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# On the cohomological equation of a hyperbolic automorphism

Abdellatif ZEGGAR

## Abstract

In this paper, we study the discrete cohomological equation (e) :  $f - f \circ \gamma = g$  associated with a hyperbolic affine automorphism  $\gamma : \mathbb{T}^d \rightarrow \mathbb{T}^d$  of the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ .  $g$  being a given  $C^\infty$  function and  $f$  an unknown function. More precisely, if we denote by  $E$  the Fréchet space of all  $C^\infty$  functions on  $\mathbb{T}^d$  and by  $\delta$  the operator defined on  $E$  by  $\delta(f) = f - f \circ \gamma$ , we show that the space  $\delta(E)$  is a closed subspace of  $E$  and we explicitly determine a continuous linear operator  $L : \delta(E) \rightarrow E$  such that for every element  $g$  of  $\delta(E)$ , the function  $f = L(g)$ , given by its Fourier series, is a solution of the equation (e).

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**Keywords.** Hyperbolic automorphism, Fourier series.

## 1 Introduction

Let  $M$  be a connected differentiable manifold. The space  $E = C^\infty(M)$  of  $C^\infty$  functions on  $M$  is a Fréchet space for the  $C^\infty$  topology (the topology of the uniform convergence of all the derivatives on compact subsets). A differentiable action of a discrete group  $\Gamma$  (supposed of finite presentation) on  $M$  induces a natural action on  $E$  given by:

$$\forall \gamma \in \Gamma, \forall f \in E, \gamma.f = f \circ \gamma^{-1}$$

which makes  $E$  a  $\Gamma$ -module. Then one can consider the cohomology  $H^*(\Gamma, E)$  of the discrete group  $\Gamma$  with values in  $E$ . In the case  $\Gamma$  is generated by a single element  $\gamma$  we can easily show that the space  $H^1(\Gamma, E)$  that we will denote  $H^1(\gamma, E)$  is the Cokernel of the cobord operator:

$$\delta : E \rightarrow E, f \mapsto \delta(f) = f - f \circ \gamma$$

The calculation of the Cokernel  $E/\delta(E)$  of  $\delta$  amounts to solving the following equation [4]:

$$f - f \circ \gamma = g \quad \text{where} \quad \begin{cases} f \in E \text{ is unknown} \\ \text{and} \\ g \in E \text{ is given} \end{cases}$$

called the cohomological equation associated with the diffeomorphism  $\gamma : M \rightarrow M$ . We say that the automorphism  $\gamma : M \rightarrow M$  is cohomologically  $C^\infty$ -stable when  $\delta(E)$  is a closed subspace of  $E$ , that is when the topological space  $H^1(\gamma, E) = E/\delta(E)$  is Hausdorff [1].

In the following, we will study this equation in the case of a hyperbolic automorphism of a torus using the fundamental tool of Fourier series.

Let  $A$  be a hyperbolic matrix of  $GL(d, \mathbb{Z})$  (every eigenvalue  $\lambda \in \mathbb{C}$  satisfies  $|\lambda| \neq 1$ ) and  $b$  an element of  $\mathbb{R}^d$ . The affine automorphism  $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $x \mapsto \gamma(x) = Ax + b$  of  $\mathbb{R}^d$  induces an affine hyperbolic automorphism of the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  that we still denote  $\gamma$ . The cohomological equation associated to the hyperbolic dynamical system  $\gamma$  is the following:

$$(e) \quad f - f \circ \gamma = g \quad \text{where} \quad \begin{cases} f \in E := C^\infty(\mathbb{T}^d) \text{ is unknown} \\ \text{and} \\ g \in E \text{ is given} \end{cases}$$

The purpose of this work is to establish the following theorem:

## The main theorem

*With the notations and the conditions above, we have the following result:*

*The range  $\delta(E)$  of the cobord operator  $\delta : E \rightarrow E$ ,  $h \mapsto \delta(h) = h - h \circ \gamma$  is a closed subspace of the Fréchet space  $E$  and there is a continuous linear operator  $L : \delta(E) \rightarrow E$  such that for any element  $g$  of  $\delta(E)$  the function  $f = L(g)$ , given by its Fourier series, is a solution of the cohomological equation (e).*

This result is a generalization of the Theorem 5.1 of [3] proved by A. Dehghan-Nezhad and A. El Kacimi in the case  $b = 0$  and  $A$  is a diagonalizable matrix whose all eigenvalues are real and positive.

Let us first give some examples of hyperbolic automorphisms and recall some notions that we will use in the proof of the main theorem.

