A sufficient condition to design unknown input observers for nonlinear systems with arbitrary relative degree

Pedro Coutinho, Iury Bessa, Wen-bo Xie, Tran Anh-Tu Nguyen, Reinaldo Martínez Palhares

A Sufficient Condition to Design Unknown Input Observers for Nonlinear Systems with Arbitrary Relative Degree

Article *in* International Journal of Robust and Nonlinear Control · June 2022

DOI: 10.1002/rnc.6273

5 authors, including:

Pedro Henrique Silva Coutinho
Federal University of Minas Gerais
47 PUBLICATIONS **250** CITATIONS

SEE PROFILE

Iury Bessa
Federal University of Amazonas
111 PUBLICATIONS **636** CITATIONS

SEE PROFILE

Wenbo Xie
Shanghai University
65 PUBLICATIONS **493** CITATIONS

SEE PROFILE

Anh-Tu Nguyen
Université Polytechnique Hauts-de-France
140 PUBLICATIONS **2,084** CITATIONS

SEE PROFILE

ORIGINAL ARTICLE

Journal Section

A Sufficient Condition to Design Unknown Input Observers for Nonlinear Systems with Arbitrary Relative Degree

Pedro H. S. Coutinho[1] | Iury Bessa[1,2] | Wen-Bo Xie[3] | Anh-Tu Nguyen[4] | Reinaldo Martínez Palhares[5]

[1]Graduate Program in Electrical Engineering, Federal University of Minas Gerais, Belo Horizonte, Minas Gerais, Brazil

[2]Department of Electricity, Federal University of Amazonas, Manaus, Amazonas, Brazil

[3]School of Mechatronic Engineering and Automation, Shanghai University, Shangai, China

[4]Univ. Polytechnique Hauts-de-France, LAMIH CNRS UMR 8201, INSA Hauts-de-France, Valenciennes, France

[5]Department of Electronics Engineering, Federal University of Minas Gerais, Belo Horizonte, Minas Gerais, Brazil

Correspondence
Reinaldo Martínez Palhares, Department of Electronics Engineering, Federal University of Minas Gerais, Belo Horizonte, Minas Gerais, 31270-901, Brazil
Email: rpalhares@ufmg.br

Funding information
CNPq (Grant Numbers: 307933/2018-0 and 164692/2020-7), Brazil; PROPG-CAPES/FAPEAM Scholarship Program (Grant Number: 88887.217045/2018-00), Brazil; and FAPEMIG (Grant number: PPM-00053-17), Brazil.

This paper addresses the problem of state and unknown inputs (UIs) estimation for nonlinear systems with arbitrary relative degree with respect to the UIs. For this purpose, a novel nonlinear unknown input observer (UIO) is proposed, which is able to decouple the UIs by using the derivatives of the output signal. The error dynamics is attained by an exact handling and a factorization of its gradient to obtain a local polytopic representation suitable for input-affine nonlinear systems. For that representation, a novel design condition based on convex optimization and linear matrix inequalities is proposed to exponentially stabilize the estimation error and to guarantee the validity of the proposed nonlinear UIO. Numerical simulations indicate the effectiveness of the proposed approach for different classes of nonlinear systems, for which the UIs could be totally decoupled from the state estimation.

KEYWORDS
Unknown input observers, Nonlinear system, Parameter-varying systems, Linear matrix inequalities

1 | INTRODUCTION

The state estimation of nonlinear systems is a basic concern in control theory, which becomes more challenging in the presence of uncertainties. The uncertainties are unavoidable since they are related, for example, to disturbances, inaccurate models, faults and cyber-attacks. This context motivates the design of observers that are able to estimate simultaneously the system states and uncertainties. In this sense, the reconstruction of disturbances plays an important role to support, for example, secure estimation [1], fault diagnosis [2], and fault-tolerant control [3] strategies. However, while the disturbance estimation problem is well addressed for linear systems under usual conditions, such as unitary relative degree, the literature is still incipient for nonlinear systems with arbitrary relative degree.

The main approaches to estimate the states of nonlinear systems in the presence of disturbances are disturbance observers (DOBs) and unknown input observers (UIOs). DOBs are filters using input-output models to estimate the external disturbances and model discrepancies. They usually require the measurement of the states but they can also provide simultaneous state and disturbance estimation by means of the so-called extended state observer. Conversely, UIOs are able to decouple the state estimation from the disturbances. Therefore, UIOs estimate the states even in the presence of unknown inputs (UIs), without explicitly estimating them, although the unknown input estimation is also possible. The UIO design is usually based on: *(i)* decoupling of UIs, and *(ii)* stabilization of error dynamics. The UI decoupling is typically enabled by matching conditions, *i.e.*, the effect of the UIs in the measured outputs allows for a direct cancellation of their effects on the estimation error dynamics. Moreover, the stabilization of the error dynamics decoupled from UIs is possible if it is observable. However, even observable dynamics may present challenges which hampers the error convergence, as in the non-minimum-phase dynamics.

Given those aspects, a number of important results for designing UIOs have been recently presented. To name a few, the UIO design with arbitrary error dynamics eigenvalues assignment in the presence of invariant zeros is addressed for discrete-time [4] and continuous-time [5] linear time-invariant (LTI) systems, which allows the improvement of the UIO convergence performance. In [6, 7], the UIO equipped with reset laws is proposed to improve the performance and robustness. Regarding the decoupling of UIs, the conservative assumption on the unitary relative degree of outputs with respect to the UIs is usually sufficient to ensure the decoupling, as shown in [8, 9, 10]. However, the matching condition may not be satisfied in some practical systems, which precludes the decoupling. When the UIs cannot be totally decoupled from the error dynamics, the set-theoretic [11, 12] and interval [13, 14] UIOs are alternatives to handle coupled UIs [15]. Another alternative for the case where the UIs are not completely decoupled is the $\mathcal{L}_2$-norm attenuation [16, 17]. For discrete-time LTI systems, the matching conditions are avoided at the expense of delayed estimations in [18]. For linear parameter-varying (LPV) systems, in [19] the UIOs are generalized for systems with arbitrary relative degree, since the high-order derivatives of the outputs are available.

Although there are relevant advances on the UIO design for nonlinear systems, the achievements regarding relaxation of the UI decoupling and error stabilization conditions are still incipient, and most of them are applicable for nonlinear systems with specific structures. For example, in [20, 17, 21, 22], UIO design approaches are proposed for systems with Lipschitz nonlinearities. The UIO design for a more general class of nonlinear systems is achieved using quasi-LPV representations with local error convergence guarantees [9, 23]. The aforementioned Lipschitzian and quasi-LPV approaches provide sufficient design conditions based on semidefinite programming with LMIs or sum-of-squares constraints, which make easier to obtain solutions and simplify the observability analysis. However, they are not able to deal with nonlinear systems that do not meet the first-order matching condition. Moreover, the design of those observers must consider the domain of validity of local models, as discussed in the control design context by [24, 25, 26, 27]. An alternative to overcome the first-order matching condition for nonlinear systems is the use of sliding-mode UIOs with high-order differentiators, as proposed in [28, 29]. In [30], a descriptor Takagi-Sugeno (T-S)

fuzzy representations for discrete-time nonlinear systems with arbitrary relative degree is used to estimate the states and the UIs with a guaranteed $\mathcal{L}_\infty$ performance.

Considering the lack of constructive design conditions for nonlinear systems with generic structure and arbitrary relative degree, this paper proposes a nonlinear Luenberger-like UIO structure to address this problem. As suggested in [19], for LPV systems, the proposed nonlinear UIO uses the derivatives of the output, which can be obtained by a sliding-mode differentiator, to avoid the mismatch condition and allow the UI decoupling even for systems with an arbitrary relative degree greater than one. The nonlinear error dynamics is obtained by employing an exact handling of the gradient which factorizes the error vector, as proposed in [31] for nonlinear observers. It allows the use of a quasi-LPV representation where the measured and non-measured nonlinear terms are embedded in scheduling parameters. That factorization eases the decoupling between the measurable and non-measurable signals and enables to obtain LMI-based constructive conditions to design the proposed nonlinear UIO for any input-affine nonlinear analytical system. Compared to previous works, the proposed approach ensures local exponential convergence of the state and UI estimates even for the non-unitary relative degree case. Therefore, the main contributions are listed as follows:

- a novel nonlinear Luenberger-like UIO to simultaneously estimate states and UIs of nonlinear systems with arbitrary relative degree;
- a factorization-based procedure to describe the error dynamics by using local polytopic quasi-LPV models;
- constructive LMI-based conditions for nonlinear UIO design that ensure the exponential convergence of the state and UIs estimates and the inclusion of the system and UIO trajectories within a predefined region of validity of the proposed local quasi-LPV model.

Compared to previous works, the proposed approach ensures local exponential convergence of the state and UI estimates even for the non-unitary relative degree case. To the best of our knowledge, there are no reports on LMI-based constructive conditions for designing nonlinear UIOs for nonlinear systems with an arbitrary relative degree even to ensure local exponential convergence.

This paper is organized as follows. Section 2 describes the state and UI system description and presents the key definitions and assumptions to support the proposed approach as well as the problem formulation. Section 3 presents the proposed nonlinear UIO, the procedure to obtain the quasi-LPV error dynamics as well as the sufficient conditions to design an UIO that ensures the exponential stability of the error dynamics and the validity of the quasi-LPV model. Section 4 provides simulation examples to show the effectiveness of the proposed approach for different nonlinear systems. Finally, Section 5 draws the conclusions and future works.

