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Tran Anh-Tu Nguyen, Juntao Pan, Thierry-Marie Guerra, Zhenhua Wang

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Avoiding Unmeasured Premise Variables in Designing Unknown Input Observers for Takagi-Sugeno Fuzzy Systems

Anh-Tu Nguyen*, *Member, IEEE*, Juntao Pan, Thierry-Marie Guerra, and Zhenhua Wang

***Abstract*—This paper investigates the design of unknown input (UI) observers for a large class of nonlinear systems using Takagi-Sugeno (TS) fuzzy modeling. To avoid the well-known issue on the *unmeasured* premise variables in fuzzy observer design, we reformulate the nonlinear systems in a TS fuzzy form with local nonlinear models. A particular feature of these so-called N-TS fuzzy models is that all the unmeasured nonlinearities are *isolated* in a nonlinear consequent. Together with a judicious use of the differential mean value theorem, the N-TS fuzzy reformulation enables an effective framework to design fuzzy UI observers. Based on an UI decoupling technique, no specific information on the UI is required for fuzzy observer design. The asymptotic estimations of both the state and the UI are guaranteed with fuzzy Lyapunov arguments. The observer gains can be effectively computed following an LMI-based design procedure. Numerical illustrations are given to demonstrate the interests of the proposed method over related existing results.**

***Index Terms*—Takagi-Sugeno models, fuzzy observers, unmeasured premise variables, unknown inputs, Lyapunov method.**

I. Introduction

STATE estimation of nonlinear systems is a basic concern in control theory and applications. Since the pionner work on linear time-invariant systems [1], numerous extensions have been studied for nonlinear cases, see [2] for an overview. Moreover, unmodeled dynamics, uncertain disturbances and faults in engineering systems, or attack signals in secure communication or cyber-physical systems can be all considered as unknown inputs (UIs) [3]. Hence, observer design for nonlinear systems affected by UIs has played a key role for fault diagnosis [4]–[7], robust and fault-tolerant control [8], [9]. Motivated by these issues, we are interested in designing UI observers for nonlinear systems, described as

$$\begin{aligned} x_{k+1} &= \Psi(x_k, u_k) + Dd_k, \\ y_k &= Cx_k, \end{aligned} \tag{1}$$

where $x_k \in \mathscr{D}_x \subseteq \mathbb{R}^{n_x}$ is the system state, $u_k \in \mathscr{D}_u \subseteq \mathbb{R}^{n_u}$ is the *known* input, $d_k \in \mathbb{R}^{n_d}$ is the *unknown* input, $y_k \in \mathbb{R}^{n_y}$ is the system output. The nonlinear function $\Psi : \mathscr{D}_x \times \mathscr{D}_u \to \mathbb{R}^{n_x}$ is differentiable with respect to the state x_k. The matrices of appropriate dimensions C and D are constant. The problem is to design a dynamical observer, generating a state estimate $\hat{x}_k$ and an UI estimate $\hat{d}_k$ of system (1), such that

$$\lim_{k\to\infty} |x_k - \hat{x}_k| = 0, \quad \lim_{k\to\infty} |d_k - \hat{d}_k| = 0.$$

Takagi-Sugeno (TS) fuzzy model-based technique [10] has been shown as a promising framework to deal with this problem [5], [8], [11]. The major interests of TS fuzzy approaches come from two main factors. First, TS fuzzy modeling can be used to approximate any smooth nonlinear system with any preciseness [12]. In particular, the sector nonlinearity approach [10, Ch. 2] offers a systematic way to derive an *equivalent* TS fuzzy representation of the original nonlinear system in a compact set of the state space. Second, based on Lyapunov method, design conditions for fuzzy observers/controllers are derived in the form of linear matrix inequalities (LMIs), which are efficiently solved via convex optimization techniques, see for instance [13]–[16]. To design a fuzzy observer for the nonlinear system (1), existing approaches have been mainly based on the following classical TS fuzzy representation:

$$x_{k+1} = \sum_{i=1}^{n} \eta_i(z_k) \left(A_i x_k + B_i u_k + D d_k\right), \tag{2}$$

where the matrices of appropriate dimensions (A_i, B_i) are constant, n is the number of fuzzy rules, z_k is the vector of premise variables. The membership functions (MFs) are such that $\eta_i(z_k) \geq 0$, $\sum_{i=1}^{n} \eta_i(z_k) = 1$.

The existing TS fuzzy approaches to design fuzzy UI observers usually require some *a priori* information on the UIs. First, assuming that the UIs are of polynomial forms [17], fuzzy proportional-integral observers [18] are only suitable to estimate low-frequency UI signal. Second, the forms of the UIs are assumed to be known and generated by an exogenous system [19]. Third, norm-bound conditions on the UI and/or its time derivatives are also required in [8], [20]. However, such *a priori* information on the UIs is not always available for many practical situations. To avoid this drawback, UI decoupling approaches have been proposed [7], [11], [21]. However, the system output in [11] must be of the form $y_k = Cx_k + Fd_k$, with a full-row rank matrix F, leading to a restrictive design [5]. Moreover, the approaches in [7], [21] can be only applied to TS fuzzy systems with measurable premise variables.

