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# EXTRINSICALLY HOMOGENEOUS LAGRANGIAN SUBMANIFOLDS OF THE PSEUDO-NEARLY KÄHLER $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$

MATEO ANARELLA

**ABSTRACT.** We consider the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and we study its Lagrangian submanifolds. We provide examples of Lagrangian submanifolds which do not have an analogue in  $\mathbb{S}^3 \times \mathbb{S}^3$ . We also provide an expression for the isometry group of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  with the pseudo-Riemannian nearly Kähler metric. The main result is a complete classification of extrinsically homogeneous Lagrangian submanifolds in this space.

## 1. INTRODUCTION

Kähler geometry can be seen as the intersection of three areas of differential geometry: symplectic, Riemannian and complex geometry. That is, a Kähler manifold carries a Riemannian metric  $g$ , a symplectic form  $\omega$  and a complex structure  $J$  such that

$$g(JX, JY) = g(X, Y), \quad \omega(X, Y) = g(JX, Y).$$

Equivalently, an almost Hermitian manifold  $(M, g, J)$  is Kähler if and only if  $\nabla J \equiv 0$ , where  $\nabla$  is the Levi-Civita connection associated to  $g$ .

There are only two spheres admitting an almost complex structure:  $\mathbb{S}^2$  and  $\mathbb{S}^6$ . Surprisingly, the unit six-sphere  $\mathbb{S}^6$  with its canonical almost complex structure, inherited from the octonion product, is not Kähler. Instead, it is what is known as a nearly Kähler manifold.

A nearly Kähler manifold is an almost Hermitian manifold  $(M, J, g)$  such that  $\nabla J$  is a skew symmetric  $(2, 1)$ -tensor. If moreover,  $\nabla_X J \neq 0$  for *all*  $X \in \mathfrak{X}(M)$ , then we say that  $M$  is strict nearly Kähler. In six dimensions, this is equivalent to being nearly Kähler but not Kähler. In general, we can think of strict nearly Kähler manifolds as nearly Kähler manifolds from which we cannot extract Kähler factors. Indeed, Gray [9] showed that any complete, simply connected nearly Kähler manifold can be written as  $M_1 \times M_2$  where  $M_1$  is strict nearly Kähler and  $M_2$  is Kähler.

Strict nearly Kähler manifolds turned out to be quite rare. For instance, in [9] Gray also showed that there are no eight-dimensional strict nearly Kähler manifolds. Also, it can be easily proved that two- and four-dimensional nearly Kähler manifolds are automatically Kähler. Later on, Nagy [16] showed that any complete 10-dimensional nearly Kähler manifold is either the product of a six-dimensional nearly Kähler manifold and a Kähler surface, or the twistor space over an eight-dimensional quaternionic Kähler manifold with positive Einstein constant (or positive quaternionic Kähler, for short). All eight-dimensional positive quaternionic Kähler manifolds are classified, being the symmetric spaces  $\mathbb{H}P^2$ ,  $Gr_2(\mathbb{C}^4)$  and  $G_2/SO(4)$ . Therefore, all 10-dimensional strict nearly Kähler manifolds are classified, as described in [12].

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Six-dimensional strict nearly Kähler manifolds are of particular interest, since they are the lowest-dimensional non-Kähler examples we encounter. In fact, Nagy [15] showed that any nearly Kähler manifold is a Riemannian product whose factors are six-dimensional nearly Kähler manifolds, certain homogeneous nearly Kähler manifolds or twistor spaces over positive quaternionic Kähler manifolds.

All six-dimensional Riemannian *homogeneous* strict nearly Kähler manifolds were classified by Butruille in [4], these being

- $\mathbb{S}^6 = G_2/SU(3)$ ,
- $\mathbb{S}^3 \times \mathbb{S}^3 = (SU(2) \times SU(2) \times SU(2))/\Delta SU(2)$ ,
- $\mathbb{C}P^3 = Sp(2)/(U(1) \times SU(2))$ ,
- $F(\mathbb{C}^3) = SU(3)/(U(1) \times U(1))$ .

Recently, Foscolo and Haskins [7] showed the existence of six-dimensional strict nearly Kähler manifolds which are not homogeneous.

By means of a  $T$ -dual construction, Kath [10] and Schäfer [18] provided six pseudo-Riemannian analogues of the spaces in the list above. However, this list does not provide a full classification of six-dimensional homogeneous pseudo-Riemannian nearly Kähler manifolds (or pseudo-nearly Kähler for short). In fact, Alekseevsky et al. constructed in [1] an example of a homogeneous pseudo-nearly Kähler six-manifold which is not a  $T$ -dual of a Riemannian one.

In this article, we focus on the analogue of  $\mathbb{S}^3 \times \mathbb{S}^3$ , which is the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . In [2], the authors studied Lagrangian submanifolds of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and gave a classification up to congruence of all totally geodesic Lagrangian submanifolds. There, the authors divided Lagrangian submanifolds into four types, which depend on their behavior with respect to a specific almost product structure.

