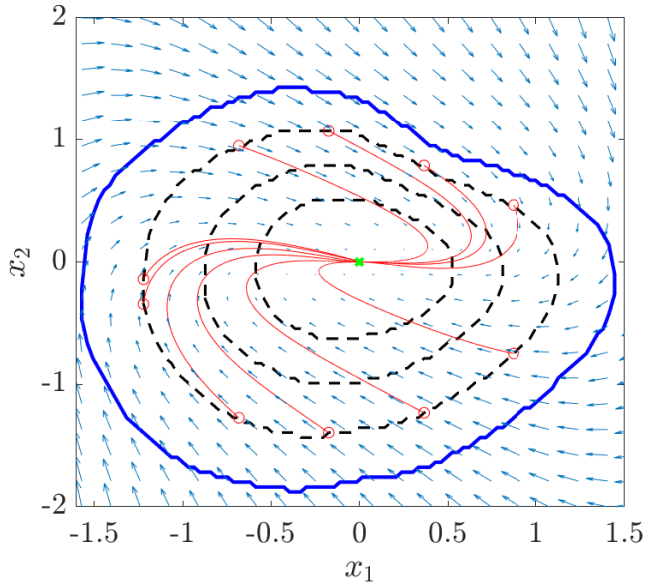

Fig. 5: Nonquadratic Lyapunov level sets and trajectories in the phase plane of Mamdani-like fuzzy model in Example 1.

in (46) are defined as

$$h_1(x) = \frac{(1 - \cos x_1)(1 - \sin x_2)}{4}$$
$$h_2(x) = \frac{(1 - \cos x_1)(\sin x_2 - 1)}{4}$$
$$h_3(x) = \frac{(\cos x_1 - 1)(1 - \sin x_2)}{4}$$
$$h_4(x) = \frac{(\cos x_1 - 1)(\sin x_2 - 1)}{4}.$$

Fig. 6 shows the feasibility regions obtained with the four following stability analysis results.

- TS fuzzy model-based stability result with a quadratic Lyapunov function [41, Theorem 4.3].
- TS fuzzy model-based stability result with a nonquadratic Lyapunov function [42, adapted Theorem 5].
- Mamdani-like fuzzy stability result in Theorem 1 with a quadratic Lyapunov function, *i.e.*, $P_k = P$, for $k \in \mathcal{K}_v$.
- Fuzzy stability result in Theorem 1.

We can see that larger feasibility regions can be obtained with Mamdani-like fuzzy model-based approaches. As expected, for both TS fuzzy and Mamdani-like fuzzy approaches, the stability results derived from nonquadratic Lyapunov functions, including the information of the MFs in their constructions, are less conservative than those based on a common quadratic Lyapunov function. Note that using Mamdani-like fuzzy approaches, we can deal with nonlinear systems in the form (11), without requiring a classical state-space parameterization $x(t+1) = A(z)x(t)$, where $z(t)$ is the vector of premise variables, as for TS fuzzy approaches or related ones. Such a state-space parameterization is not unique, which may be a source of conservativeness [13]. As discussed in Remark 7, although the stability of Mamdani-like fuzzy models does not directly imply the stability of the nonlinear system (11)-(45), the proposed comparison provides an idea on the conservativeness of the stability analysis conditions in Theorem 1,

especially when the constructed Mamdani-like fuzzy models accurately approximate system (45).

For illustrations, Fig. 7 presents the nonquadratic Lyapunov level sets obtained with Theorem 1 and some trajectories in the phase plane of the Mamdani-like fuzzy model of system (45) with $a = 1.84$ and $b = -5$. We can see that the corresponding fuzzy model is stable within the state-space set $\mathscr{R}$. Moreover, note that the trajectories of the nonlinear system (45) and those of its Mamdani-like fuzzy model are superposed in Fig. 7, which confirms an accurate approximation of the fuzzy modeling in this case.