Two cases are distinguished for fuzzy observer design depending on the real-time availability of vector z_k [22]. For

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A.-T. Nguyen and T.-M. Guerra are with the laboratory LAMIH UMR CNRS 8201, Université Polytechnique Hauts-de-France, France (email: nguyen.trananhtu@gmail.com; thierry.guerra@uphf.fr).

J. Pan is with the School of Electrical and Information Engineering, North Minzu University, China (email: jtpan@aliyun.com).

Z. Wang is with the School of Astronautics, Harbin Institute of Technology, China (email: zhenhua.wang@hit.edu.cn).

*Corresponding author: Anh-Tu Nguyen.

the first case, z_k can be fully measured from the output y_k. Then, the MFs $\eta_i(z_k)$, for $i = 1, 2, \ldots, n$, are directly used to construct the fuzzy observers. The main goal here is to reduce the design conservativeness via various matrix transformations and fuzzy Lyapunov functions [3], [15], [23]. However, the design approaches in this case can be only applied to a restrictive class of nonlinear systems. To overcome this major drawback, most of research efforts have been devoted to the second case, much more challenging, for which the premise variables z_k are unmeasured or only partially measured. The technical difficulty consists in dealing with a *mismatch* nonlinear term, function of $\eta_i(z_k) - \eta_i(\hat{z}_k)$, where $\hat{z}_k$ is an estimate of z_k. Up to now, the most common approach is based on the Lipschitz property of the MFs [4], [11], [22]. Despite its simplicity, this approach generally leads to over-conservative results [24]. Differential mean value theorem (DMVT) has been exploited in [25] to handle the unmeasurable MFs. However, the norm-bounded uncertainty approach used to deal with the unknown time-varying terms issued from the application of DMVT may lead to a complex and conservative design framework.

It is important to note that existing results on TS fuzzy observer design are mostly concerned with *known* inputs, see [24], [25] and related works. Moreover, fuzzy UI observers and fuzzy fault detection/diagnosis have been mainly proposed for TS systems with measurable premise variables [3], [5], [8], [18], [21], [23], [26]. Some few extensions based on the conservative Lipschitz assumption to deal with the unmeasured premise variables can be found in [4], [11]. DMVT has been recently exploited in [27] to design UI observers for continuous-time systems with unmeasured nonlinearities. Unfortunately, the proposed observer structure requires the information of the time-derivative of the measured premise variables, which can be unmeasurable. In particular, asymptotic estimation convergence cannot be guaranteed in [27]. Motivated by the above practical and theoretical issues, this letter proposes a new method to design fuzzy UI observers while avoiding the issue of unmeasured premise variables. Specifically, our main contributions are as follows.

i) We reformulate system (1) as a N-TS fuzzy model, for which all the unmeasured nonlinearities in (1) are isolated in a local nonlinear consequent. With a judicious use of the differential mean value theorem, this particular feature of the resulting N-TS model enables an effective framework to design UI observers with reduced conservatism.
ii) The asymptotic estimations of both the state and the UI are guaranteed using fuzzy Lyapunov-based arguments. No specific information on the UI is required for UI observer design as in most of existing results [8], [18]–[20]. Moreover, the observer gains can be effectively computed following an LMI-based design procedure.

Notation. The set of nonnegative integers is denoted by $\mathbb{Z}_+$ and $\mathcal{I}_r = \{1, 2, \ldots, r\} \subset \mathbb{Z}_+$. For $i \in \mathcal{I}_r$, we denote $\sigma_r(i) = [0, \ldots, 0, \overbrace{1}^{i\text{th}}, 0, \ldots, 0]^\top \in \mathbb{R}^r$ a vector of the canonical basis of $\mathbb{R}^r$. For a vector x, x_i denotes its ith entry. For two vectors $x, y \in \mathbb{R}^n$, the convex hull of these vectors is denoted as $\mathrm{co}(x, y) = \{\lambda x + (1-\lambda)y : \ \lambda \in [0, 1]\}$. For a matrix X, $X^\top$ denotes its transpose, $X \succ 0$ means X is positive definite, and $\mathrm{He}X = X + X^\top$. When the existence is guaranteed, $X^\dagger$ denotes the Moore–Penrose pseudo-inverse of matrix X, *i.e.*, $X^\dagger = \left(X^\top X\right)^{-1} X^\top$. $\mathrm{diag}(X_1, X_2)$ denotes a block-diagonal matrix composed of X_1, X_2. I denotes the identity matrix of appropriate dimension. In block matrices, the symbol $\star$ stands for the terms deduced by symmetry.