In this paper, we provide an expression of the isometry group of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  which, to the knowledge of the author, cannot be found in the literature.

**Theorem 1.** *The isometry group of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  is  $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) \rtimes (\mathbb{Z}_2 \times S_3)$ , where  $S_3$  is the symmetric group of order 6.*

Moreover, we study extrinsically homogeneous Lagrangian submanifolds of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . That is, those Lagrangian submanifolds  $f : M \rightarrow SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  such that there exist a Lie group  $H$  acting transitively by isometries on  $f(M)$ . To simplify things we take  $H$  to be a Lie subgroup of the connected component of the identity of the isometry group, i.e.

$$\text{Iso}_o(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}).$$

Orbits of Lie subgroups of the isometry group are among the most natural submanifolds of homogeneous spaces. They are tightly linked to submanifolds with constant sectional curvature, totally geodesic submanifolds and submanifolds with parallel second fundamental form (see for instance [5, 6]). In particular, Lagrangian submanifolds of six-dimensional (pseudo-)nearly Kähler manifolds are automatically minimal.

In Theorem 2 we provide a full classification of extrinsically Lagrangian homogeneous submanifolds of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Among the submanifolds in the classification, we can find the three totally geodesic examples of [2] and two submanifolds with constant sectional curvature. These five submanifolds have analogues in the classification of extrinsically homogeneous Lagrangian submanifolds of  $\mathbb{S}^3 \times \mathbb{S}^3$ , found in [3].

In addition, we obtain three more Lagrangian submanifolds without an analogue in the Riemannian case. The first two are immersions of a space form with constant sectional curvature  $-\frac{3}{2}$ .

Moreover, the second example is actually an infinite family of Lagrangian immersions. Finally, the last example does not seem to have any intrinsic or extrinsic invariant besides of being Lagrangian and minimal.

**Theorem 2.** *Let  $f : (M, g) \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  be an extrinsically homogeneous Lagrangian immersion into the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Then  $f(M)$  is congruent to an open subset of the image of one of the following embeddings, whose image is the orbit of  $(\mathrm{Id}_2, \mathrm{Id}_2)$  by  $H \subset \mathrm{Iso}_o(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}))$ :*

$(M, g)$	$f$	$H$	Isotropy	Remarks
$(\mathrm{SL}(2, \mathbb{R}), \frac{2}{3}\langle, \rangle)$	$u \mapsto (u, u)$	$\mathrm{SL}(2, \mathbb{R})$	0	Totally geodesic $K = -\frac{3}{2}$
$(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}^+)$	$u \mapsto (u, \mathbf{i}u\mathbf{i})$	$\mathrm{SL}(2, \mathbb{R})$	0	Totally geodesic
$(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}^-)$	$u \mapsto (u, -\mathbf{k}u\mathbf{k})$	$\mathrm{SL}(2, \mathbb{R})$	0	Totally geodesic
$(\mathrm{PSL}(2, \mathbb{R}), \frac{8}{3}\langle, \rangle)$	$[u] \mapsto (\mathbf{i}u\mathbf{i}u^{-1}, \mathbf{j}u\mathbf{j}u^{-1})$	$\mathrm{SL}(2, \mathbb{R})$	$\mathbb{Z}_2$	$K = -\frac{3}{8}$
$\mathbb{R}_1^3/\mathbb{Z}$	$(u, v, w) \mapsto (e^{vi}e^{-uk}, e^{wj}e^{-uk})$	$\mathbb{R}^2 \times \mathbb{S}^1$	0	$K = 0$
$(\mathbb{R}^3, \hat{g})$	$\iota$	$\mathbb{R} \times_{\varphi_o} \mathbb{R}^2$	0	$K = -\frac{3}{2}$
$(\mathbb{R}^3/H_\lambda, g_\lambda)$	$f_\lambda$	$(\mathbb{R} \times_{\varphi_1} \mathbb{R}^2)/H_\lambda$	0	$K = -\frac{3}{2}$
$(\mathbb{R}^3, \tilde{g})$	$j$	$\mathbb{R} \times_{\varphi_2} \mathbb{R}^2$	0	

Here  $K$  is the sectional curvature of  $f(M)$  and  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the matrices

$$\mathbf{i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore,  $\langle, \rangle$  is the metric given in (1),  $g_{\kappa, \tau}^+$  and  $g_{\kappa, \tau}^-$  are Berger-like metrics stretched in a spacelike and timelike direction, respectively;  $\hat{g}, \iota$  and  $\varphi_o$  are given in Example 25;  $g_\lambda, f_\lambda, H_\lambda$  and  $\varphi_1$  are given in Example 26;  $\tilde{g}, j$  and  $\varphi_2$  are given in Example 29.

Conversely, the maps listed in the table above are extrinsically homogeneous Lagrangian submanifolds of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Moreover, all immersions are not congruent to each other, including the different immersions of the family  $f_\lambda$ .