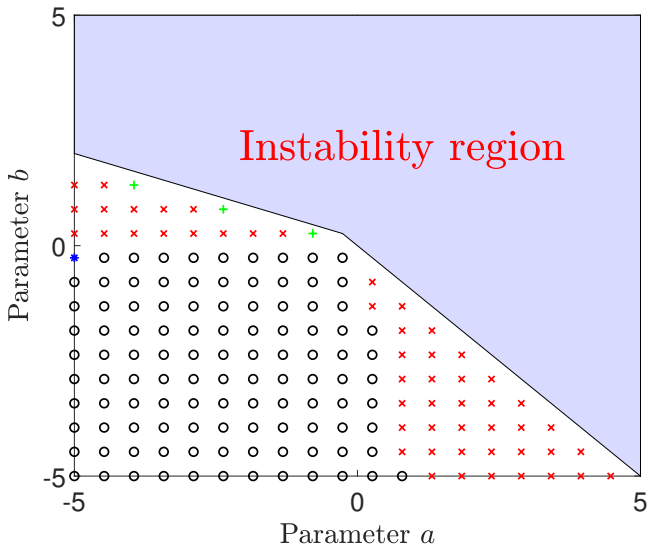

Fig. 6: Feasibility regions obtained with [41, Theorem 4.3] (∘), [42, adapted Theorem 5] (∘,∗), adapted Theorem 1 with a quadratic Lyapunov function (∘,∗,×), Theorem 1 (∘,∗,×,+).

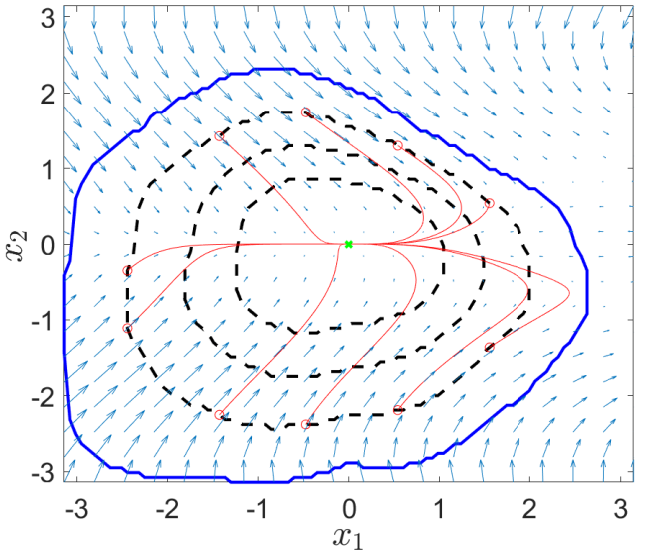

Fig. 7: Nonquadratic Lyapunov level sets and trajectories in the phase plane of Mamdani-like fuzzy model in Example 2 with $a = 1.84$ and $b = -5$.

Example 3. To further study the stability conservatism, we consider a physically motivated two-tank system, whose dynamics can be described as [43]

$$\begin{aligned}\dot{h}_1(t) &= \frac{1}{A}\left(\bar{k}v(t) - a_1\sqrt{2g(h_1(t)-h_2(t))}\right)\\ \dot{h}_2(t) &= \frac{1}{A}\left(a_1\sqrt{2g(h_1(t)-h_2(t))} - a_2\sqrt{2gh_2(t)}\right)\end{aligned} \tag{47}$$

where $h_1(t)$ and $h_2(t)$ denote the water level of the two tanks, $v(t)$ is the flow rate of the pump, $A = 100$ cm^2 is the horizontal section, $\bar{k} = 0.01$ is a constant, $a_1 \in [1, 50]$ cm^2 is the section of the valve connecting the tanks, $a_2 \in [0.7, 35]$ cm^2 is the section of the outlet valve, $g = 981$ cm/s^2 is the gravitational acceleration constant. Considering the equilibrium of system (47) as $\begin{bmatrix} h_{1e} & h_{2e}\end{bmatrix}^\top = \begin{bmatrix}15 & 6\end{bmatrix}^\top$, then it follows that

$$\begin{aligned}0 &= \frac{1}{A}\left(\bar{k}v_e - a_1\sqrt{2g(h_{1e}-h_{2e})}\right)\\ 0 &= \frac{1}{A}\left(a_1\sqrt{2g(h_{1e}-h_{2r})} - a_2\sqrt{2gh_{2e}}\right).\end{aligned}$$

Let us define $x_1 = h_1 - h_{1e}$, $x_2 = h_2 - h_{2e}$, $\Delta v = v - v_e$. Then, the following incremental model can be obtained:

$$\begin{aligned}\dot{x}_1 &= \frac{a_1}{A}\frac{2g}{\phi(x)}(x_2 - x_1) + \frac{\bar{k}}{A}\Delta v\\ \dot{x}_2 &= \frac{a_1}{A}\frac{2g}{\phi(x)}(x_1 - x_2) - \frac{2ga_2x_2}{A}\varphi(x_2)\end{aligned} \tag{48}$$

with

$$\begin{aligned}\phi(x) &= \sqrt{2g(x_1 - x_2 + h_{1e} - h_{2e})} + \sqrt{2g(h_{1e} - h_{2e})}\\ \varphi(x_2) &= \frac{1}{\sqrt{2g(x_2 + h_{2e})} + \sqrt{2gh_{2e}}}.\end{aligned}$$