II. Problem Statement

This section formulates a new framework to design UI observers for fuzzy systems with unmeasured premise variables.

A. Fuzzy System Description

For fuzzy observer design, inspired by the N-TS fuzzy modeling [28], [29], we propose to reformulate the nonlinear system (1) as

$$x_{k+1} = A(\xi_k)x_k + Dd_k + f(\xi_k, u_k) + G(\xi_k)\phi(x_k, u_k), \quad (3)$$

where $\xi_k \in \mathbb{R}^{n_\xi}$ is the vector of *measured* premise variables. The nonlinear functions $f : \mathscr{D}_x \times \mathscr{D}_u \to \mathbb{R}^{n_x}$ and $\phi : \mathscr{D}_x \times \mathscr{D}_u \to \mathbb{R}^{n_\phi}$ are differentiable with respect to the state x_k. The elements of the vector-valued function $f(\cdot)$ and the matrix-valued functions $A(\cdot)$ and $G(\cdot)$ are measurable whereas those of $\phi(\cdot)$ cannot be measured from the output. Using the fuzzy modeling technique [10, Ch. 2], system (3) can be represented by the following TS fuzzy model with nonlinear consequents:

$$\begin{aligned} &\textsc{Rule}\ \mathcal{R}_i: \ \textsc{If}\ \xi_1 \text{ is } \mathcal{M}_1^i \text{ and } \ldots \text{ and } \xi_{n_\xi} \text{ is } \mathcal{M}_{n_\xi}^i \\ &\textsc{Then}\ x_{k+1} = A_i x_k + Dd_k + f(\xi_k, u_k) + G_i\phi(x_k, u_k) \quad (4) \end{aligned}$$

where the constant matrices (A_i, G_i) are known. $\mathcal{R}_i$ denotes the ith fuzzy inference rule, and r is the number of inference rules. $\mathcal{M}_j^i$, with $i \in \mathcal{I}_r$ and $j \in \mathcal{I}_{n_\xi}$, is the fuzzy set. The fuzzy MFs are given by $h_i(\xi_k) = \frac{\prod_{j=1}^{n_\xi} \mu_j^i(\xi_{jk})}{\sum_{i=1}^{r} \prod_{j=1}^{n_\xi} \mu_j^i(\xi_{jk})}$, $i \in \mathcal{I}_r$, where $\mu_j^i(\xi_{jk})$ represents the membership grade of ξ_{jk} in the respective fuzzy set $\mathcal{M}_j^i$. Note that the MFs $h_i(\xi_k)$, for $i \in \mathcal{I}_r$, satisfy the convex sum property

$$\sum_{i=1}^{r} h_i(\xi_k) = 1, \quad 0 \le h_i(\xi_k) \le 1, \quad \forall i \in \mathcal{I}_r. \quad (5)$$

Let Ω be the set of membership functions satisfying (5), *i.e.*, $h = \left[h_1(\xi_k), h_2(\xi_k), \ldots, h_r(\xi_k)\right]^\top \in \Omega$. We denote also $h_+ = \left[h_1(\xi_{k+1}), h_2(\xi_{k+1}), \ldots, h_r(\xi_{k+1})\right]^\top \in \Omega$. Applying the center-of-gravity method for defuzzification, the fuzzy system (4) can be represented in the compact form [10]

$$x_{k+1} = A(h)x_k + Dd_k + f(\xi_k, u_k) + G(h)\phi(x_k, u_k), \quad (6)$$

where $\begin{bmatrix} A(h) & G(h) \end{bmatrix} = \sum_{i=1}^{r} h_i(\xi_k) \begin{bmatrix} A_i & G_i \end{bmatrix}$.

Remark 1. All the *unmeasurable* premise variables of system (1) are *isolated* in the nonlinear term $\phi(x, u)$ of (3). Then, the MFs $h_i(\xi_k)$, $i \in \mathcal{I}_r$, of system (6) are measurable, and can be directly used to construct fuzzy observers. This particular feature of the N-TS fuzzy modeling (3)–(6) enables an effective fuzzy UI observer design for nonlinear system (1). Moreover, through the system reformulation (3), a part

of the nonlinearities of system (1) are retained in $f(\xi_k, u_k)$ and $\phi(x_k, u_k)$. Hence, the resulting N-TS fuzzy model (6) requires fewer fuzzy rules than the classical TS fuzzy model (2), *i.e.*, $r < n$. In many situations, especially for complex nonlinear systems, this can lead to several advantages: (i) simpler observer structure, (ii) less computational burden and less conservative results for fuzzy observer design [28].

The following assumptions are considered for system (6).

Assumption 1. We assume that matrix C is of full row rank, matrix D is of full column rank and $\mathrm{rank}(CD) = \mathrm{rank}(D)$.