Notation

The following notations are adopted in this work. $\mathbb{N}$ denotes the set of natural numbers (or positive integers) and $\mathbb{N}_{\leq p}$ denotes the set of integers less than or equal to $p \in \mathbb{N}$. $\mathbb{R}$ is the field of real numbers and $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers. Given a matrix A, $A^\top$, A^{-1} and $A^\dagger$ respectively represent the transpose, the inverse and the pseudo-inverse of A, and $\mathrm{He}[A] = A + A^\top$. $L_f h_j(x)$ denotes the Lie derivative of the scalar function h_j with respect to the vector field $f(x)$ and $L_f^k h_j(x) = L_f L_{f-1}^k h_j(x)$. The symbol $y^{(k)}(t)$ denotes the k-th time-derivative of the function y at time t. For N matrices X_i of appropriate orders, it is defined $(X_1, \ldots, X_N) = [X_1^\top, \ldots, X_N^\top]^\top$ and $\mathrm{col}(X_1, \ldots, X_N) = [X_1, \ldots, X_N]$. The Euclidean norm of a real vector x is denoted by $\|x\|$. A multi-index is denoted as $\mathbf{i} = (i_1, \ldots, i_p) \in \mathbb{B}^p$, where $\mathbb{B}^p = \{\mathbf{i} : i_j \in \mathbb{B}, j \in \mathbb{N}_{\leq p}\}$. It is defined $\mathbb{B}^{p+} = \{\mathbf{i} : i_j \leq i_{j+1}, i_j \in \mathbb{B}, j \in \mathbb{N}_{\leq p-1}\}$ as the set of "upper-triangle" indexes and $\mathscr{P}(\mathbf{i})$ is the set of permutations of the entries of $\mathbf{i}$. The functions $\lambda_{\min}(X)$ ($\lambda_{\max}(X)$) denotes the minimum (maximum) eigenvalue of matrix X. Finally, the Minkowski sum and the Pontryagin difference

of two sets $\mathscr{X}$ and $\mathscr{Y}$ are defined respectively by $\mathscr{X} \oplus \mathscr{Y} \triangleq \{x + y | x \in \mathscr{X}, y \in \mathscr{Y}\}$ and $\mathscr{X} \ominus \mathscr{Y} \triangleq \{x | x \oplus \mathscr{Y} \subseteq \mathscr{X}\}$.

2 | SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the following class of input-affine nonlinear systems subject to disturbances:

$$\begin{aligned} \dot{x} &= f(x) + g(x)d \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathscr{D} \subset \mathbb{R}^n$ is the state, $\mathscr{D}$ is a convex polytope containing the origin with n_f hyperplanes:

$$\mathscr{D} = \{x \in \mathbb{R}^n : a_j^\top x \leq b_j,\ j \in \mathbb{N}_{\leq n_f},\ a_j \in \mathbb{R}^n,\ b_j \in \mathbb{R}\}, \tag{2}$$

$d \in \mathbb{R}^m$ is the vector of bounded disturbances, and $y \in \mathbb{R}^p$ is the output. The vector fields $f : \mathscr{D} \to \mathbb{R}^n$, $g : \mathscr{D} \to \mathbb{R}^{n \times m}$, with $g(x) = \text{col}(g_1(x), \ldots, g_m(x))$ and $h : \mathscr{D} \to \mathbb{R}^p$, $h(x) = (h_1(x), \ldots, h_p(x))$ are assumed to be sufficiently smooth on $\mathscr{D}$. The following assumptions are considered for UIO design.

Assumption 1 *The nonlinear system* (1) *has uniform UI relative degree $\rho = \{\rho_1, \ldots, \rho_p\}$ on $\mathscr{D}$. It means that*

$$L_{g_j} L_f^{k-1} h_i(x) = 0, \quad k \in \mathbb{N}_{\leq \rho_i - 1},\ i \in \mathbb{N}_{\leq p},\ j \in \mathbb{N}_{\leq m},$$

and

$$\Gamma(x) = \begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{\rho_1 - 1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\rho_p - 1} h_p(x) & \cdots & L_{g_m} L_f^{\rho_p - 1} h_p(x) \end{bmatrix}, \tag{3}$$

is full column rank for all $x \in \mathscr{D}$.

Remark 1 The nonlinear UIO design approaches in the literature are often based on the assumption of unitary UI relative degree (UIRD), that is, $\rho_i = 1$, $\forall i \in \mathbb{N}_{\leq p}$, and $L_{g_j} h_i(x) \neq 0$, $\forall i \in \mathbb{N}_{\leq p}, j \in \mathbb{N}_{\leq m}$. It ensures that sufficient information on the UI vector d is available directly from the output channel y. Unfortunately, it is not direct to employ those conditions to deal with nonlinear systems with arbitrary UIRD as considered in this work. Thus, Assumption 1 ensures that sufficient information on d is available from the high-order derivative terms. The definition of UIRD can be found in [32].

Assumption 2 *The ρ_i-th time-derivative of the output signal $y_i(t)$, denoted as $y_i^{(\rho_i)}(t)$, is available for all $i \in \mathbb{N}_{\leq p}$.*

Remark 2 In this paper, the signals $y_i^{(\rho_i)}$ required in Assumption 2 are computed with Levant's differentiators [33], which are able to provide exact and finite-time estimates. It is worth noting that it is common to use sliding-mode differentiators for control and estimation problems. As a matter of fact, the use of high-order differentiators in UIOs was already proposed, for instance, to linear time-invariant systems [29, 34], linear parameter-varying systems [19] and nonlinear systems (using sliding-mode UIO) [28]. Probably, one of the earliest papers that proposed the use of Levant's differentiator for unknown input estimation is [34]. In [19], the proposed nonlinear UIO for LPV systems

uses the derivatives of the output, which can be obtained by a sliding-mode differentiator, to avoid the mismatch condition and to allow the UI decoupling even for systems with an arbitrary relative degree greater than one. In [35], the Levant's differentiator is used to estimate the derivatives of the membership functions for fuzzy control design.

Assumption 3 *There exist convex sets containing the origin $\mathscr{W} \subset \mathbb{R}^m$ and $\mathscr{R} \subset \mathscr{D}$, whose support function is $\phi_{\mathscr{R}} : \mathbb{R}^n \to \mathbb{R}$, such that $x(t) \in \mathscr{R}$ and $d(t) \in \mathscr{W}$, $\forall t \geq t_0$.*

Remark 3 The support function $\phi_{\mathscr{R}}(z)$ of the convex set $\mathscr{R}$ is defined as $\phi_{\mathscr{R}}(z) = \sup_{x \in \mathscr{R}} z^\top x$ [36]. For example, if an ellipsoidal set $\mathscr{R} = \{x \in \mathbb{R}^n : x^\top \Lambda x \leq \mu^2\}$ is considered, its support function is given by $\phi_{\mathscr{R}}(z) = \mu\sqrt{z^\top \Lambda^{-1} z}$; if a convex polytope $\mathscr{R} = \{x \in \mathbb{R}^n : Fx \leq 1\}$ is considered, its support function is given by $\phi_{\mathscr{R}}(z) = \min\{1^\top w : F^\top w = z, w \in \mathbb{R}^{\varphi}_{\geq 0}\}$, where φ is the number of hyperplanes of the polytope $\mathscr{R}$.

Under Assumptions 1, 2 and 3, the following augmented observation vector is considered

$$\begin{bmatrix} y \\ \mathcal{Y} \end{bmatrix} = \begin{bmatrix} h(x) \\ \Psi(x) + \Gamma(x)d \end{bmatrix}, \tag{4}$$

where $\mathcal{Y} = \left(y_1^{(\rho_1)}, \ldots, y_p^{(\rho_p)}\right)$ and $\Psi(x) = (L_f^{\rho_1} h_1(x), \ldots, L_f^{\rho_p} h_p(x))$. Then, the system dynamics (1) can be written as follows:

$$\begin{aligned} \dot{x} &= f(x) + Q(x)\Gamma(x)d \\ y &= h(x) \\ \mathcal{Y} &= \Psi(x) + \Gamma(x)d \end{aligned} \tag{5}$$

where $\Gamma(x)$ is given in (3) and, for $\text{rank}(\Gamma(x)) = \text{rank}(g(x))$ (i.e., $p \geq m$) and $\forall x \in \mathscr{D}$ (cf. Assumption 1), $Q(x)$ is defined by

$$Q(x) = g(x)\Gamma^{\dagger}(x). \tag{6}$$

Based on Assumptions 1–3, the following unknown input and state estimation problem for the nonlinear system (1) is proposed.

Problem statement

Consider a nonlinear system (1) satisfying Assumptions 1–3. Design a UIO such that the estimates of state and UI, respectively $\hat{x}(t)$ and $\hat{d}(t)$, converge exponentially to $x(t)$ and $d(t)$, respectively, for $t \geq t_0$.

3 | MAIN RESULTS

This section presents a novel nonlinear UIO structure to estimate both the state and UIs of nonlinear systems with an arbitrary UIRD as well as a constructive condition to design the proposed UIO.