The state $x = \begin{bmatrix}x_1 & x_2\end{bmatrix}^\top$ of system (48) belongs to the set $\mathscr{R} = [-4, 4] \times [-5, 5]$. For stability analysis, we consider the case without control input, *i.e.*, $\Delta v = 0$. Moreover, we perform a forward Euler discretization with a sampling period $T_e = 0.5$ [s], the discrete-time model of system (47) can be represented by (11) with

$$f(x) = \begin{bmatrix}\frac{a_1T_e}{A}\frac{2g}{\phi(x)}(x_2 - x_1) + x_1\\ \frac{a_1T_e}{A}\frac{2g}{\phi(x)}(x_1 - x_2) + x_2 - \frac{2ga_2T_ex_2}{A}\varphi(x_2)\end{bmatrix} \tag{49}$$

The fuzzy model (10) is constructed with 16 piecewise rectangular regions, partitioned on $\mathscr{R}$ in a similar way as for Example 2. Moreover, using the sector nonlinearity approach [11] with two premise variables $\frac{1}{\phi(x)}$ and $\varphi(x_2)$, a TS fuzzy model can be directly obtained, whose details are omitted here for conciseness. For comparison purposes, we also consider four stability analysis results as in Example 2 for this system. Fig. 8 shows the feasibility regions corresponding to the four stability analysis results. We can see that compared to TS fuzzy based methods, Mamdani-like fuzzy based results yield a much larger feasibility region. For this physically motivated example, quadratic and nonquadratic approaches for both TS fuzzy and Mamdani-like fuzzy cases lead to the same level of conservativeness with the same feasibility regions. Fig. 9 illustrates the nonquadratic Lyapunov level sets in the phase

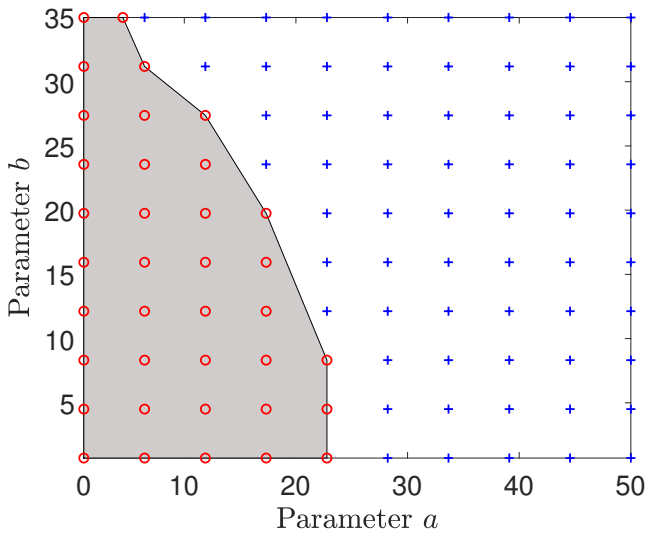

Fig. 8: Feasibility regions obtained with [41, Theorem 4.3] and [42, adapted Theorem 5] (○), adapted Theorem 1 with a quadratic Lyapunov function and Theorem 1 (○,+).

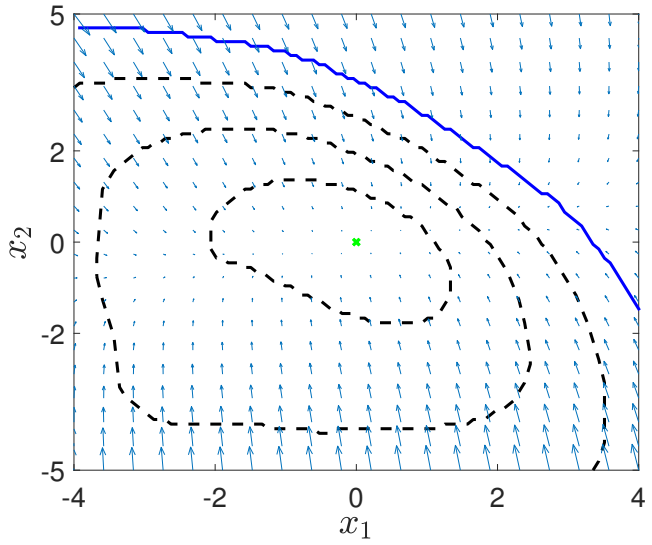

Fig. 9: Nonquadratic Lyapunov level sets in the phase plane of Mamdani-like fuzzy model in Example 3 with $a_1 = 6.44$ and $a_2 = 35$.

plane obtained with Theorem 1 and the Mamdani-like fuzzy model of system (49) with $a_1 = 6.44$ and $a_2 = 35$.