The rank condition in Assumption 1 is common in UI observer design framework for UI decoupling [8], [17], [21].

Assumption 2. The function $\phi(x_k, u_k)$ satisfies the condition

$$\underline{\theta}_{ij} \leq \frac{\partial \phi_i}{\partial x_j}(x, u) \leq \overline{\theta}_{ij}, \text{ for } \forall (i,j) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x}, \quad (7)$$

with

$$\underline{\theta}_{ij} = \min_{\mu \in \mathscr{D}_x \times \mathscr{D}_u} \left(\frac{\partial \phi_i}{\partial x_j}(\mu) \right), \quad \overline{\theta}_{ij} = \max_{\mu \in \mathscr{D}_x \times \mathscr{D}_u} \left(\frac{\partial \phi_i}{\partial x_j}(\mu) \right).$$

Remark 2. The boundedness condition (7) is not restrictive for fuzzy observer design since TS fuzzy models are generally defined within a compact set $\mathscr{D}_x$ [10]. Moreover, the input u_k of engineering systems is physically bounded, *i.e.*, $u_k \in \mathscr{D}_u$. Note that in many situations the bounds of u_k are not required for the proposed UI observer design, *i.e.*, when the Jacobian of $\phi(x_k, u_k)$ is *independent* to u_k. This is not the case of the recent DMVT-based TS fuzzy observer design in [25], where the bounds of u_k are always required for observer design.

B. Problem Definition

We consider the following UI observer structure:

$$z_{k+1} = N(h) z_k + L(h) y_k + M \hat{\Phi}(\xi_k, u_k, \hat{x}_k), \quad (8)$$

$$\hat{x}_k = z_k - F y_k, \quad z_0 = 0, \quad (9)$$

$$\hat{d}_k = (CD)^\dagger \left(y_{k+1} - C\hat{\Phi}(\xi_k, u_k, \hat{x}_k) - CA(h)\hat{x}_k \right), \quad (10)$$

with $\hat{\Phi}(\xi_k, u_k, \hat{x}_k) = f(\xi_k, u_k) + G(h)\phi(\hat{x}_k, u_k)$ and

$$M = I + FC. \quad (11)$$

The existence of the pseudo-inverse matrix $(CD)^\dagger$ is guaranteed by the rank condition in Assumption 1. The MF-dependent matrices $N(h) \in \mathbb{R}^{n_x \times n_x}$, $L(h) \in \mathbb{R}^{n_x \times n_y}$, $M \in \mathbb{R}^{n_x \times n_x}$ and $F \in \mathbb{R}^{n_x \times n_y}$ are to be designed such that

$$\begin{bmatrix} L(h) & N(h) \end{bmatrix} = \sum_{i=1}^{N} h_i(\xi_k) \begin{bmatrix} L_i & N_i \end{bmatrix}.$$

Let us define $e_k = \hat{x}_k - x_k$. It follows from (9) and (11) that $e_k = z_k - F y_k - x_k$, or

$$e_k = z_k - M x_k. \quad (12)$$

Then, the estimation error dynamics can be defined from (6), (8), (9) and (12) as

$$\begin{aligned} e_{k+1} &= z_{k+1} - M x_{k+1} \\ &= N(h) e_k + \mathscr{M}(h) x_k + MG(h)\Delta_\phi - MDd_k, \end{aligned} \quad (13)$$

where $\Delta_\phi = \phi(x_k, u_k) - \phi(\hat{x}_k, u_k)$ and $\mathscr{M}(h) = N(h)M + L(h)C - MA(h)$. The "mismatch" term Δ_ϕ caused by the unmeasured nonlinearities raises major challenge for fuzzy observer design [11], [25]. To effectively deal with this term and guarantee an asymptotic error convergence, the following differential mean value theorem is useful to convert Δ_ϕ into a function of the estimation error e_k.

Lemma 1. [30] Let $h(x) : \mathbb{R}^{n_x} \to \mathbb{R}^q$ and $a, b \in \mathbb{R}^{n_x}$. If $h(x)$ is differentiable on $\mathrm{co}(a, b)$, then there exist constant vectors $c_i \in \mathrm{co}(a, b)$, $c_i \neq a$, $c_i \neq b$, for $\forall i \in \mathcal{I}_q$, such that

$$h(a) - h(b) = \left(\sum_{i=1}^{q} \sum_{j=1}^{n_x} \sigma_q(i) \sigma_{n_x}^\top(j) \frac{\partial h_i}{\partial x_j}(c_i) \right) (a - b).$$