3.1 | Nonlinear unknown input observer

Since the dynamics of the nonlinear system (1) satisfying Assumptions 1 and 2 can be written as in (5), the following Luenberger-like observer is proposed:

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}) + Q(\hat{x})\,(\mathcal{Y} - \Psi(\hat{x})) + L(\hat{x}, y)(y - h(\hat{x}))\\ \hat{d} &= \Gamma^{\dagger}(\hat{x})\,(\mathcal{Y} - \Psi(\hat{x}))\,,\end{aligned} \tag{7}$$

where $L(\hat{x}, y)$ is a gain to be designed that only depends on the measurement vector y, and it does not depend on the observation vector $\mathcal{Y}$.

The big picture of the estimation problem handled in this paper is depicted in Figure 1, where it illustrates the proposed scheme to estimate the UIs and states by using the nonlinear UIO (7), such that the observation vector $\mathcal{Y}$ used in the UIO is provided by a Levant's differentiator.

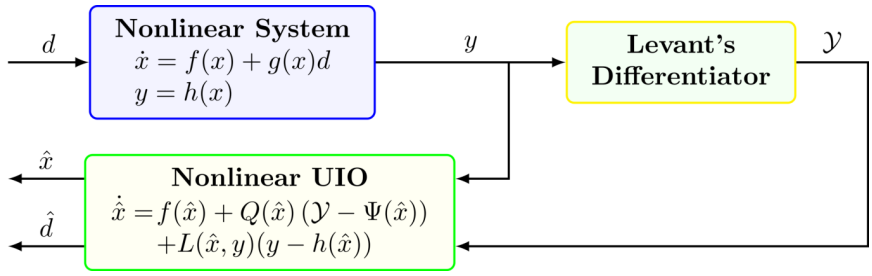

FIGURE 1 Block diagram of the proposed strategy for estimating the UIs and states by using the nonlinear UIO (7).

Remark 4 Notice that the estimation dynamics in (7) depends on the matrix valued function $Q(\hat{x})$, which can be determined only if $\hat{x} \in \mathscr{D}$, as one can see from (6) and Assumption 1. Thus, it is required to constrain the trajectories of $\hat{x}$ on $\mathscr{D}$ to ensure the rank condition and the correct operation of the nonlinear UIO.

Let the state estimation error be defined as $e = x - \hat{x}$, such that its dynamics is given as follows:

$$\dot{e} = f(x) - f(\hat{x}) - L(\hat{x}, y)(h(x) - h(\hat{x})) - Q(\hat{x})(\Psi(x) - \Psi(\hat{x})) + (Q(x) - Q(\hat{x}))\Gamma(x)d. \tag{8}$$

Note that if the UIO is appropriately designed such that $\hat{x}(t) \to x(t)$ as $t \to +\infty$, then it follows from (8) that $e \to 0$. Let $\alpha = (\hat{x}, y)$ and $\beta = (x, d)$ be the vectors of available and non-available signals, respectively.

Remark 5 Regardless of the fact that both vectors of available and non-available signals are within polytopic sets, they are also separated into distinguished vectors α and β because the proposed nonlinear UIO (7) is able to use only the α signals, although both must be considered in its design.

Following the exact factorization arguments in [31], the estimation error dynamics can be written as follows:

$$\dot{e} = [A(\alpha, \beta) - L(\alpha)C(\alpha, \beta)]\,e, \tag{9}$$

where $A(\alpha,\beta) = A_1(\alpha,\beta) + A_2(\alpha,\beta) + A_3(\alpha,\beta)$, with

$$
\begin{aligned}
A_1(\alpha,\beta)e &= f(x) - f(\hat{x}),\\
A_2(\alpha,\beta)e &= -Q(\hat{x})(\Psi(x) - \Psi(\hat{x})),\\
A_3(\alpha,\beta)e &= (Q(x) - Q(\hat{x}))\Gamma(x)d,\\
C(\alpha,\beta)e &= h(x) - h(\hat{x}).
\end{aligned}
$$

Remark 6 In [31], it is shown that the observation error of nonlinear observers can be written by exactly handling of its gradient using systematic factorization procedures. In this paper, we propose using the same factorization procedures for rewriting the error dynamics (8) as (9). As shown in [31, Lemma 1], for any multivariate polynomial function $p(x)$, the difference $p(x) - p(\hat{x})$ can be factorized as $q(x,\hat{x})(x - \hat{x})$. The same conclusion is extended for any analytic function $p(x)$, by using the truncated Taylor's series. In this paper, the terms $A_1(\alpha,\beta)$, $A_2(\alpha,\beta)$, $A_3(\alpha,\beta)$, and $C(\alpha,\beta)$ are obtained by following the same results.

The following example illustrates the procedure to obtain a factorized form (9) for the error dynamics of a nonlinear system described as (1).

Example 1

Consider the following nonlinear system adapted from [19]:

$$
\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}(1+x_1^2)x_2\\ -x_1^3 - 2x_2\end{bmatrix} + \begin{bmatrix}0\\ 1+x_1^2\end{bmatrix} d\\
y &= x_1.
\end{aligned}
\tag{10}
$$

This system has an uniform UIRD of $\rho = 2$ on $\mathbb{R}^2$. To define the nonlinear UIO, the following maps are computed:

$$
\begin{aligned}
\Psi(x) &= L_f^2 h(x) = (1+x_1^2)(-x_1^3 + 2x_1x_2^2 - 2x_2),\\
\Gamma(x) &= L_g L_f h(x) = (1+x_1^2)^2.
\end{aligned}
$$

Note that the matrix $\Gamma(x)$ is nonsingular for all $x \in \mathbb{R}^2$ and $\text{rank}(\Gamma(x)) = \text{rank}(g(x)) = 1$, $\forall x \in \mathbb{R}^2$. The matrix $Q(x)$ is thus obtained by solving (6), which admits the solution

$$
Q(x) = \begin{bmatrix}0\\ \dfrac{1}{1+x_1^2}\end{bmatrix}.
$$

As suggested in [31], the following factorizations are obtained considering that the state x_1 is measured

$$
\begin{aligned}
f(x) - f(\hat{x}) &= \begin{bmatrix}(1+x_1^2)(x_2 - \hat{x}_2)\\ -2(x_2 - \hat{x}_2)\end{bmatrix},\\
Q(x) - Q(\hat{x}) &= 0,\\
Q(\hat{x})(\Psi(x) - \Psi(\hat{x})) &= \begin{bmatrix}0\\ 2x_1(x_2^2 - \hat{x}_2^2) - 2(x_2 - \hat{x}_2)\end{bmatrix}.
\end{aligned}
$$

Thus, the error dynamics is given by

$$\dot{e} = \begin{bmatrix} (1+x_1^2)e_2 \\ -2x_1(x_2+\hat{x}_2)e_2 \end{bmatrix},$$

which can be rewritten as in (9) with

$$A(\alpha,\beta) = A(x,\hat{x}) = \begin{bmatrix} 0 & 1+x_1^2 \\ 0 & -2x_1(x_2+\hat{x}_2) \end{bmatrix}. \tag{11}$$

3.2 | Local polytopic modeling

To obtain a convex structure for the exact rewriting of the state estimation error dynamics (9), the nonlinear terms in the state-dependent matrices, which depend on available signals α, are collected in the vector-valued function $z(\alpha) = (z_1(\alpha), \ldots, z_q(\alpha))$, and the nonlinear terms depending on the non-available signals, β, are collected in $\zeta(\beta) = (\zeta_1(\beta), \ldots, \zeta_\sigma(\beta))$. Assuming that $x \in \mathscr{R}$, $\hat{x} \in \mathscr{D}$, and $d \in \mathscr{W}$, it is possible to determine bounds $z_j(\alpha) \in [z_j^0, z_j^1]$, $j \in \mathbb{N}_{\leq q}$, and $\zeta_k(\beta) \in [\zeta_k^0, \zeta_k^1]$, $k \in \mathbb{N}_{\leq \sigma}$. As a result, each nonlinear term can be equivalently written as

$$\begin{aligned} z_j(\alpha) &= w_0^j(\alpha) z_j^0 + w_1^j(\alpha) z_j^1, \\ \zeta_k(\beta) &= \omega_0^k(\beta)\zeta_k^0 + \omega_1^k(\beta)\zeta_k^1, \end{aligned}$$

where

$$\begin{aligned} w_0^j(\alpha) &= \tfrac{z_j^1 - z_j(\alpha)}{z_j^1 - z_j^0}, \quad w_1^j(\alpha) = 1 - w_0^j(\alpha), \\ \omega_0^k(\beta) &= \tfrac{\zeta_k^1 - \zeta_k(\beta)}{\zeta_k^1 - \zeta_k^0}, \quad \omega_1^k(\beta) = 1 - \omega_0^k(\beta). \end{aligned}$$