Example 4. One of our key contributions compared to the seminal work [3] and related references [14], [31], [32] is the possibility to study the stability of discrete-time high-dimensional nonlinear systems, *i.e.*, $n \geq 3$. To illustrate this contribution, we consider the following electric circuit system, modeled by a Hopfield artificial neural network [33]:

$$\begin{aligned} \dot{x}_i &= \Lambda_i \left(\sum_{i=1}^{3} S_{ij} x_j - \frac{2V_m}{\lambda \pi R_i} \tan\left(\frac{\pi x_i}{2V_m}\right) \right) \\ \Lambda_i &= \frac{\lambda}{C_i} \frac{1}{1 + \tan^2\left(\frac{\pi x_i}{2V_m}\right)} \end{aligned} \tag{50}$$

where the state variables x_i, for $i \in \mathcal{I}_3$, are the voltages at the amplifier outputs. These voltages can only take values in the set $\mathscr{R} = [-V_m, V_m] \times [-V_m, V_m] \times [-V_m, V_m]$. The parameter values are given by $\lambda = 7.0 \times 10^{-2}$, $C_1 = 3.3$ [μF], $C_2 = 1.5$ [μF], $C_2 = 5.6$ [μF], $R_1 = 1.0$ [kΩ], $R_2 = 3.3$ [kΩ], $R_3 = 1.2$ [kΩ], $V_m = 5$ [V], and

$$S_{ij} = 10^{-3} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}.$$

Performing a forward Euler discretization with a sampling period $T_e = 0.001$ [s], a discrete-time version of system (50) can be represented by (11). The Mamdani-like fuzzy model (10) of this system is constructed with 800 points of a $5 \times 4 \times 4$ hyper-rectangular grid on the state space $\mathscr{R}$. Solving LMI conditions in Theorem 1, a membership-functions-dependent Lyapunov function (20) can be found to prove the stability of the discrete-time fuzzy model of system (50). Fig. 10 shows the behaviors of system (50) in the phase plane and the time evolution of the obtained Lyapunov function corresponding to the system trajectories with four different initial conditions: $x(0) = \begin{bmatrix} 1.667 & -2.778 & -5 \end{bmatrix}^\top$, $x(0) = \begin{bmatrix} 1.667 & -2.778 & 3.889 \end{bmatrix}^\top$, $x(0) = \begin{bmatrix} -5 & -5 & 5 \end{bmatrix}^\top$ and $x(0) = \begin{bmatrix} -5 & -5 & -5 \end{bmatrix}^\top$. These results indicates a stable behavior of the fuzzy model of system (50). It is important to note that the trajectories of the nonlinear system (50) and those of its Mamdani-like fuzzy model are superposed in Fig. 10. This again confirms that the considered Mamdani-like fuzzy modeling can accurately approximate the complex nonlinear system (50).

V. Concluding Remarks

A numerical method to check the stability of discrete-time fuzzy systems with singleton consequents has been proposed. We leverage parametric expressions, specific to this class of Mamdani-like fuzzy systems, together with Finsler's lemma to derive stability analysis conditions for systems of any order. In particular, using the $S-$procedure and Lyapunov stability tools, the information of the piecewise regions of fuzzy systems is fully exploited to further reduce the conservatism of the stability conditions, expressed in terms of linear matrix inequalities. Several numerical and physically motivated examples are provided to show the interests of the new stability method for Mamdani-like fuzzy systems. As discussed in Remark 7, future works focus on characterizing the approximation errors between the original nonlinear system and its Mamdani-like fuzzy model, which is crucial to derive fuzzy-model-based conditions for stability analysis of general nonlinear systems. Another promising research topic is to develop stabilizing Mamdani-like fuzzy controllers in