Applying Lemma 1 to the nonlinear function $\phi(x_k, u_k)$, it follows that there exist $\chi_i \in \mathrm{co}(x_k, \hat{x}_k)$, $i \in \mathcal{I}_{n_\phi}$, such that

$$\Delta_\phi = \left(\sum_{i=1}^{n_\phi} \sum_{j=1}^{n_x} \sigma_{n_\phi}(i) \sigma_{n_x}^\top(j) \frac{\partial \phi_i}{\partial x_j}(\chi_i, u_k) \right) (x_k - \hat{x}_k). \quad (14)$$

Let us denote $\rho_{ij} = \frac{\partial \phi_i}{\partial x_j}(\chi_i, u_k)$, for $\forall (i,j) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x}$, and

$$\rho = \left[\rho_{11}, \ldots, \rho_{1n_x}, \ldots, \rho_{n_\phi n_x} \right]. \quad (15)$$

Due to condition (7), the parameter ρ belongs to a bounded convex set $\mathscr{S}_\phi$, whose set of $2^{n_\phi n_x}$ vertices is given by

$$\mathscr{V}_\phi = \{ \rho \doteq \left[\rho_{11}, \ldots, \rho_{1n_x}, \ldots, \rho_{n_\phi n_x} \right] : \ \rho_{ij} \in [\underline{\theta}_{ij}, \overline{\theta}_{ij}] \}.$$

From (13) and (14), the error dynamics can be rewritten as

$$e_{k+1} = \mathscr{N}(h, \rho) e_k + \mathscr{M}(h) x_k - MDd_k, \quad (16)$$

where $\mathscr{N}(h, \rho) = \sum_{i=1}^{r} h_i(\xi_k) \mathscr{N}_i(\rho_{lj})$ and

$$\mathscr{N}_i(\rho_{lj}) = N_i + \sum_{l=1}^{n_\phi} \sum_{j=1}^{n_x} \sigma_{n_\phi}(l) \sigma_{n_x}^\top(j) \rho_{lj} M G_i. \quad (17)$$

The observer design problem of interest is stated as follows.

Problem 1. Consider a nonlinear system in the N-TS fuzzy form (6). Determine the matrices $N(h)$, $L(h)$, M and F of the UI observer (8)–(10) such that the state estimate $\hat{x}_k$ and the UI estimate $\hat{d}_k$ converge *asymptotically* to the system state x_k and the unknown input d_k, respectively.

Using Lyapunov-based arguments, this paper proposes a tractable solution for the above observer problem.

III. UI Observer Design for Fuzzy Systems with Unmeasured Premise Variables

This section develops a new framework to design UI observers for nonlinear systems in the N-TS fuzzy form (6).

A. On Fuzzy Unknown Input Observer Design

The following theorem provides the theoretical basis to achieve our UI observer design goal.

Theorem 1. Consider the N-TS fuzzy system (6), there is an asymptotic UI observer in the form (8)–(10) if there exist MF-dependent positive definite matrix $P(h) \in \mathbb{R}^{n_x \times n_x}$,

MF-dependent matrices $N(h) \in \mathbb{R}^{n_x \times n_x}$, $L(h) \in \mathbb{R}^{n_x \times n_y}$, matrices $R \in \mathbb{R}^{n_x \times n_x}$, $M \in \mathbb{R}^{n_x \times n_x}$ and $F \in \mathbb{R}^{n_x \times n_y}$ with $M = I + FC$, and the following conditions are satisfied for $h, h_+ \in \Omega$, $\rho \in \mathscr{S}_\phi$:

$$N(h)M + L(h)C - MA(h) = 0, \tag{18}$$

$$MD = 0, \tag{19}$$

$$\begin{bmatrix} P(h) & \star \\ R\mathscr{N}(h,\rho) & R + R^\top - P(h_+) \end{bmatrix} \succ 0, \tag{20}$$

with $P(h_+) = \sum_{l=1}^{r} h_l(\xi_{k+1}) P_l$.

Proof. From (18) and (19), the error dynamics (16) can be represented as

$$e_{k+1} = \mathscr{N}(h,\rho) e_k. \tag{21}$$

Pre- and postmultiplying (20) with $\begin{bmatrix} I & -\mathscr{N}(h,\rho) \end{bmatrix}^\top$, one has

$$\mathscr{N}(h,\rho)^\top P(h_+) \mathscr{N}(h,\rho) - P(h) \prec 0. \tag{22}$$

For the asymptotic stability analysis of the error dynamics (21), we consider the fuzzy Lyapunov function candidate

$$\mathcal{V}(e_k) = e_k^\top P(h) e_k, \quad P(h) = \sum_{i=1}^{r} h_i(\xi_k) P_i, \tag{23}$$

with $P_i \succ 0$, for $i \in \mathcal{I}_r$. The variation of the fuzzy Lyapunov function (23) along the trajectory of system (21) is defined as

$$\begin{aligned} \Delta\mathcal{V} &= \mathcal{V}(e_{k+1}) - \mathcal{V}(e_k) \\ &= e_k^\top \left(\mathscr{N}(h,\rho)^\top P(h_+) \mathscr{N}(h,\rho) - P(h) \right) e_k. \end{aligned} \tag{24}$$

It follows from (22) and (24) that $\Delta\mathcal{V} < 0$, for $\forall e_k \neq 0$.