Thus, $z(\alpha)$ and $\zeta(\beta)$ belong to convex polytopes in $\mathbb{R}^q$ and $\mathbb{R}^\sigma$ with 2^q and 2^σ vertices, respectively. As a result, the state-dependent matrices in (9) can be given as convex combinations of their vertices, providing the following quasi-LPV model for the error dynamics:

$$\dot{e} = \sum_{\mathbf{i}\in\mathbb{B}^q} \sum_{\mathbf{j}\in\mathbb{B}^q} \sum_{\mathbf{k}\in\mathbb{B}^\sigma} \mathrm{w}_\mathbf{i}(\alpha)\mathrm{w}_\mathbf{j}(\alpha)\mathrm{v}_\mathbf{k}(\beta)\left(A_{\mathbf{ik}} - L_\mathbf{j} C_\mathbf{ik}\right) e, \tag{12}$$

where the functions

$$\mathrm{w}_\mathbf{i}(\alpha) = \prod_{j=1}^{q} w_{i_j}^j(\alpha), \quad \mathrm{v}_\mathbf{k}(\beta) = \prod_{j=1}^{\sigma} \omega_{k_j}^j(\beta),$$

satisfy the following properties:

$$\begin{aligned} &0 \leq \mathrm{w}_\mathbf{i}(\alpha) \leq 1, \ 0 \leq \mathrm{v}_\mathbf{k}(\beta) \leq 1, \\ &\sum_{\mathbf{i}\in\mathbb{B}^q} \mathrm{w}_\mathbf{i}(\alpha) = 1, \ \sum_{\mathbf{k}\in\mathbb{B}^\sigma} \mathrm{v}_\mathbf{k}(\beta) = 1. \end{aligned}$$

Example 2

Consider the following modified version of system (10) from Example 1:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}(1+x_1^2)x_2\\ -x_1^3-2x_2\end{bmatrix} + \begin{bmatrix}0\\ 1+x_1^2\end{bmatrix} d\\ y &= \text{atan}\,(x_1).\end{aligned} \tag{13}$$

This system also has an uniform UIRD of $\rho = 2$ on $\mathbb{R}^2$, and the following maps are computed:

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix}\frac{1}{1+x_1^2} & 0\end{bmatrix}\begin{bmatrix}(1+x_1^2)x_2\\ -x_1^3-2x_2\end{bmatrix} = x_2.$$

$$\Psi(x) = L_f^2 h(x) = \frac{\partial L_f h(x)}{\partial x} f(x) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}(1+x_1^2)x_2\\ -x_1^3-2x_2\end{bmatrix} = -x_1^3-2x_2$$

$$\Gamma(x) = L_g L_f h(x) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\ 1+x_1^2\end{bmatrix} = 1+x_1^2.$$

The following matrix $Q(x)$ is obtained from (6):

$$Q(x) = \begin{bmatrix}0\\ 1\end{bmatrix}.$$

In this case, neither x_1 nor x_2 are measured, therefore the following factorizations are performed:

$$f(x) - f(\hat{x}) = \begin{bmatrix}(1+x_1^2)(x_2-\hat{x}_2) - (x_1+\hat{x}_1)\hat{x}_2(x_1-\hat{x}_1)\\ -(x_1^3-\hat{x}_1^3) - 2(x_2-\hat{x}_2)\end{bmatrix},$$

$$Q(x) - Q(\hat{x}) = 0,$$

$$Q(\hat{x})(\Psi(x) - \Psi(\hat{x})) = \begin{bmatrix}0\\ -(x_1^3-\hat{x}_1^3) - 2(x_2-\hat{x}_2)\end{bmatrix}.$$

Thus, the matrix $A(\alpha,\beta) = A_1(\alpha,\beta) + A_2(\alpha,\beta) + A_3(\alpha,\beta)$ is given by

$$A(\alpha,\beta) = \begin{bmatrix}-(x_1+\hat{x}_1)\hat{x}_2 & 1+x_1^2\\ 0 & 0\end{bmatrix}.$$

Considering the Taylor's series expansion of atan (x_1) for $|x_1| \leq 1$, the difference of the output maps is:

$$\begin{aligned}h(x) - h(\hat{x}) &= \text{atan}\,(x_1) - \text{atan}\,(\hat{x}_1)\\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} x_1^{2i+1} - \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \hat{x}_1^{2i+1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \left(x_1^{2i+1} - \hat{x}_1^{2i+1}\right)\end{aligned}$$

Using the factorization $x_1^{2i+1} - \hat{x}_1^{2i+1} = (x_1 - \hat{x}_1)\sum_{k=0}^{2n} x_1^k \hat{x}_1^{2n-k}$, one has

$$h(x) - h(\hat{x}) = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \sum_{k=0}^{2n} x_1^k \hat{x}_1^{2n-k}\right)(x_1 - \hat{x}_1),$$

such that the following parameter-dependent output matrix is obtained:

$$C(\alpha, \beta) = \left[\sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \sum_{k=0}^{2n} x_1^k \hat{x}_1^{2n-k} \quad 0\right].$$

Assuming that $\mathscr{D} = \{x \in \mathbb{R}^2 : |x_i| \leq 1\}$ and $\mathscr{R} = \{x \in \mathbb{R}^2 : |x_i| \leq 0.5\}$, we obtain the following bounds:

$$z_1^0 = z_2^0 = -1,\ z_1^1 = z_2^1 = 1,\ \zeta_1^0 = -0.5,\ \zeta_2^0 = 0,$$
$$\zeta_1^1 = 0.5,\ \zeta_2^1 = 0.25,\ \zeta_3^0 = 0.2216,\ \zeta_3^1 = 0.9763.$$

Therefore, the matrices $A(\alpha, \beta)$, $L(\alpha)$, and $C(\beta)$ are

$$A(\alpha, \beta) = \sum_{\mathbf{i} \in \mathbb{B}^q} \sum_{\mathbf{k} \in \mathbb{B}^\sigma} \mathrm{w}_{\mathbf{i}}(\alpha)\mathrm{v}_{\mathbf{k}}(\beta) A_{\mathbf{ik}},\ L(\alpha) = \sum_{\mathbf{j} \in \mathbb{B}^q} \mathrm{w}_{\mathbf{j}}(\alpha) L_{\mathbf{j}},\ C(\alpha, \beta) = \sum_{\mathbf{i} \in \mathbb{B}^q} \sum_{\mathbf{k} \in \mathbb{B}^\sigma} \mathrm{w}_{\mathbf{i}}(\alpha)\mathrm{v}_{\mathbf{k}}(\beta) C_{\mathbf{ik}},$$
$$A_{\mathbf{ik}} = \begin{bmatrix} -\left(\zeta_1^{k_1} + z_1^{i_1}\right) z_2^{i_2} & 1 + \zeta_2^{k_2} \\ 0 & 0 \end{bmatrix},\ C_{\mathbf{ik}} = \begin{bmatrix} \zeta_3^{k_3} & 0 \end{bmatrix}.$$

3.3 | UIO design condition

The following result allows to design the gain of the nonlinear UIO in (7) ensuring that the equilibrium $e = 0$ of system (8) is exponentially stable.

Lemma 1 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, and matrices $\widetilde{L}_{\mathbf{j}} \in \mathbb{R}^{n \times p}$, $\mathbf{j} \in \mathbb{B}^q$, such that the following inequalities hold*

$$\sum_{(\mathbf{i},\mathbf{j}) \in \mathscr{P}(\mathbf{m},\mathbf{n})} \Upsilon_{\mathbf{ijk}} < 0, \quad \forall\, \mathbf{m}, \mathbf{n} \in \mathbb{B}^{q+},\ \mathbf{k} \in \mathbb{B}^\sigma, \tag{14}$$

with $\Upsilon_{\mathbf{ijk}} = \mathrm{He}[P A_{\mathbf{ik}} - \widetilde{L}_{\mathbf{j}} C_{\mathbf{ik}} + \eta P]$. Then, the origin of system (9) is exponentially stable with a decay rate $\eta \in \mathbb{R}_{>0}$ and the observer gain is given by

$$L(\hat{x}, y) = L(\alpha) = \sum_{\mathbf{j} \in \mathbb{B}^q} \mathrm{w}_{\mathbf{j}}(\alpha) L_{\mathbf{j}}, \quad L_{\mathbf{j}} = P^{-1}\widetilde{L}_{\mathbf{j}},\ \mathbf{j} \in \mathbb{B}^q. \tag{15}$$

Proof Assume that condition (14) is verified. Similar as in [37, 31, 26], it is possible to conclude that the parameter-dependent matrix

$$\Upsilon(\alpha, \beta) = \sum_{\mathbf{i} \in \mathbb{B}^q} \sum_{\mathbf{j} \in \mathbb{B}^q} \sum_{\mathbf{k} \in \mathbb{B}^\sigma} \mathrm{w}_{\mathbf{i}}(\alpha)\mathrm{w}_{\mathbf{j}}(\alpha)\mathrm{v}_{\mathbf{k}}(\beta)\Upsilon_{\mathbf{ijk}} \tag{16}$$

can be rewritten as

$$\Upsilon(\alpha,\beta) = \sum_{\mathbf{m}\in\mathbb{B}^{\rho+}} \sum_{\mathbf{n}\in\mathbb{B}^{\rho+}} \sum_{\mathbf{k}\in\mathbb{B}^{\sigma}} w_{\mathbf{m}}(\alpha) w_{\mathbf{n}}(\alpha) v_{\mathbf{k}}(\beta) \left(\sum_{(\mathbf{i},\mathbf{j})\in\mathscr{P}(\mathbf{m},\mathbf{n})} \Upsilon_{\mathbf{ijk}} \right).$$

Then, it is clear that condition (14) is sufficient to ensure that $\Upsilon(\alpha,\beta) < 0$. Since P is a positive definite matrix, defining $L_{\mathbf{j}} = P^{-1}\widetilde{L}_{\mathbf{j}}$, it follows that

$$\mathrm{He}[P\,(A(\alpha) - L(\alpha)C(\alpha,\beta))] + 2\eta P < 0, \tag{17}$$

where $L(\alpha)$ is defined in (8). Based on the error dynamics (9), and given that $P > 0$, define the following Lyapunov candidate function

$$V(e) = e^{\top} P e. \tag{18}$$

Pre- and post-multiplying inequality (17) with $e^{\top}$ and e, it implies that

$$\dot{V}(e) < -2\eta V(e). \tag{19}$$

By the comparison lemma, it follows from (19) that $V(e) \leq V(e_0)e^{-2\eta(t-t_0)}$, thus

$$\|e\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\|e_0\|e^{-\eta(t-t_0)},$$

with $e(t_0) = e_0$. Thus, $V(e)$, defined in (18), is a Lyapunov function that certifies the exponential stability of the zero equilibrium of system (9) and, consequently, of system (8). ■

Although Lemma 1 provides conditions to guarantee the exponential stability of the origin of system (9), it is a valid representation for the error dynamics (8) only if the state x and its estimation $\hat{x}$ remains confined in $\mathscr{D}$. Otherwise, the UIO (7) is not valid for the nonlinear system (1). The following theorem provides conditions to ensure that all the trajectories of $\hat{x}(t)$ are confined in $\mathscr{D}$.