## 2 Examples of hyperbolic matrix

### 2.1 Diagonalizable matrix with eigenvalues all real

A well-known example on the torus  $\mathbb{T}^2$  is the hyperbolic automorphism (called Arnold's cat) defined by the matrix  $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  whose eigenvalues are :  $\lambda_1 = \frac{3-\sqrt{5}}{2}$  and  $\lambda_2 = \frac{3+\sqrt{5}}{2}$ .

$A_1$  is a diagonalizable hyperbolic matrix whose eigenvalues are positive.

The matrix  $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ , whose eigenvalues are :  $\lambda_1 = \frac{1-\sqrt{5}}{2}$  and  $\lambda_2 = \frac{1+\sqrt{5}}{2}$ , defines a hyperbolic automorphism on the torus  $\mathbb{T}^2$ . It is diagonalizable and its eigenvalues are not all positive.

### 2.2 Diagonalizable matrix with complex eigenvalues

Consider the matrix  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ . Its characteristic polynomial  $P_A(X)$  is given by

$$P_A(X) = \det(A - X id) = -X^3 + X^2 + X + 1$$

Its derivative is factorized as follows:

$$P'_A(x) = -3x^2 + 2x + 1 = (x^2 + 2x + 1) - 4x^2 = (x + 1)^2 - (2x)^2 = (3x + 1)(1 - x)$$

A study of variation allows to see that the polynomial function  $P_A$  admits only one real root  $\mu$  in the interval  $[\frac{3}{2}, 2]$ .

Denote by  $\lambda = a + ib$  and  $\bar{\lambda} = a - ib$  the other two roots of  $P_A$  in  $\mathbb{C}$  (the polynomial being with real coefficients). We have :  $b \neq 0$  and  $1 = \det(A) = \mu\lambda\bar{\lambda} = \mu|\lambda|^2$ .

So we have three eigenvalues  $\mu, \lambda, \bar{\lambda}$  such that  $\mu > 1$  and  $|\lambda| = |\bar{\lambda}| = \frac{1}{\sqrt{\mu}} < 1$ .

### 2.3 Non-diagonalizable matrix with complex eigenvalues

Consider  $Q(X) = (P_A(X))^2$  where  $P_A$  is the polynomial of the example above. Explicitly we have  $Q(X) = X^6 - 2X^5 - X^4 + 3X^2 + 2X + 1 \in \mathbb{Z}[X]$ ; it is both the characteristic polynomial and the minimal polynomial of its companion matrix:

$$C_Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$C_Q$  is therefore a matrix of  $GL(6, \mathbb{Z})$  admitting  $\mu, \lambda$  and  $\bar{\lambda}$  as eigenvalues each of them with multiplicity equal 2.  $C_Q$  is therefore a non-diagonalizable hyperbolic matrix whose eigenvalues are not all real.

## 3 Preliminary results

Suppose that the matrix  $A$  of  $GL(d, \mathbb{Z})$  is hyperbolic. Then the transpose  $B$  of  $A^{-1}$  is also hyperbolic since its eigenvalues are the inverse of those of  $A$ .

Denote by  $\lambda_1, \dots, \lambda_r$  the eigenvalues of  $B$  in  $\mathbb{C}$  with:

$$\begin{cases} 0 < |\lambda_j| < 1 & \text{for } 1 \leq j \leq q < r \\ |\lambda_j| > 1 & \text{for } q + 1 \leq j \leq r \end{cases}$$

For  $j \in \{1, \dots, r\}$ , the eigenvalue  $\lambda_j$  and its conjugate  $\bar{\lambda}_j$  are roots of the characteristic polynomial  $P_B(X)$  of  $B$  with the same modulus  $|\lambda_j|$  and the same multiplicity  $k_j \in \mathbb{N}^*$ . So we have :

$$P_B(X) = S_B(X)U_B(X) \text{ where } \begin{cases} S_B(X) = (X - \lambda_1)^{k_1} \dots (X - \lambda_q)^{k_q} \in \mathbb{R}[X] \\ U_B(X) = (X - \lambda_{q+1})^{k_{q+1}} \dots (X - \lambda_r)^{k_r} \in \mathbb{R}[X] \end{cases}$$

The polynomials  $S_B(X)$  and  $U_B(X)$  are relatively prime in the ring  $\mathbb{R}[X]$  and the kernel lemma gives the following decomposition of the vector space  $V = \mathbb{R}^d$ :

$V = V_s \oplus V_u$  where  $V_s = Ker S(B)$  (the stable subspace) and  $V_u = Ker U(B)$  (the unstable subspace)

The spaces  $V_s$  and  $V_u$  being invariant by  $B$ , we note by  $B_s$  and  $B_u$  the automorphisms induced by  $B$  on these respective spaces.

The eigenvalues of the automorphism  $B_s$  all have modulus  $< 1$  and those of  $B_u$  all have modulus  $> 1$ .