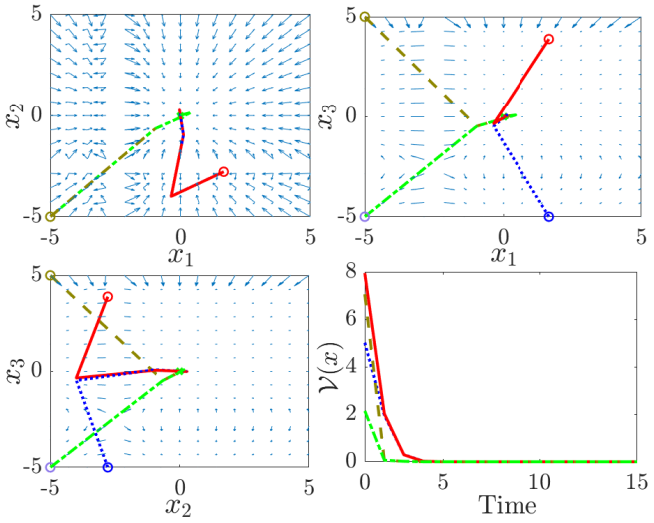

Fig. 10: Behavior in the phase plane of both the nonlinear system and its Mamdani-like fuzzy model in Example 4, and the time evolution of the obtained nonquadratic Lyapunov function with respect to four different initial conditions.

parametric expression forms, conveniently implemented with look-up tables for industrial interests.

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Anh-Tu Nguyen (M'18, SM'21) is an Associate Professor at the INSA Hauts-de-France, Université Polytechnique Hauts-de-France, Valenciennes, France. He received the degree in engineering and the M.Sc. degree in automatic control from Grenoble Institute of Technology, Grenoble, France, in 2009, and the Ph.D. degree in automatic control from the University of Valenciennes, Valenciennes, France, in 2013. He is an Associate Editor for the IEEE Transactions on Intelligent Transportation Systems, the IFAC journal Control Engineering Practice, the IET Journal of Engineering, the SAE International Journal of Vehicle Dynamics, Stability, and NVH, the Springer Automotive Innovation, Frontiers in Control Engineering, and a Guest Editor for special issues in various international journals. His research interests include robust control and estimation, cybernetics control systems, human-machine shared control with a strong emphasis on mechatronics applications (see more information at https://sites.google.com/view/anh-tu-nguyen).

Amine Dehak received the B.Sc. degree from the École Nationale Supérieur des Arts & Métiers, Morocco, in 2017, and the M.Sc. degree from École Central de Nantes, France, in 2018. He received a Ph.D. degree in automatic control at Université Polytechnique des Hauts-de-France, Valenciennes, France, in 2022. He was a Temporary Lecturer and Research Assistant at the LAMIH laboratory, UMR CNRS 8201, Valenciennes. His research interests included reduced-complexity modeling and control of nonlinear systems and their applications.

Thierry-Marie Guerra is a Full Professor at the Polytechnic University Hauts-de-France, Valenciennes, France. He received his PhD degree in automatic control in 1991 and the HDR in 1999. From 2009 to 2019, he was head of the Laboratory of Industrial and Human Automation, Mechanics and Computer Science (LAMIH CNRS UMR 8201) (148 researchers and staff, 110 PhD students and postdocs) http://www.univ-valenciennes.fr/LAMIH/. He is chair of the Technical Committee 3.2 "Computational Intelligence in Control" for IFAC, member of the IFAC TC 7.1 "Automotive Control", Area Editor of the international journals: Fuzzy Sets & Systems and IEEE Transactions on Fuzzy Systems. His research fields and topics of interest are wine, hard rock, stamps, nonlinear control, LPV, Takagi-Sugeno models control and observation, LMI constraints, nonquadratic Lyapunov functions and applications to mobility, soft robotics and disabled persons.

Michio Sugeno (LM'18) received the B.Sc. degree from the Department of Physics, University of Tokyo, Tokyo, Japan, in 1962.

He was with the company for three years and then with the Tokyo Institute of Technology, Tokyo, Japan, as a Research Associate, an Associate Professor, and a Professor from 1965 to 2000. After retiring from the Tokyo Institute of Technology, he was a Laboratory Head with the Brain Science Institute, RIKEN, from 2000 to 2005 and then as a Distinguished Visiting Professor with the Doshisha University from 2005 to 2010. Finally, he was an Emeritus Researcher with the European Centre for Soft Computing, Spain, from 2010 to 2015. He is currently an Emeritus Professor with the Tokyo Institute of Technology.

Dr. Sugeno was the President of the Japan Society for Fuzzy Theory and Systems from 1991 to 1993, and also the President of the International Fuzzy Systems Association from 1997 to 1999. He is the first recipient of the IEEE CIS Pioneer Award in Fuzzy Systems with Zadeh in 2000. He was the recipient of the 2010 IEEE Frank Rosenblatt Award. He was also the recipient of the IEEE International Conference on Systems, Man, and Cybernetics 2017 Lotfi A. Zadeh Pioneer Award.