Theorem 1 *Let the nonlinear system* (1) *be subject to Assumptions 1, 2, and 3. If the conditions of Lemma 1 and*

$$\begin{bmatrix} 1 & s_j^{\top} \\ s_j & P \end{bmatrix} \geq 0, \quad \forall\, j \in \mathbb{N}_{\leq n_f}, \tag{20}$$

with

$$s_j = \frac{1}{b_j - \phi_{\mathscr{R}}(a_j)}\, a_j, \quad \forall j \in \mathbb{N}_{\leq n_f}, \tag{21}$$

are satisfied, then the origin of the error dynamics (8) *is exponentially stable with a decay rate* $\eta \in \mathbb{R}_{>0}$*, and the trajectories*

of $\hat{x}(t)$ are confined in $\mathscr{D}$ for all $t \geq t_0$ and $e(t_0) \in \mathscr{E}$, where

$$\mathscr{E} = \{e \in \mathbb{R}^n : e^\top P e \leq 1\}, \tag{22}$$

which ensures the nonlinear UIO dynamics in (7) *is valid with the observer gain given by* (15)*.*

Proof First, consider the Pontryagin difference between the sets $\mathscr{D}$ and $\mathscr{R}$ given by [38, Th. 2.3]

$$\mathscr{D} \ominus \mathscr{R} = \{x \in \mathbb{R}^n : a_j^\top x \leq b_j - \phi_{\mathscr{R}}(a_j),\ j \in \mathbb{N}_{\leq n_f},\ a_j \in \mathbb{R}^n,\ b_j \in \mathbb{R}\}, \tag{23}$$

which, for s_j defined as in (21), is equivalent to

$$\mathscr{D} \ominus \mathscr{R} = \{x \in \mathbb{R}^n : s_j^\top x \leq 1,\ j \in \mathbb{N}_{\leq n_f},\ s_j \in \mathbb{R}^n\}. \tag{24}$$

If condition (20) is satisfied, then

$$\begin{bmatrix} -1 \\ e \end{bmatrix}^\top \begin{bmatrix} 1 & s_j^\top \\ s_j & P \end{bmatrix} \begin{bmatrix} -1 \\ e \end{bmatrix} \geq 0,\ j \in \mathbb{N}_{\leq n_f}, \tag{25}$$

which implies that

$$1 - 2s_j^\top e + e^\top P e \geq 0,\ j \in \mathbb{N}_{\leq n_f}.$$

Thus, if condition (20) holds, it guarantees the inclusion $\mathscr{E} \subset \mathscr{D} \ominus \mathscr{R}$, for $\mathscr{D} \ominus \mathscr{R}$ given by (24). Moreover, as proved in Lemma 1, the ellipsoid $\mathscr{E}$ is positively invariant and ensures the exponential convergence of the error dynamics with a decay rate of η since it is a level set of the Lyapunov function $V(e) = e^\top P e$ and (14) guarantees $\dot{V}(e) < -2\eta V(e)$. Therefore, for all $e(t_0) \in \mathscr{E}$, the trajectories of the error will stay confined in $\mathscr{E}$.

Finally, since $\hat{x} = x + e$, with $e \in \mathscr{E}$ and $x \in \mathscr{R} \subset \mathscr{D}$ for $t \geq t_0$, then $\hat{x} \in \mathscr{R} \oplus \mathscr{E} \subset \mathscr{D}$, which ensures the validity, for all $t \geq t_0$, of the nonlinear UIO given in (7) with observer gain in (15). This concludes the proof. ■

Remark 7 Lemma 1 and Theorem 1 provide sufficient conditions to design the proposed nonlinear UIO. If the LMI constraints are not feasible, it is not possible to conclude anything about the system properties. However, notice that it has already been discussed in [39] that similar LMI-based design conditions for observers cannot be feasible if the usual rank-based detectability conditions are not met. It means that the detectability is an underlying condition for the existence of the nonlinear UIO which satisfies the inequalities in Lemma 1 and Theorem 1.

3.4 | Enlargement of the region of admissible initial error

Theorem 1 guarantees that the nonlinear UIO (7) is valid for the nonlinear system (1) and its error is exponentially stable if the initial error $e(0)$ belongs to the admissible initial error set $\mathscr{E} \subset \mathscr{D} \ominus \mathscr{R}$. For this reason, it is worthy enlarging the set $\mathscr{E}$ which ensures the UIO validity and convergence. The following convex optimization problem with

LMI constraints enlarges the ellipsoid $\mathscr{E}$ within the convex polytope $\mathscr{D} \ominus \mathscr{R}$.

$$\begin{aligned} \min_{P,\, \widetilde{L}_{\mathbf{j}}, \forall \mathbf{j} \in \mathbb{B}^q} \quad & \text{trace}(P) \\ \text{s.t.} \quad & P > 0,\ (14),\ (20). \end{aligned} \tag{26}$$

It is clear that minimizing the trace of P enlarges the set $\mathscr{E}$, while the constraints $P > 0$ and (14) ensure that the $\mathscr{E}$ is a positively invariant set, finally, the constraint (20) ensures the inclusion $\mathscr{E} \subset \mathscr{D} \ominus \mathscr{R}$. The numerical results of this paper are obtained by solving[1] the optimization problem (26) for obtaining the UIO gains $L_{\mathbf{j}} = P^{-1}\widetilde{L}_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{B}^q$ and an enlarged set of admissible initial error $\mathscr{E}$.

Notice that enlarging the set of admissible initial error $\mathscr{E}$ is important to guarantee a reasonable estimation solution. For this purpose, the set difference $\mathscr{D} \ominus \mathscr{R}$ should be significant, i.e., the domain of validity must be sufficiently larger than the set $\mathscr{R}$ itself (recall that $\mathscr{R}$ contains the system trajectories). But it can be difficult for some applications where the set $\mathscr{D} \ominus \mathscr{R}$ cannot enlarge enough. In this context, some solutions, as the one proposed by [40], can be adopted to redesign the observer to ensure that $\hat{x} \in \mathscr{D}$. That kind of alternative will be investigated in further researches.

4 | NUMERICAL EXAMPLES

In this section, numerical simulations are performed to illustrate the effectiveness of the proposed UIO design methodology to estimate the UIs and the states of different classes of nonlinear systems.

4.1 | Example 1 (continued)

In this example, the UIO is designed for the nonlinear system (10). By assuming that $\mathscr{D} = \{x \in \mathbb{R}^2 : |x_i| \leq 2\}$ and $\mathscr{R} = \{x \in \mathbb{R}^2 : |x_i| \leq 1\}$, then $\mathscr{D} \ominus \mathscr{R} = \{x \in \mathbb{R}^2 : |x_i| \leq 1\}$. Based on (11), a polytopic representation for the error estimation dynamics in (12) can be obtained by selecting the available and non-available nonlinear functions as

$$\begin{aligned} &z_1(\alpha) = x_1^2 \in [0, 1],\ z_2(\alpha) = x_1 \in [-1, 1], \\ &z_3(\alpha) = \hat{x}_2 \in [-2, 2],\ \zeta_1 = x_2 \in [-1, 1]. \end{aligned}$$

Then, the estimation error dynamics can be rewritten as in (9) with the following matrix-valued functions:

$$A(\alpha, \beta) = \begin{bmatrix} 0 & 1 + z_1(\alpha) \\ 0 & -2z_2(\alpha)(\zeta_1(\beta) + z_3(\alpha)) \end{bmatrix}, \quad C(\alpha, \beta) = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

[1] The optimization problem is solved by using the YALMIP parser and the MOSEK solver in the MATLAB environment.

which can be written as (12) following the procedure in Section 3.2. For $\eta = 1$, the optimization problem (26) in Section 3.4 leads to the following solution:

$$L_{000} = L_{011} = \begin{bmatrix} 18.0293 \\ 236.5305 \end{bmatrix}, \quad L_{001} = L_{010} = \begin{bmatrix} 9.2654 \\ 107.3512 \end{bmatrix},$$
$$L_{100} = L_{111} = \begin{bmatrix} 32.0396 \\ 433.9209 \end{bmatrix}, \quad L_{101} = L_{110} = \begin{bmatrix} 24.0144 \\ 320.6193 \end{bmatrix},$$
$$P = \begin{bmatrix} 196.0051 & -13.9303 \\ -13.9303 & 1.9900 \end{bmatrix}.$$

Consider the initial conditions $x(0) = [0.0961 \;\; 0.8850]^\top$ and $\hat{x}(0) = 0$, such that the initial estimation error is $e(0) = x(0) \in \mathscr{E}$. Figure 2(a) shows the simulation results obtained with the nonlinear system (10), the proposed UIO and the unknown input given by

$$d = \begin{cases} \sin\left(\pi t + \frac{\pi}{6}\right), & t \leq 15 \text{ s}, \\ 0.3, & t > 15 \text{ s}. \end{cases} \tag{27}$$

The time-series of the states, UI and their estimates are shown in Figure 2(a), and they indicate the ability of the proposed UIO to simultaneously estimate the states and the UI . As shown in Figure 2(b), the error estimates of the states and the UI converge to zero after few seconds and the error remains negligible even after the discontinuous change in the UI at $t = 15$ s.