Moreover, we have the following proposition:

### 3.1 proposition

We can construct on the vector space  $V$  a norm  $\|\cdot\|_*$  adapted to the hyperbolic automorphism  $B$  in the following meaning :

- (i) for all  $x_s \in V_s$  and  $x_u \in V_u$ ,  $\|x_s + x_u\|_* = \max(\|x_s\|_*, \|x_u\|_*)$ .
- (ii)  $\|B_s\|_* < 1$  and  $\|B_u^{-1}\|_* < 1$ .

PROOF. Let  $L : W \rightarrow W$  be a linear automorphism of a real vector subspace  $W$  of  $V = \mathbb{R}^d$ . We consider a basis  $(w_j)_{1 \leq j \leq m}$  of  $W$  ( $\dim W = m$ ) and we denote by  $\widetilde{W}$  the complex vector subspace of  $\mathbb{C}^d$  generated by the vectors  $w_1, \dots, w_m$ :  $\widetilde{W} = \text{Vect}_{\mathbb{C}}(w_1, \dots, w_m)$ . The automorphism  $L$  induces on  $\widetilde{W}$  the linear complex automorphism :

$$\widetilde{L} : \widetilde{W} \rightarrow \widetilde{W}, x = \sum_{j=1}^m x_j w_j \mapsto \widetilde{L}(x) = \sum_{j=1}^m x_j L(w_j)$$

$\widetilde{W}$  is of complex dimension  $m$ ,  $W \subset \widetilde{W}$  and  $\forall x \in W$ ,  $\widetilde{L}(x) = L(x)$ . Now consider a basis  $(v_j)_{1 \leq j \leq m}$  of  $\widetilde{W}$  in which the matrix  $T = (t_{i,j})$  of  $\widetilde{L}$  is upper triangular ( $t_{i,j} = 0$  for  $i > j$ ) and for a real number  $\alpha$  fixed in  $]0, 1]$ , we consider the new base  $(u_j)_{1 \leq j \leq m}$  of  $\widetilde{W}$  defined by:  $u_j = \alpha^{j-1} v_j$ . We have :

$$\forall x = \sum_{j=1}^m x_j u_j \in \widetilde{W}, \widetilde{L}(x) = \sum_{j=1}^m x_j \widetilde{L}(u_j) = \sum_{j=1}^m \alpha^{j-1} x_j \sum_{i=1}^m t_{i,j} v_i = \sum_{j=1}^m \sum_{i=1}^m \alpha^{j-i} x_j t_{i,j} u_i = \sum_{i=1}^m \left( \sum_{j=1}^m \alpha^{j-i} x_j t_{i,j} \right) u_i$$

Thus, for the norm  $\|\cdot\|_{L,\alpha} : x = \sum_{j=1}^m x_j u_j \mapsto \|x\|_{L,\alpha} = \sum_{j=1}^m |x_j|$  we have, for all  $x \in \widetilde{W}$ :

$$\|\widetilde{L}(x)\|_{L,\alpha} = \sum_{i=1}^m \left| \sum_{j=1}^m \alpha^{j-i} x_j t_{i,j} \right| \leq \sum_{i=1}^m \sum_{j=1}^m \alpha^{j-i} |x_j| |t_{i,j}| \leq \left( \max_{1 \leq i \leq m} |t_{i,i}| \right) \|x\|_{L,\alpha} + \alpha \left( \sum_{1 \leq i < j \leq m} |t_{i,j}| \right) \|x\|_{L,\alpha}$$

Which implies that  $\|\widetilde{L}\|_{L,\alpha} \leq \rho(L) + \alpha \left( \sum_{1 \leq i < j \leq m} |t_{i,j}| \right)$  where  $\rho(L) = \max_{1 \leq i \leq m} |t_{i,i}|$  is the spectral radius of  $L$ . We thus obtain by restriction a norm on the real vector space  $W$ , again denoted  $\|\cdot\|_{L,\alpha}$ , such that

$$\|L\|_{L,\alpha} \leq \rho(L) + \alpha \left( \sum_{1 \leq i < j \leq m} |t_{i,j}| \right)$$

In addition, if  $\rho(L) < 1$ , we can choose  $\alpha$  in  $]0, 1]$  so that  $\rho(L) + \alpha \left( \sum_{1 \leq i < j \leq m} |t_{i,j}| \right) < 1$ . In this case, we will have  $\|L\|_{L,\alpha} < 1$ .