Now, we prove that the UI estimate $\hat{d}_k$ in (10) asymptotically converges to d_k. To this end, note from (6) that

$$d_k = (CD)^\dagger \left(y_{k+1} - C\Phi_k - CA(h)x_k \right), \tag{25}$$

with $\Phi_k = f(\xi_k, u_k) + G(h)\phi(x_k, u_k)$. The UI estimation error $\varepsilon_k = d_k - \hat{d}_k$ can be obtained from (10) and (25) as

$$\varepsilon_k = (CD)^\dagger C \left(A(h) e_k + G(h) \Delta_\phi \right).$$

Using expression (14), the UI estimation error ε_k is rewritten as

$$\varepsilon_k = (CD)^\dagger C \mathscr{A}(h,\rho) e_k, \tag{26}$$

where $\mathscr{A}(h,\rho) = \sum_{i=1}^{r} h_i(\xi_k) \mathscr{A}_i(\rho_{lj})$ and

$$\mathscr{A}_i(\rho_{lj}) = A_i + \sum_{l=1}^{n_\phi} \sum_{j=1}^{n_x} \sigma_{n_\phi}(l) \sigma_{n_x}^\top(j) \rho_{lj} G_i. \tag{27}$$

Note from the *algebraic* equation (26) that if $e_k \to 0$, then $\varepsilon_k \to 0$, when $k \to \infty$. This concludes the proof. □

Due to the nonlinear matrix equality (18), the nonlinear matrix inequality (20), and their MF-dependency, Theorem 1 cannot be solved directly for UI observer design. Based on this result, we derive hereafter tractable conditions to design UI observer (8)–(10) for system (6). The following technical lemma is useful for the theoretical developments.

Lemma 2. [31] Consider matrix $\mathbf{W} \in \mathbb{R}^{m \times n}$ with $m \geq n$, and matrix $\mathbf{Z} \in \mathbb{R}^{k \times n}$. Matrix $\mathbf{X}$ of the form $\mathbf{X} = \mathbf{Z}\mathbf{W}^\dagger + \mathbf{Y}\left(I - \mathbf{W}\mathbf{W}^\dagger\right)$, with an arbitrary matrix $\mathbf{Y} \in \mathbb{R}^{k \times m}$, is a solution of $\mathbf{X}\mathbf{W} = \mathbf{Z}$ when the condition $\mathbf{Z}\mathbf{W}^\dagger\mathbf{W} = \mathbf{Z}$ holds.

B. LMI-Based Fuzzy Unknown Input Observer Design

The following theorem provides an LMI-based solution for the N-TS fuzzy UI observer design in Problem 1.

Theorem 2. Consider the N-TS fuzzy system (6), there is an asymptotic UI observer in the form (8)–(10) if there exist positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, matrices $R \in \mathbb{R}^{n_x \times n_x}$, $Z \in \mathbb{R}^{n_x \times n_y}$, $X_i \in \mathbb{R}^{n_x \times n_y}$, $i \in \mathcal{I}_r$, such that

$$\begin{bmatrix} P_i & \star \\ (R + (RU + ZV)C)\mathscr{A}_i(\rho_{lj}) - X_i C & R + R^\top - P_k \end{bmatrix} \succ 0, \tag{28}$$

for $i, k \in \mathcal{I}_r$ and $\rho_{lj} \in \mathscr{V}_\phi$, $l \in \mathcal{I}_{n_\phi}$, $j \in I_{n_x}$, where

$$U = -D(CD)^\dagger, \tag{29}$$

$$V = I - (CD)(CD)^\dagger, \tag{30}$$

$$Z = RY, \tag{31}$$

$$X_i = RH_i. \tag{32}$$

Furthermore, the matrices $M \in \mathbb{R}^{n_x \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, $N_i \in \mathbb{R}^{n_x \times n_x}$, and $L_i \in \mathbb{R}^{n_x \times n_y}$, for $i \in \mathcal{I}_r$, of the fuzzy UI observer (8)–(10) are computed as

$$F = U + YV, \tag{33}$$

$$N_i = MA_i - H_i C, \tag{34}$$

$$L_i = H_i(I + CF) - MA_i F, \tag{35}$$

and matrix M given by (11).