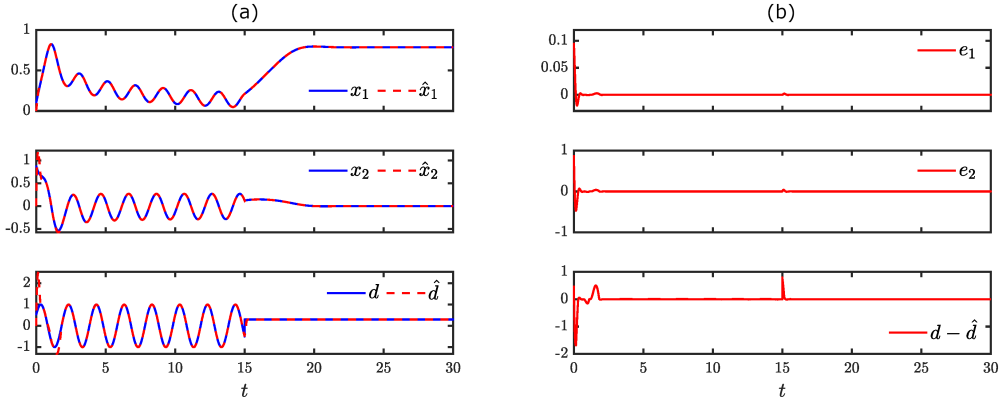

FIGURE 2 Time-series of the numerical simulations for the system (10) with the proposed UIO: (a) system states x_1 and x_2, the unknown input d, and their estimates $\hat{x}_1$, $\hat{x}_2$ and $\hat{d}$; and (b) the estimation errors of the states e_1 and e_2, and of the unknown input $d - \hat{d}$.

Remark 8 In real world applications it is usual to deal with noisy measurements, which pose an additional challenge to the decoupling and estimation of states and disturbances. To illustrate the sensitivity of the proposed approach with respect to measurement noises, the system (10) in Example 1 is considered here under the same conditions of those considered in Figure 2, but adding a measurement noise at the measurement channel (y).

Figure 3 depicts the results of the simulation. One can notice that the states estimations $\hat{x}$ have converged to the correct values of x with bounded estimation errors. The disturbance estimation exhibits some significant bounded error, which becomes negligible after $t = 15$ s. Those errors are not unlikely, since the strategy proposed in this paper is not designed to deal with measurement noises and disturbances at the output channel. However, it can be further adapted to handle this problem by using, for example, the robust filtering differentiators presented in [41] for estimating $\mathcal{Y}$ under noisy conditions. Moreover, since those output disturbances are bounded, uniformly ultimate boundedness (UUB) conditions can be derived to circumvent their effect on the UIO estimations.

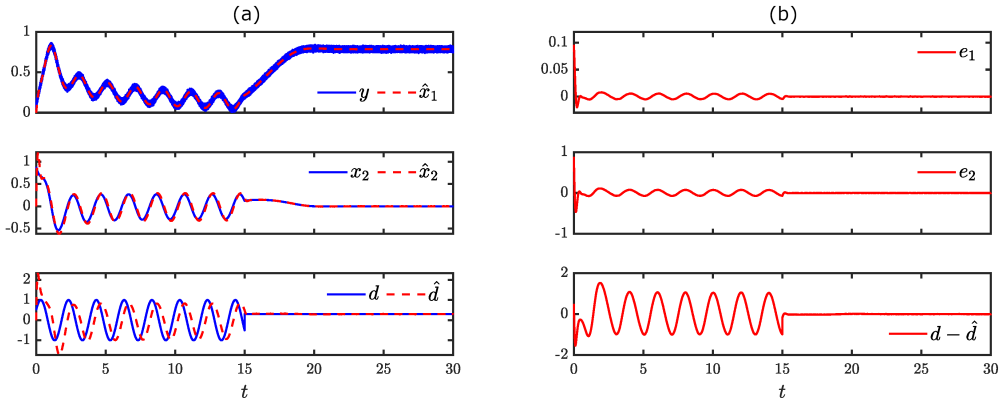

FIGURE 3 Time-series of the numerical simulations for the system (10) with the proposed UIO under sensor noise : (a) output y, state x_2, the unknown input d, and their estimates $\hat{x}_1$, $\hat{x}_2$ and $\hat{d}$; and (b) the estimation errors of the states e_1 and e_2, and of the unknown input $d - \hat{d}$.

4.2 | Nonlinear systems in normal form

Consider the following class of nonlinear systems:

$$\begin{aligned} \dot{x} &= A_\rho x + B_\rho(\psi(x) + \gamma(x)d) \\ y &= C_\rho x \end{aligned} \tag{28}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $\gamma : \mathbb{R}^n \to \mathbb{R}$ is a continuous and nonsingular function, and

$$A_\rho = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \; B_\rho = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C_\rho = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{29}$$

Note that this system has an uniform UIRD of $\rho = n$ on $\mathbb{R}^n$, which leads to $\Psi(x) = L_f^n h(x) = \psi(x)$, $\Gamma(x) = L_g L_f^{n-1} h(x) = \gamma(x)$, where $\Gamma(x)$ is nonsingular for all $x \in \mathbb{R}^n$ and $\text{rank}(\Gamma(x)) = \text{rank}(g(x)) = 1$, $\forall x \in \mathbb{R}^n$. The

matrix $Q(x)$ is obtained from (6) as $Q(x) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. Thus, the error dynamics is given by

$$\dot{e} = (A_\rho - LC_\rho)e, \tag{30}$$

where the pair (A_ρ, C_ρ) is observable. Thus, a constant observer gain $L(\alpha) = L$ is necessary and sufficient to ensure that the origin of the error dynamics is globally exponentially stable. It means that the error dynamics (30) is valid for $\mathscr{D} = \mathbb{R}^n$, and the UIO (7) is globally valid as well, since its gain does not depend on the states or its estimates. Therefore, it is not necessary to compute the set of admissible initial error in this case because it is $\mathscr{E} = \mathbb{R}^n$, and the gain L of the UIO is computed by simply evaluating the feasibility of the LMIs described in (14) for some symmetric positive-definite matrix P.

To exemplify the case of nonlinear systems in normal form, consider the Van der Pol oscillator system described as follows and subject to an unknown input d

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -x_1 + \mu(1 - x_1^2)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \\ y &= x_1, \end{aligned} \tag{31}$$

where $\mu = 2$. Note that this system can be written in the form (28) with $\psi(x) = -x_1 + \mu(1 - x_1^2)x_2$ and $\gamma(x) = 1$. In this example, as a constant observer gain is necessary to ensure the global exponential stability of the error dynamics equilibrium, only the LMI condition in (14) is solved with $\eta = 2$, which leads to

$$L = \begin{bmatrix} 9.3684 \\ 30.888 \end{bmatrix}, \quad P = \begin{bmatrix} 1.7032 & -0.3949 \\ -0.3949 & 0.1238 \end{bmatrix}. \tag{32}$$

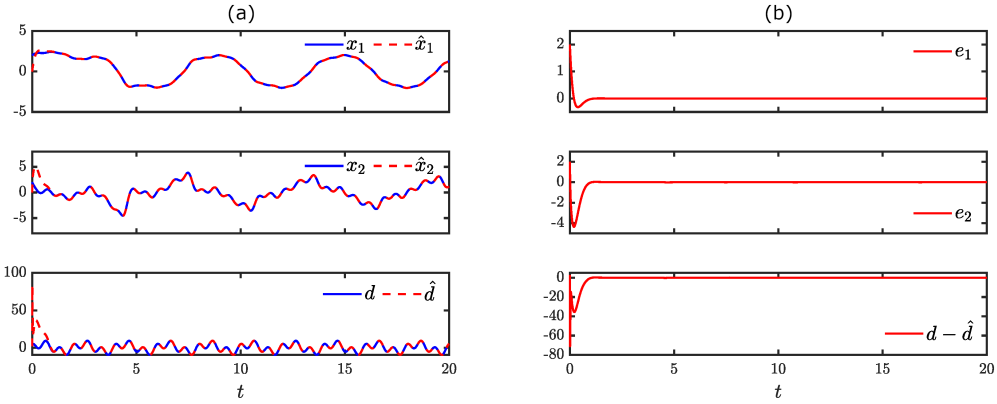

FIGURE 4 Time-series of the numerical simulations for the system (31) with the proposed UIO: (a) the system states x_1 and x_2, the unknown input d, and their estimates $\hat{x}_1$, $\hat{x}_2$ and $\hat{d}$; (b) the estimation errors of the states e_1 and e_2, and of the unknown input $d - \hat{d}$.