In this way, we obtain a norm  $\|\cdot\|_s$  on  $V_s$  such that  $\|B_s\|_s < 1$  (if  $L = B_s$ ) and a norm  $\|\cdot\|_u$  on  $V_u$  such that  $\|B_u^{-1}\|_u < 1$  (if  $L = B_u^{-1}$ ). The norm  $\|\cdot\|_*$  defined on  $V = V_s \oplus V_u$  by:

$$\forall x = x_s + x_u \in V = V_s \oplus V_u, \|x\|_* = \max(\|x_s\|_s, \|x_u\|_u)$$

clearly verifies the desired properties. □

### 3.2 remark

- (i) For any eigenvalue  $\lambda_j$  of  $B$ ,  $\lambda'_j = \frac{1}{\lambda_j}$  is an eigenvalue of the hyperbolic automorphism  $B' = B^{-1}$  having for multiplicity  $k_j$ .
- (ii) For any eigenvalue  $\lambda_j$  of  $B$ ,

$$(B - \lambda_j id)^{k_j} = (-\lambda_j)^{k_j} B^{k_j} \left( B^{-1} - \frac{1}{\lambda_j} id \right)^{k_j} = (-\lambda_j)^{k_j} B^{k_j} (B' - \lambda'_j id)^{k_j}$$

Consequently, we have:

$$\begin{cases} S_B(B) = \tau_s B^l U_{B'}(B') \text{ where } l = \dim(V_s) \text{ and } \tau_s \in \mathbb{R} \setminus \{0\} \\ U_B(B) = \tau_u B^{d-l} S_{B'}(B') \text{ where } d-l = \dim(V_u) \text{ and } \tau_u \in \mathbb{R} \setminus \{0\} \end{cases}$$

- (iii)  $V'_u = V_s$ ,  $V'_s = V_u$  and for all  $x \in V$ ,  $x'_u = x_s$  and  $x'_s = x_u$
- (iv) The norm  $\|\cdot\|_*$  is adapted to the hyperbolic automorphism  $B' = B^{-1}$ .

We also have the following lemma

### 3.3 lemma

Let  $m \in \mathbb{Z}^d \setminus \{0\}$ .

- (i)  $m_s \neq 0$  and  $m_u \neq 0$ .
- (ii) If  $\|m\|_* = \|m_u\|_*$ , then the numerical sequence  $(\|B^k m\|_*)_{k \geq 0}$  is strictly increasing.
- (iii) If  $\|m\|_* = \|m_s\|_*$ , then the numerical sequence  $(\|B^{-k} m\|_*)_{k \geq 0}$  is strictly increasing.

PROOF. Let  $m \in \mathbb{Z}^d \setminus \{0\}$ .

- (i) If  $m_u = 0$ , then  $\lim_{k \rightarrow +\infty} \|B^k(m)\|_* = \lim_{k \rightarrow +\infty} \|B_s^k(m_s)\|_* = 0$  ( $\|B_s\|_* < 1$ ) and by the equivalence of norms  $\|\cdot\|$  and  $\|\cdot\|_*$  we also have  $\lim_{k \rightarrow +\infty} \|B^k(m)\|^2 = 0$ . The sequence of strictly positive integers  $(\|B^k m\|^2)$  is therefore zero from a certain rank. which is not possible. Hence  $m_u \neq 0$ . Likewise  $m_s = m'_u \neq 0$  from Remark 3.2.

- (ii) If  $\|m\|_* = \|m_u\|_*$ , then

$$\|m\|_* = \|m_u\|_* = \|B_u^{-1}(Bm_u)\|_* \leq \underbrace{\|B_u^{-1}\|_*}_{<1} \cdot \underbrace{\|Bm_u\|_*}_{\neq 0} < \|Bm_u\|_* \leq \|Bm\|_*$$

$$\|Bm_s\|_* \leq \|B_s\|_* \cdot \|m_s\|_* < \|m_s\|_* \leq \|m\|_* \leq \|Bm_u\|_* \text{ and so } \|Bm\|_* = \|Bm_u\|_*$$

So we have  $\|m\|_* < \|Bm\|_*$  and  $\|Bm\|_* = \|Bm_u\|_*$  which allows to prove by induction that, for any integer  $k \in \mathbb{N}$ ,  $\|B^k m\|_* < \|B^{k+1} m\|_*$ . The sequence  $(\|B^k m\|_*)_{k \geq 0}$  is then strictly increasing.

- (iii) If  $\|m\|_* = \|m_s\|_*$ , by Remark 3.2. we have  $\|m\|_* = \|m'_u\|_*$  and therefore according to the previous case, the sequence  $(\|B^k m\|_*)_{k \geq 0} = (\|B^{-k} m\|_*)_{k \geq 0}$  is strictly increasing.