Proof. Note that condition (28) implies $R + R^\top \succ 0$, which guarantees the existence of R^{-1}. Moreover, from (11) and (19), we can obtain

$$F(CD) = -D. \tag{36}$$

Under the matrix rank condition in Assumption 1, according to Lemma 2, the solution F of the matrix equation (36) is directly obtained in (33) with U and V defined in (29) and (30), respectively. We define

$$H_i = L_i + N_i F, \quad \forall i \in \mathcal{I}_r. \tag{37}$$

From (11), (18) and (37), we can obtain (34). Note also from (34) and (37) that

$$\begin{aligned} L_i &= H_i - N_i F = H_i - (MA_i - H_i C)F \\ &= H_i(I + CF) - MA_i F, \quad \forall i \in \mathcal{I}_r, \end{aligned}$$

which is exactly expression (35). Moreover, it follows from (11) and (34) that

$$\begin{aligned} N_i &= MA_i - H_i C = (I + FC)A_i - H_i C \\ &= (I + (U + YV)C)A_i - H_i C, \quad \forall i \in \mathcal{I}_r. \end{aligned} \tag{38}$$

Using expressions (11), (27), (33), (38), followed by the variable changes (31)–(32), inequality (28) is rewritten as

$$\begin{bmatrix} P_i & \star \\ R\mathscr{N}_i(\rho_{lj}) & R + R^\top - P_k \end{bmatrix} \succ 0, \tag{39}$$

for $i, k \in \mathcal{I}_r$ and $\rho_{lj} \in \mathscr{V}_\phi$, $l \in \mathcal{I}_{n_\phi}$, $j \in \mathcal{I}_{n_x}$, with $\mathscr{N}_i(\rho_{lj})$ defined in (17). Since $h, h_+ \in \Omega$ and due the convexity property of $\mathscr{S}_\phi$, it follows that condition (39) implies (20). The proof can be concluded via the result of Theorem 1. □

Remark 3. Condition (28) in Theorem 2 is expressed in an LMI form, which can be effectively solved with numerical solvers, *e.g.*, YALMIP toolbox with SDPT3 solver [32].

The UI observer design algorithm is summarized as follows.

Observer Design Procedure

Input: Nonlinear system in the N-TS fuzzy form (6).

Output: Fuzzy UI observer (8)–(10) such that $\hat{x}_k \to x_k$ and $\hat{d}_k \to d_k$, when $k \to \infty$.

Step 1: Check the matrix rank conditions in Assumption 1.
- If YES, then go to Step 2.
- If NO, then the proposed algorithm is unapplicable to the considered system.

Step 2: Compute the matrices U and V from (29) and (30).

Step 3: Solve LMI-based condition (28) to get R, Z, P_i and X_i, for $i \in \mathcal{I}_r$.

Step 4: Compute Y from (31) as $Y = R^{-1}Z$, and H_i from (32) as $H_i = R^{-1}X_i$, for $i \in \mathcal{I}_r$.

Step 5: Compute the observer gains M, F, N_i and L_i from (11), (33), (34) and (35), respectively.

To illustrate the proposed fuzzy UI observer design, a numerical example is presented in the following section.

IV. Numerical Discussions

We revisit the following nonlinear discrete-time system [28]:

$$\begin{aligned} x_{1(k+1)} &= (1+T)x_{1k} + Tx_{2k} + \frac{\sin b}{b}Tx_{3k} \\ &\quad - 0.1Tx_{4k} + T(1+x_{1k}^2)u_k + T\eta(x_k) \\ x_{2(k+1)} &= Tx_{1k} + (1-2T)x_{2k} \\ x_{3(k+1)} &= Tx_{1k} + (1-0.3T)x_{3k} + Tx_{1k}^2x_{2k} \\ x_{4(k+1)} &= (1-T)x_{4k} + \frac{\sin b}{b}Tx_{3k} + T\eta(x_k) + Td_k \\ y_k &= x_{1k} \end{aligned} \tag{40}$$

where $\eta(x_k) = \sin(x_{3k}) - \frac{\sin b}{b}x_{3k}$, $x_{1k} \in [-a, a]$, $x_{3k} \in [-b, b]$ and $b = \frac{\pi}{2}$. The fixed step of discretization is $T = 0.5$. The premise variable is defined as $\xi_k = x_{1k}^2$. Then, system (40) can be rewritten in the form (3) with

$$A(\xi_k) = \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & T\xi_k & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix},$$

$$f(\xi_k, u_k) = \begin{bmatrix} T(1+\xi_k)u_k \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad G(\xi_k) = G = \begin{bmatrix} T \\ 0 \\ 0 \\ T \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \phi(x_k) = \sin(x_{3k}).$$

Using the sector nonlinearity approach [10, Ch. 2] with the measurable premise variable $\xi_k = x_{1k}^2$, the following 2-rule N-TS fuzzy model can be obtained:

$$x_{k+1} = \sum_{i=1}^{2} h_i(\xi_k)A_ix_k + f(\xi_k, u_k) + G\phi(x_k), \tag{41}$$

where

$$A_1 = \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & Ta^2 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & 0 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix}.$$