For the UI given by $d = 5\sin(\pi t) + 5\sin\left(3\pi t + \frac{\pi}{2}\right)$ and the initial conditions $x(0) = [2 \;\; 2]^\top$, $\hat{x}(0) = [0 \;\; 0]^\top$, the simulation results with the system's states, the UI, and their respective estimations are depicted in Figure 4(a) and the

related estimation errors are depicted in Figure 4(b). We can observe the occurrence of peaking in the UI estimate due to the use of Levant's differentiators to provide the second-order derivative of the output. However, the estimation errors converges exponentially to zero as shown in Figure 4(b).

The state trajectory and its estimate provided by the nonlinear UIO in the state-space are shown in Figure 5. Notice that, after few seconds, the state estimate trajectory starting at the origin converges to the correct state trajectory of the Van der Pol system subject to the unknown input.

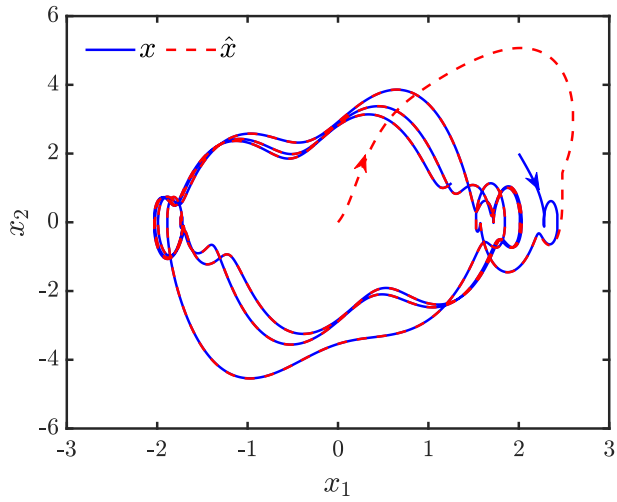

FIGURE 5 Trajectories of state of the Van der Pol system (31) subject to the unknown input d and the respective state estimate provided by the UIO (7).

4.3 | The case of non-invertible $\Gamma(x)$

Consider the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2\theta(x_1) - x_4 \\ x_4 \\ -x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \theta(x_1) & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{33}$$

$$y_1 = x_1,\ y_2 = x_3,$$

where $\mathscr{D} = \{x \in \mathbb{R}^4 : |x_i| \leq 4\pi, i \in \mathbb{N}_{\leq 4}\}$, $\mathscr{R} = \{x \in \mathbb{R}^4 : |x_i| \leq 2\pi, i \in \mathbb{N}_{\leq 4}\}$, and $\theta(x_1) = 3 + 2\sin(x_1)$. It is clear that the system has UIRD $\rho = \{\rho_1, \rho_2\} = \{2, 1\}$. According to Assumption 1, the matrix $\Gamma(x)$ is given by

$$\Gamma(x) = \begin{bmatrix} \theta(x_1) & 0 \\ 1 & 0 \end{bmatrix}. \tag{34}$$

Although (33) is algebraically observable, $\Gamma(x)$ does not admit pseudo-inverse. Such a phenomenon is already reported in [19], which indicates that the output y_2 can be replaced by a virtual output $\tilde{y}_2$ that is a function of the system output and its derivatives in order to ensure that the system has uniform UIRD $\rho = \{\rho_1, \rho_2\} = \{2, 1\}$ on $\mathscr{D}$ and a new matrix $\tilde{\Gamma}(x)$ which admits pseudo-inverse can be obtained. For instance, the virtual output $\tilde{y}_2$ can be defined as

$$\tilde{y}_2 = \ddot{y}_1 + \theta(x_1)(\dot{y}_1 - \dot{y}_2), \tag{35}$$

which leads to $\mathcal{Y} = (\ddot{y}_1, \dot{\tilde{y}}_2)$. Therefore, substituting y_2 by $\tilde{y}_2$, the new matrices $\tilde{\Psi}(x)$ and $\tilde{\Gamma}(x)$ can be computed as

$$\tilde{\Psi}(x) = \begin{bmatrix} -x_2\theta(x_1) - x_4 \\ (1+\theta(x_1))x_3 - 2\cos(x_1)x_2x_4 \end{bmatrix},$$
$$\tilde{\Gamma}(x) = \begin{bmatrix} \theta(x_1) & 0 \\ 0 & -(1+\theta(x_1)) \end{bmatrix}.$$

In this case, the matrix $\tilde{\Gamma}(x)$ is non-singular for all $x \in \mathscr{D}$. Then, according to (6), the following matrix $Q(x)$ is obtained:

$$Q(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \frac{1}{\theta(x_1)} & 0 \\ 0 & -\frac{1}{1+\theta(x_1)} \end{bmatrix}.$$

Thus, the error dynamics in (9) is written with the following matrix-valued functions:

$$A(\alpha, \beta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 + \frac{1}{\theta(x_1)} \\ 0 & \frac{2\cos(x_1)\hat{x}_4}{1+\theta(x_1)} & 0 & \frac{2\cos(x_1)x_2}{1+\theta(x_1)} \end{bmatrix}, \tag{36}$$

$$C(\alpha, \beta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{37}$$

Selecting the nonlinear functions as

$$z_1(\alpha) = \frac{1}{\theta(x_1)} \in [0.2, 1],\ z_2(\alpha) = \frac{1}{1+\theta(x_1)} \in [0.1667, 0.5],$$
$$z_3(\alpha) = \cos(x_1) \in [-1, 1],\ z_4(\alpha) = \hat{x}_4 \in [-4\pi, 4\pi],$$
$$\zeta_1(\beta) = x_2 \in [-2\pi, 2\pi],$$

the quasi-LPV representation (12) can be then constructed. For the decaying rate $\eta = 1$, the optimization problem in Theorem 1 is feasible and the observer gains $L_{\mathbf{j}}, \mathbf{j} \in \mathbb{B}^q$, and the matrix P are obtained.

The simulation is performed considering the initial conditions $x(0) = \begin{bmatrix} 0.5 & 0.88 & -0.25 & 1 \end{bmatrix}^\top$ and $\hat{x}(0) = 0$, which ensures that $e(0) = x(0) \in \mathscr{R} \subset \mathscr{D}_e$. The state trajectories and their estimations are depicted in Figure 6(a) while the considered UIs that drives the state trajectories and the obtained estimations are depicted in Figure 7(a). It can be verified that the UIO with the virtual output $\tilde{y}_2$ can effectively provide the estimations of the states and the UIs,

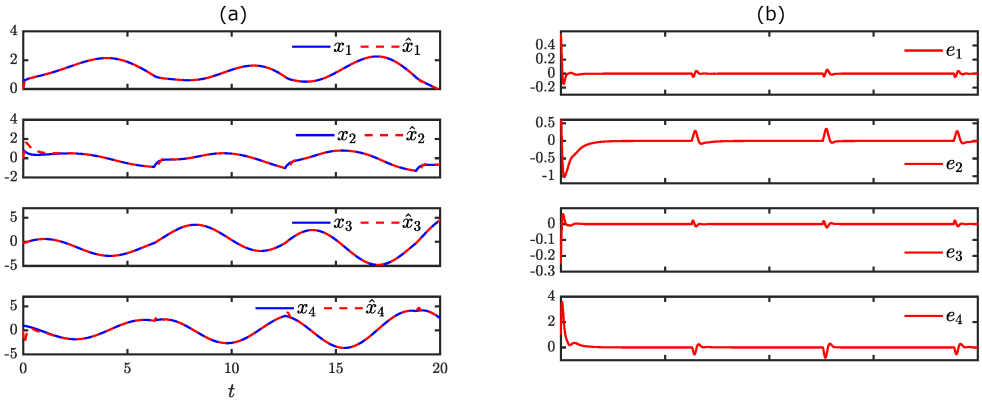

FIGURE 6 Time-series of numerical simulations for the system (33) with the proposed nonlinear UIO: (a) the states of the system and their estimates; and (b) the estimation errors.

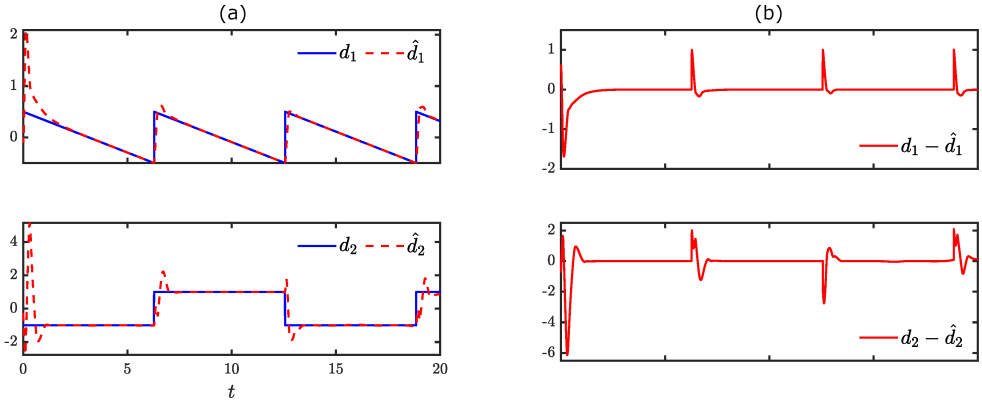

FIGURE 7 Time-series of numerical simulations for the system (33): (a) the UIs applied to system (33) and their estimates; and (b) the UIs estimation errors.

as confirmed with the convergence of the state estimation errors shown in Figure 6(b) and the UI estimation errors in Figure 7(b). It can be observed the existence of small peaks when the discontinuities of the UIs occur, however in few seconds the estimation errors converge exponentially to zero again.