□

## 4 The space $E := C^\infty(\mathbb{T}^d)$

The vector space  $\mathbb{R}^d$  being equipped with its usual scalar product  $\langle \cdot, \cdot \rangle$  and the associated Euclidean norm  $\|\cdot\|$ , we denote by  $C_{per}^\infty(\mathbb{R}^d)$  the space of  $C^\infty$ -functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$  which are  $\mathbb{Z}^d$ -periodic, that is:

$$\forall m \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d : \varphi(x + m) = \varphi(x).$$

If  $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  is the canonical projection, we have the bijection:

$$C^\infty(\mathbb{T}^d) \rightarrow C_{per}^\infty(\mathbb{R}^d), h \mapsto h \circ \pi$$

which identifies the elements of  $E = C^\infty(\mathbb{T}^d)$  to those of  $C_{per}^\infty(\mathbb{R}^d)$ . For more detail, we can refer to [2] for instance.

Any function  $h \in E$  can be represented by its Fourier expansion:

$$h = \sum_{m \in \mathbb{Z}^d} \widehat{h}(m) \Theta_m$$

where  $\widehat{h}(m)$  is the  $m$ -th Fourier coefficient of  $h$  given by:

$$\widehat{h}(m) = \int_{\mathbb{T}^d} h(x) e^{-2i\pi \langle x, m \rangle} dx = \int_{[0,1]^p} h(t) e^{-2i\pi \langle t, m \rangle} dt_1 \cdots dt_p$$

and  $\Theta_m : \mathbb{R}^d \rightarrow \mathbb{C}$  is the  $\mathbb{Z}^d$ -periodic function defined by  $\Theta_m(x) = e^{2i\pi \langle x, m \rangle}$ . It is well known that the family of complex numbers  $(\widehat{h}(m))_{m \in \mathbb{Z}^d}$  belongs to the vector space  $\mathcal{S}(\mathbb{Z}^d, \mathbb{C})$  consisting of families  $(a_m)_{m \in \mathbb{Z}^d}$  of complex numbers, with rapid decay, that is:

$$\forall r \in \mathbb{N}, \lim_{\|m\| \rightarrow +\infty} \|m\|^r |a_m| = 0$$

Moreover, the map:

$$C^\infty(\mathbb{T}^d) \rightarrow \mathcal{S}(\mathbb{Z}^d, \mathbb{C}), h \mapsto (\widehat{h}(m))_{m \in \mathbb{Z}^d}$$

is a linear bijection identifying  $E$  to the space  $\mathcal{S}(\mathbb{Z}^d, \mathbb{C})$  which can be also expressed as:

$$\mathcal{S}(\mathbb{Z}^d, \mathbb{C}) = \bigcap_{r \in \mathbb{N}} W^{1,r}.$$

Here  $W^{1,r}$  is the space of families  $(a_m)_{m \in \mathbb{Z}^d}$  of complex numbers satisfying the condition:

$$\sum_{m \in \mathbb{Z}^d} \|m\|^r |a_m| < \infty$$

equipped with the norm:

$$\|(a_m)\|_{1,r} = |a_0| + \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \|m\|^r |a_m|$$

The topology of the vector space  $\mathcal{S}(\mathbb{Z}^d, \mathbb{C})$  being that defined by the family of norms  $(\|\cdot\|_{1,r})_{r \in \mathbb{N}}$ .

## 5 Necessary conditions for solving equation (e)

We consider the affine hyperbolic automorphism  $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $x \mapsto \gamma(x) = Ax + b$ . For all  $k \in \mathbb{Z}$ , we note:  $\gamma^0 = id$ ,  $\gamma^k = \underbrace{\gamma \circ \dots \circ \gamma}_{k \text{ times}}$  for  $k > 0$  and  $\gamma^k = (\gamma^{-1})^{|k|}$  for  $k < 0$ .

Let  $B \in GL(d, \mathbb{Z})$  be the transpose of  $A^{-1}$  and set  $b_k = \gamma^k(0)$  for  $k \in \mathbb{Z}$ . We have:

### 5.1 lemma

For  $k \in \mathbb{Z}$ ,  $m \in \mathbb{Z}^d$  and  $h \in E$ , we have:

$$b_{-k} = -A^{-k}b_k \quad \text{and} \quad \widehat{h \circ \gamma^k}(m) = e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m).$$

PROOF. We have:  $0 = \gamma^{-k}[\gamma^k(0)] = \gamma^{-k}(b_k) = A^{-k}b_k + b_{-k}$ . Hence  $b_{-k} = -A^{-k}b_k$ . We also have :

$$\begin{aligned} \widehat{h \circ \gamma^k}(m) &= \int_{\mathbb{T}^d} h \circ \gamma^k(x) e^{-2i\pi \langle x, m \rangle} dx \\ &= \int_{\mathbb{T}^d} h[\gamma^k(x)] e^{-2i\pi \langle x, m \rangle} dx \\ &= \int_{\mathbb{T}^d} h(u) e^{-2i\pi \langle \gamma^{-k}(u), m \rangle} du \\ &= \int_{\mathbb{T}^d} h(u) e^{-2i\pi \langle A^{-k}(u) + b_{-k}, m \rangle} du \\ &= e^{-2i\pi \langle b_{-k}, m \rangle} \int_{\mathbb{T}^d} h(u) e^{-2i\pi \langle A^{-k}(u), m \rangle} du \\ &= e^{2i\pi \langle A^{-k}b_k, m \rangle} \int_{\mathbb{T}^d} h(u) e^{-2i\pi \langle A^{-k}(u), m \rangle} du \\ &= e^{2i\pi \langle b_k, B^k m \rangle} \int_{\mathbb{T}^d} h(u) e^{-2i\pi \langle u, B^k m \rangle} du \\ &= e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m) \end{aligned}$$