The corresponding membership functions are given by $h_1(\xi_k) = \frac{x_1^2}{a^2}$ and $h_2(\xi_k) = \frac{a^2-x_1^2}{a^2}$. We define the parameter vector ρ as in (15) with $\rho_{1i} = 0$, for $i \in \{1, 2, 4\}$, and $\rho_{13} = \cos(x_{3k})$. Since $x_{3k} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then the bounded convex set $\mathscr{S}_\phi$ has two vertices, *i.e.*, $\mathscr{V}_\phi = \{\rho_{13} : \ \rho_{13} \in \{0, 1\}\}$. The design procedure in Section III can be now applied to design a N-TS fuzzy UI observer for system (40), equivalently represented as (41).

Note that the nonlinearity $\sin(x_{3k})$ cannot be measured, which raises technical difficulties for TS fuzzy observer design. For instance, the recent fuzzy UI observer designs with measured premise variables in [8], [21] cannot be used. Due to restrictive rank conditions, the fuzzy UI observer method in [11] is also not applicable in this case. Indeed, TS fuzzy UI observer design, guaranteeing an asymptotic estimation performance for the discrete-time nonlinear system (40), is not yet available in the literature. To provide an idea on the conservatism of the proposed UI observer design, we compare the following methods.

- Method 1: proposed result in Theorem 2.
- Method 2: similar as Method 1 using Lipschitz property in place of DMVT, which can be considered as an extension of [28, Theorem 1].
- Method 3: extension to the discrete-time case of the TS fuzzy UI observer design using Lipschitz property in [4].
- Method 4: extension to the discrete-time case with UI of the TS fuzzy observer design using DMVT in [24].

For each method, the goal is to search for the largest value of parameter a, denoted by a^*, for which an UI observer can be found for system (40). Table I summarizes the numerical comparisons between four considered methods. Remark that the proposed method provides a significant improvement in reducing the conservatism over other ones. Note also that using N-TS fuzzy modeling in Theorem 2 and [28], the number of fuzzy rules is decreased from four to two, leading to a reduction on the computational burden, represented by the number of scalar decision variables $\mathcal{N}_{var}$ and the number of rows $\mathcal{N}_{row}$ of all involved LMIs, see Remark 1.

TABLE I
Numerical comparison between different methods.

Design	Method 1	Method 2	Method 3	Method 4
a^*	9301.0	6341.7	371.7	7406.2
Nb. rules	2	2	4	4
$\mathcal{N}_{var}$	56	57	93	92
$\mathcal{N}_{row}$	72	44	784	1040

For illustrations, we consider the UI observer design for system (40) with $a = 9301$. Fig. 1 depicts the response of the

proposed observer with respect to $x_0 = \begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}^\top$ and a randomly chosen UI signal $d_k = 5e^{-0.1k}\sin(7k)$. Remark that both the states and the UI are asymptotically estimated with a high accuracy. Moreover, differently from the UI estimation approach in [3], the proposed observer does not lead to an *excessive* initial UI estimation error ε_0.

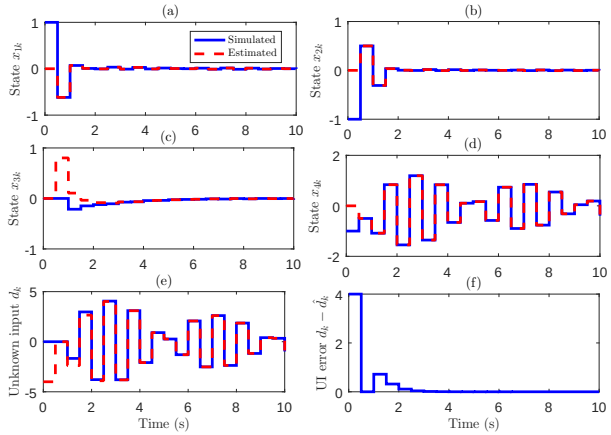

Fig. 1. Asymptotic estimation performance obtained with system (40).

V. Concluding Remarks

An UI observer design has been proposed for nonlinear systems represented in a N-TS fuzzy form. All the unmeasured nonlinearities are contained in a nonlinear consequent part. Via the differential mean value theorem, this N-TS fuzzy reformulation allows for an effective framework to design fuzzy UI observers while avoiding the major issue on unmeasured premise variables. Both the state and the UI can be asymptotically estimated using UI decoupling technique and fuzzy Lyapunov arguments. No *a priori* information on the UI is required for fuzzy UI observer design. The observer gains can be easily computed from an LMI-based design procedure. Numerical illustrations are given to demonstrate the interests of the proposed fuzzy UI observer design. Future works focus on designing N-TS fuzzy UI observers for nonlinear systems, whose output is also affected by unmeasured nonlinearities.

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