5 | CONCLUSIONS

This paper presented a novel condition to design a nonlinear UIO for nonlinear systems with an arbitrary UIRD. Based on an effective factorization procedure for representing the nonlinear error dynamics by means of a polytopic quasi-LPV model, the UIO design condition has been proposed to guarantee the exponential convergence of the estimates of the states and UIs. Numerical examples have demonstrated the effectiveness of the new approach with different classes of nonlinear systems. Future works focus on fault detection and fault-tolerant control of nonlinear systems

using the proposed UIO, and on adopting strategies to ensure $\hat{x} \in \mathscr{D}$ without constraining the error dynamics [40].

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] Kazemi Z, Safavi A, Naseri F, Urbas L, Setoodeh P. A secure hybrid dynamic-state estimation approach for power systems under false data injection attacks. IEEE Transactions on Industrial Informatics 2020;6(12):275–286.

[2] Bernardi E, Adam E. Observer-based fault detection and diagnosis strategy for industrial processes. Journal of the Franklin Institute 2020;357(14):10054–10081.

[3] Rabaoui B, Hamdi H, Braiek NB, Rodrigues M. Descriptor observer-based sensor and actuator fault tolerant tracking control design for linear parameter varying systems. International Journal of Robust and Nonlinear Control 2020;31(17):8329–8352.

[4] Ichalal D, Mammar S. Asymptotic unknown input decoupling observer for discrete-time LTI systems. IEEE Control Systems Letters 2020;4(2):361–366.

[5] Ichalal D, Mammar S. On unknown input observers of linear systems: Asymptotic unknown input decoupling approach. IEEE Transactions on Automatic Control 2020;65(3):1197–1202.

[6] Hosseini I, Khayatian A, Karimaghaee P, Fiacchini M, Davo Navarro M. LMI-based reset unknown input observer for state estimation of linear uncertain systems. IET Control Theory & Applications 2019;13(12):1872–1881.

[7] Hosseini I, Fiacchini M, Karimaghaee P, Khayatian A. Optimal reset unknown input observer design for fault and state estimation in a class of nonlinear uncertain systems. Journal of the Franklin Institute 2020;357(5):2978–2996.

[8] Nguyen AT, Pan J, Guerra TM, Wang Z. Avoiding unmeasured premise variables in designing unknown input observers for Takagi–Sugeno fuzzy systems. IEEE Control Systems Letters 2020;5(1):79–84.

[9] Nguyen AT, Guerra TM, Sentouh C, Zhang H. Unknown input observers for simultaneous estimation of vehicle dynamics and driver torque: Theoretical design and hardware experiments. IEEE/ASME Transactions on Mechatronics 2019;24(6):2508–2518.

[10] Marx B, Ichalal D, Ragot J, Maquin D, Mammar S. Unknown input observer for LPV systems. Automatica 2019;100(9):67–74.

[11] Xu F, Tan J, Wang Y, Wang X, Liang B, Yuan B. Robust fault detection and set-theoretic UIO for discrete-time LPV systems with state and output equations scheduled by inexact scheduling variables. IEEE Transactions on Automatic Control 2019;64(12):4982–4997.

[12] Xu F, Tan J, Wang Y, Puig V, Wang X. Combining set-theoretic UIO and invariant sets for optimal guaranteed robust fault detection and isolation. Journal of Process Control 2019;78:155–169.

[13] Tuan H, Apkarian P, Narikiyo T, Yamamoto Y. Parameterized linear matrix inequality techniques in fuzzy control system design. IEEE Transactions on Fuzzy Systems 2001;9(2):324–332.

[14] Wang X, Tan C, Liu L, Qi Q. A novel unknown input interval observer for systems not satisfying relative degree condition. International Journal of Robust and Nonlinear Control 2021;31(7):2762–2782.

[15] Xu F, Tan J, Wang X, Liang B. Conservatism comparison of set-based robust fault detection methods: Set-theoretic UIO and interval observer cases. Automatica 2019;105:307–313.

[16] Chakrabarty A, Corless MJ, Buzzard GT, Zak SH, Rundell AE. State and unknown input observers for nonlinear systems with bounded exogenous inputs. IEEE Transactions on Automatic Control 2017;62(11):5497–5510.

[17] Tavasolipour E, Poshtan J, Shamaghdari S. A new approach for robust fault estimation in nonlinear systems with state-coupled disturbances using dissipativity theory. ISA Transactions 2021;114:31–43.

[18] Chakrabarty A, Ayoub R, Żak S, Sundaram S. Delayed unknown input observers for discrete-time linear systems with guaranteed performance. Systems & Control Letters 2017;103:9–15.

[19] Ichalal D, Mammar S. On unknown input observers for LPV systems. IEEE Transactions on Industrial Electronics 2015;62(9):5870–5880.

[20] Li T. Observer design for Lipschitz nonlinear parabolic PDE systems with unknown input. IEEE Access 2020;8:956–963.

[21] Zhang W, Su H, Zhu F, Azar M. Unknown input observer design for one-sided Lipschitz nonlinear systems. Nonlinear Dynamics 2015;79(2):1469–1479.

[22] Tian J, Ma S, Zhang C. Unknown input reduced-order observer design for one-sided Lipschitz nonlinear descriptor Markovian jump systems. Asian Journal of Control 2019;21(2):952–964.

[23] Ichalal D, Guerra TM. Decoupling Unknown Input Observer for nonlinear quasi-LPV systems. In: 58th Conference on Decision and Control (CDC) IEEE; 2019. p. 3799–3804.

[24] Klug M, Castelan EB, Leite VJS, Silva LFP. Fuzzy dynamic output feedback control through nonlinear Takagi–Sugeno models. Fuzzy Sets and Systems 2015 Mar;263:92–111.

[25] Silva LFP, Leite VJS, Castelan EB, Feng G. Delay Dependent Local Stabilization Conditions for Time-delay Nonlinear Discrete-time Systems Using Takagi-Sugeno Models. International Journal of Control, Automation and Systems 2018 May;16(3):1435–1447.

[26] Coutinho PHS, Bernal M, Palhares RM. Robust sampled-data controller design for uncertain nonlinear systems via Euler discretization. International Journal of Robust and Nonlinear Control 2020;30(18):8244–8258.

[27] Nguyen AT, Coutinho PHS, Guerra TM, Palhares RM, Pan J. Constrained output-feedback control for discrete-time fuzzy systems with local nonlinear models subject to state and input constraints. IEEE Transactions on Cybernetics 2020;51(9):4673–4684.

[28] López E, Botero H. State estimation and reconstruction of unknown inputs with arbitrary relative degree via a predefined-time algebraic solver. Journal of the Franklin Institute 2020;357(13):9083–9106.

[29] Kalsi K, Lian J, Hui S, Żak S. Sliding-mode observers for systems with unknown inputs: A high-gain approach. Automatica 2010;46(2):347–353.

[30] Nguyen AT, Campos V, Guerra TM, Pan J, Xie W. Takagi–Sugeno fuzzy observer design for nonlinear descriptor systems with unmeasured premise variables and unknown inputs. International Journal of Robust and Nonlinear Control 2021;31(7):8353–8372.

[31] Quintana D, Estrada-Manzo V, Bernal M. An exact handling of the gradient for overcoming persistent problems in nonlinear observer design via convex optimization techniques. Fuzzy Sets and Systems 2021;416:125–140.

[32] Li S, Yang J, Chen WH, Chen X. Disturbance Observer-Based Control: Methods and Applications. CRC Press; 2016.

[33] Levant A. Higher-order sliding modes, differentiation and output-feedback control. International Journal of Control 2003;76(9-10):924–941.

[34] Fridman L, Levant A, Davila J. Observation of linear systems with unknown inputs via high-order sliding-modes. International Journal of Systems Science 2007;38(10):773–791.

[35] Gonzalez T, Bernal M, Sala A, Aguiar B. Cancellation-based nonquadratic controller design for nonlinear systems via Takagi–Sugeno models. IEEE Transactions on Cybernetics 2016;47(9):2628–2638.

[36] Blanchini F, Miani S. Set-Theoretic Methods in Control. Springer International Publishing; 2015.

[37] Sala A, Ariño C. Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. Fuzzy Sets and Systems 2007;158(24):2671–2686.

[38] Kolmanovsky I, Gilbert EG. Theory and computation of disturbance invariant sets for discrete-time linear systems. Mathematical Problems in Engineering 1998;4(4):317–367.

[39] Corless M, Tu J. State and Input Estimation for a Class of Uncertain Systems. Automatica 1998 Jun;34(6):757–764.

[40] Astolfi D, Bernard P, Postoyan R, Marconi L. Constrained State Estimation for Nonlinear Systems: A Redesign Approach Based on Convexity. IEEE Transactions on Automatic Control 2022 Feb;67(2):824–839.

[41] Levant A, Livne M. Robust exact filtering differentiators. European Journal of Control 2020;55:33–44.