□

### 5.2 lemma

For  $m \in \mathbb{Z}^d \setminus \{0\}$  and  $h \in E$ , the three series:

$$\sum_{k \geq 0} \widehat{h \circ \gamma^k}(m), \quad \sum_{k < 0} \widehat{h \circ \gamma^k}(m) = \sum_{l > 0} \widehat{h \circ \gamma^{-l}}(m) \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \widehat{h \circ \gamma^k}(m)$$

are absolutely convergent and the functionals:

$$\Phi_0 : E \rightarrow \mathbb{C}, \quad h \mapsto \Phi_0(h) = \int_{\mathbb{T}^d} h(x) dx$$

$$\Phi_m : E \rightarrow \mathbb{Z}, \quad h \mapsto \Phi_m(h) = \sum_{k \in \mathbb{Z}} \widehat{h \circ \gamma^k}(m)$$

$$\Phi_m^+ : E \rightarrow \mathbb{C}, \quad h \mapsto \Phi_m^+(h) = \sum_{k \geq 0} \widehat{h \circ \gamma^k}(m)$$

$$\Phi_m^- : E \rightarrow \mathbb{C}, \quad h \mapsto \Phi_m^-(h) = - \sum_{k < 0} \widehat{h \circ \gamma^k}(m)$$

are continuous.



PROOF. Since the family  $(\widehat{h}(\alpha))_{\alpha \in \mathbb{Z}^d}$  is absolutely summable, so are each of the sub-families  $(\widehat{h}(B^k m))_{k \geq 0}$  and  $(\widehat{h}(B^k m))_{k < 0}$ . Therefore the three series:

$$\begin{aligned}\sum_{k \geq 0} \widehat{h \circ \gamma^k}(m) &= \sum_{k \geq 0} e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m) \\ \sum_{k < 0} \widehat{h \circ \gamma^k}(m) &= \sum_{k < 0} e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m) \\ \sum_{k \in \mathbb{Z}} \widehat{h \circ \gamma^k}(m) &= \sum_{k \in \mathbb{Z}} e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m)\end{aligned}$$

are absolutely convergent. Then, the functionals  $\Phi_m$ ,  $\Phi_m^+$  and  $\Phi_m^-$  are well defined for each  $m \in \mathbb{Z}^d \setminus \{0\}$ . Furthermore, We have :

$$\forall r \in \mathbb{N}, \forall h \in E, |\Phi_0(h)| = |\widehat{h}(0)| \leq \sum_{\alpha \in \mathbb{Z}^d} |\widehat{h}(\alpha)| = \|h\|_{1,0} \leq \|h\|_{1,r}$$

$$\forall r \in \mathbb{N}, \forall h \in E, |\Phi_m(h)| = \left| \sum_{k \in \mathbb{Z}} e^{2i\pi \langle b_k, B^k m \rangle} \widehat{h}(B^k m) \right| \leq \sum_{k \in \mathbb{Z}} |\widehat{h}(B^k m)| \leq \sum_{\alpha \in \mathbb{Z}^d} |\widehat{h}(\alpha)| = \|h\|_{1,0} \leq \|h\|_{1,r}$$

Also, we have:  $\forall r \in \mathbb{N}, \forall h \in E, |\Phi_m^+(h)| \leq \|h\|_{1,r}$  and  $|\Phi_m^-(h)| \leq \|h\|_{1,r}$ . This proves the continuity of the considered functionals. □

### 5.3 proposition

If the cohomological equation (e) admits a solution  $f$  for the data  $g$  then:

$$\begin{cases} \forall m \in \mathbb{Z}^d, \Phi_m(g) = 0 \\ \forall m \in \mathbb{Z}^d \setminus \{0\}, \widehat{f}(m) = \Phi_m^+(g) = \Phi_m^-(g) \end{cases}$$

PROOF. Suppose that the cohomological equation (e) admits a solution  $f$  for the data  $g$  then:

$$\Phi_0(g) = \int_{\mathbb{T}^d} g(x) dx = \int_{\mathbb{T}^d} f(x) dx - \int_{\mathbb{T}^d} f[\gamma(x)] dx = \int_{\mathbb{T}^d} f(x) dx - \int_{\mathbb{T}^d} f(t) dt = 0 \quad (\text{because } \det(A) = 1)$$

and if  $m \in \mathbb{Z}^d \setminus \{0\}$  and  $n \in \mathbb{N}^*$ , we have :

$$\sum_{k=0}^n \widehat{g \circ \gamma^k}(m) = \sum_{k=0}^n \widehat{f \circ \gamma^k}(m) - \sum_{k=0}^n \widehat{f \circ \gamma^{k+1}}(m) = \widehat{f}(m) - \widehat{f \circ \gamma^{n+1}}(m)$$

In addition, the numerical series  $\sum_{k \geq 0} \widehat{f \circ \gamma^k}(m)$  is convergent. So  $\lim_{n \rightarrow +\infty} \widehat{f \circ \gamma^{n+1}}(m) = 0$  and hence

$$\sum_{k \geq 0} \widehat{g \circ \gamma^k}(m) = \widehat{f}(m) \text{ either } \Phi_m^+(g) = \widehat{f}(m).$$

$$\text{Similarly, } \Phi_m^-(g) = - \sum_{k < 0} \widehat{g \circ \gamma^k}(m) = \widehat{f}(m); \text{ so } \Phi_m(g) = \sum_{k \in \mathbb{Z}} \widehat{g \circ \gamma^k}(m) = \Phi_m^+(g) - \Phi_m^-(g) = 0.$$

□

## 6 End of the proof of the main theorem

PROOF. For all  $m \in \mathbb{Z}^d$ , the linear form  $\Phi_m$  is continuous on  $E$ . This implies that its kernel  $\text{Ker}(\Phi_m)$  is a closed subspace of  $E$ . So the intersection  $\mathcal{H} := \bigcap_{m \in \mathbb{Z}^d} \text{Ker}(\Phi_m)$  is a closed subspace of  $E$ .

On the other hand, if  $g$  is in the range  $\delta(E)$  of the cobord operator  $\delta : E \rightarrow E$ ,  $h \mapsto \delta(h) = h - h \circ \gamma$ , the equation (e) admits at least one solution and so  $g \in \mathcal{H}$  from Proposition 5.3. Then  $\delta(E) \subseteq \mathcal{H}$ .

Conversely, if  $g \in \mathcal{H}$ , then  $\forall m \in \mathbb{Z}^d$ ,  $\Phi_m(g) = 0$  or, in an equivalent way,

$$\begin{cases} \Phi_0(g) = 0 \\ \forall m \in \mathbb{Z}^d \setminus \{0\}, \Phi_m^+(g) = \Phi_m^-(g) \end{cases}$$

Let us show that the sequence of complex numbers  $(\Phi_m^+(g))_{m \in \mathbb{Z}^d \setminus \{0\}}$  is fast decreasing and thus defines a function  $f = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \Phi_m^+(g) \Theta_m \in E = C^\infty(\mathbb{T}^d)$  which is a solution of (e).

For this, it suffices to prove that for all  $r \in \mathbb{N}$ ,  $\lim_{\|m\| \rightarrow +\infty} \|m\|^r |\Phi_m^+(g)| = 0$  or again, in an equivalent way,

$$\lim_{\|m\|_* \rightarrow +\infty} \|m\|_*^r |\Phi_m^+(g)| = 0.$$

Let  $r \in \mathbb{N}$  and let  $m \in \mathbb{Z}^d \setminus \{0\}$ .

• If  $\|m\|_* = \|m_u\|_*$  then, according to Lemma 3.3. the sequence  $(\|B^k m\|_*)_{k \geq 0}$  is strictly increasing; so we have:

$$\begin{aligned} \|m\|_*^{r+2} |\Phi_m^+(g)| &= \|m\|_*^{r+2} \left| \widehat{\sum_{k \geq 0} g \circ \gamma^k(m)} \right| \\ &\leq \sum_{k \geq 0} \|m\|_*^{r+2} |\widehat{g}(B^k m)| \\ &\leq \sum_{k \geq 0} \|B^k m\|_*^{r+2} |\widehat{g}(B^k m)| \\ &\leq \sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} \|\alpha\|_*^{r+2} |\widehat{g}(\alpha)| \end{aligned}$$

• If  $\|m\|_* = \|m_s\|_*$  then, according to Lemma 3.3. the sequence  $(\|B^{-k} m\|_*)_{k \geq 0}$  is strictly increasing; so we have:

$$\begin{aligned} \|m\|_*^{r+2} |\Phi_m^+(g)| &= \|m\|_*^{r+2} |\Phi_m^-(g)| \\ &= \|m\|_*^{r+2} \left| -\widehat{\sum_{k < 0} g \circ \gamma^k(m)} \right| \\ &= \|m\|_*^{r+2} \left| \widehat{\sum_{l > 0} g \circ \gamma^{-l}(m)} \right| \\ &\leq \sum_{l > 0} \|m\|_*^{r+2} |\widehat{g}(B^{-l} m)| \\ &\leq \sum_{l > 0} \|B^{-l} m\|_*^{r+2} |\widehat{g}(B^{-l} m)| \\ &\leq \sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} \|\alpha\|_*^{r+2} |\widehat{g}(\alpha)| \end{aligned}$$

The norms  $\|\cdot\|$  and  $\|\cdot\|_*$  being equivalents on  $V$ , there are real numbers  $\eta > 0$  and  $\mu > 0$  such as  $\eta \|x\|_* \leq \|x\| \leq \mu \|x\|_*$  for all  $x \in V$ . Consequently:

$$\forall r \in \mathbb{N}, \forall m \in \mathbb{Z}^d \setminus \{0\}, \|m\|_*^{r+2} |\Phi_m^+(g)| \leq \frac{\|g\|_{1,r+2}}{\eta^{r+2}} \quad \text{and} \quad \|m\|_*^r |\Phi_m^+(g)| \leq \frac{\|g\|_{1,r+2}}{\eta^{r+2} \|m\|_*^2}$$

where  $\|g\|_{1,r+2} = \underbrace{|\widehat{g}(0)|}_{=0} + \sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} \|\alpha\|^{r+2} |\widehat{g}(\alpha)| < \infty$ .

This implies that  $\forall r \in \mathbb{N}$ ,  $\lim_{\|m\|_* \rightarrow +\infty} \|m\|_*^r |\Phi_m^+(g)| = 0$ . So  $f = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \Phi_m^+(g) \Theta_m \in E$  and according to

the first point of Lemma 3.3. for all  $m \in \mathbb{Z}^d \setminus \{0\}$ ,

$$\begin{aligned} \widehat{f \circ \gamma}(m) &= e^{2i\pi \langle b_1, Bm \rangle} \widehat{f}(Bm) \\ &= e^{2i\pi \langle b_1, Bm \rangle} \Phi_{Bm}^+(g) \\ &= \sum_{k \geq 0} e^{2i\pi \langle b_1, Bm \rangle} \widehat{g \circ \gamma^k}(Bm) \\ &= \sum_{k \geq 0} \widehat{g \circ \gamma^{k+1}}(m) \end{aligned}$$

So:  $\forall m \in \mathbb{Z}^d \setminus \{0\}$ ,  $(f - f \circ \gamma)(m) = \sum_{k \geq 0} \widehat{g \circ \gamma^k}(m) - \sum_{k \geq 0} \widehat{g \circ \gamma^{k+1}}(m) = \widehat{g}(m)$ .

We also have  $(f - f \circ \gamma)(0) = 0 = \widehat{g}(0)$ . Which proves that  $g = \delta(f) \in \delta(E)$ . We deduce that  $\mathcal{H} \subset \delta(E)$ . Finally  $\delta(E) = \mathcal{H}$  which shows that this space is closed in the Fréchet space  $E$ .

The linear operator:  $L : \delta(E) \rightarrow E$ ,  $g \mapsto f = \sum_{m \in \mathbb{Z}^d} \Phi_m^+(g) \Theta_m$  is continuous. Indeed,

$$\forall r \in \mathbb{N}, \forall g \in \delta(E), \|L(g)\|_{1,r} = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \|m\|^r |\Phi_m^+(g)| \leq \left[ \left( \frac{\mu}{\eta} \right)^{r+2} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|m\|^2} \right] \|g\|_{1,r+2}$$

In addition, we have:  $\forall g \in \delta(E)$ ,  $\delta \circ L(g) = g$ . □

## References

- [1] A. AVILA AND A. KOCSARD, Cohomological equations and invariant distributions for minimal circle diffeomorphisms. *Duke Mathematical Journal*, Vol. 158 (2011), DOI 10.1215 / 00127094 - 1345662.
- [2] CHOU CHIN-CHENG, *Séries de Fourier et Théorie des distributions*. Editions Scientifiques (1983).
- [3] A. DEGHAN-NEZHAD AND A. EL KACIMI ALAOU, Équations cohomologiques de flots riemanniens et de difféomorphismes d'Anosov. *Journal of the Mathematical Society of Japan*, Vol. 59, N° 4 (2007), 1105 - 1134.
- [4] A. EL KACIMI ALAOU, Cohomologie des groupes discrets à valeurs dans un Fréchet. Prépublication UPHF (Mars 2022).  
<http://perso.numericable.fr/azizelkacimi/CohomologieDesGroupes.pdf>

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