As the definition indicates, classifications of extrinsically homogeneous submanifolds usually follow from classifications of Lie subgroups of the isometry group. However, as opposed to the compact case, an isometry group with non-compact semi-simple Lie algebra might present many difficulties when looking for subgroups, as maximal Lie subalgebras might not be reductive (see [14]). The condition of being reductive simplifies the process considerably, as we can obtain its maximal Lie subalgebras in a simple way (see Theorem 2.1 in [11]). Moreover, when we increase the codimension of the submanifold, the process gets more difficult, as we may have to repeat it several times.

Therefore, in this paper we classify extrinsically homogeneous Lagrangian submanifolds by using the properties of Lagrangian submanifolds of nearly Kähler spaces, and how the Lie subgroup acting on them preserves their structure.

The paper is organized as follows. In Section 2 we give a brief introduction to the pseudo-nearly Kähler structure of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  and we prove Theorem 1. In Section 3 we state some properties of Lagrangian submanifolds of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . In Section 4 we provide examples of extrinsically homogeneous Lagrangian submanifolds of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Finally, in Section 5 we prove Theorem 2.

## 2. THE PSEUDO-NEARLY KÄHLER $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$

The nearly Kähler structure of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  is given in detail in [2] and [8]. Here we recall some of the structure necessary for this article.

**2.1. The manifold  $\mathrm{SL}(2, \mathbb{R})$ .** Consider the real vector space of  $2 \times 2$  real matrices  $M(2, \mathbb{R})$  with the indefinite inner product  $\langle, \rangle$  given by

$$\langle a, b \rangle = -\frac{1}{2} \mathrm{Trace}(\mathrm{adj}(a)b). \quad (1)$$

The real special linear group  $\mathrm{SL}(2, \mathbb{R})$  can be defined as

$$\mathrm{SL}(2, \mathbb{R}) = \{a \in M(2, \mathbb{R}) : \langle a, a \rangle = -1\}. \quad (2)$$

Identifying  $(M(2, \mathbb{R}), \langle, \rangle)$  with  $\mathbb{R}_2^4$  we obtain that  $\mathrm{SL}(2, \mathbb{R})$  is isometric to the three-dimensional anti-de Sitter space  $H_1^3(-1)$ , defined as

$$H_1^3(-1) = \{x \in \mathbb{R}_2^4 : -x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1\}.$$

Hence,  $(\mathrm{SL}(2, \mathbb{R}), \langle, \rangle)$  is a Lorentzian manifold with constant sectional curvature  $-1$ . From (2) it follows that the tangent space of  $\mathrm{SL}(2, \mathbb{R})$  at a matrix  $a$  is the orthogonal space  $a^\perp$ , which can be written as

$$T_a \mathrm{SL}(2, \mathbb{R}) = a^\perp = \{a\alpha : \alpha \in \mathfrak{sl}(2, \mathbb{R})\}.$$

Here  $\mathfrak{sl}(2, \mathbb{R})$  is the Lie algebra of  $\mathrm{SL}(2, \mathbb{R})$ , consisting of all the matrices in  $M(2, \mathbb{R})$  with vanishing trace.

Similar to the tangent space of the three-sphere  $\mathbb{S}^3$ , the tangent space of  $\mathrm{SL}(2, \mathbb{R})$  at  $\mathrm{Id}_2$  is spanned by the split-quaternions  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ , given by

$$\mathbf{i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3)$$

We define the frame  $\{X_i\}_i$  on  $\mathrm{SL}(2, \mathbb{R})$  by

$$X_1(a) = a\mathbf{i}, \quad X_2(a) = a\mathbf{j}, \quad X_3(a) = a\mathbf{k}. \quad (4)$$

This is an orthogonal frame on  $\mathrm{SL}(2, \mathbb{R})$  with

$$\langle X_1, X_1 \rangle = \langle X_2, X_2 \rangle = 1, \quad \langle X_3, X_3 \rangle = -1.$$

Given  $\alpha$  and  $\beta$  in  $\mathfrak{sl}(2, \mathbb{R})$ , we have

$$\alpha\beta = \alpha \times \beta + \langle \alpha, \beta \rangle \mathrm{Id}_2, \quad (5)$$

where  $\alpha \times \beta = \frac{1}{2}(\alpha\beta - \beta\alpha)$ .

**2.2. The homogeneous nearly Kähler structure on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ .** Consider the triple product  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  with the product metric arising from  $\langle, \rangle$  given in (1). Let  $\pi: \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  be the submersion given by

$$\pi(a, b, c) = (ac^{-1}, bc^{-1}).$$

Let  $g$  be the metric on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  such that  $\pi$  is a pseudo-Riemannian submersion. Then,  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), g)$  is a pseudo-Riemannian homogeneous manifold, expressed as

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) = \frac{\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})}{\Delta \mathrm{SL}(2, \mathbb{R})},$$

where  $\Delta \mathrm{SL}(2, \mathbb{R}) = \{(a, a, a) : a \in \mathrm{SL}(2, \mathbb{R})\}$ .

We define an almost complex structure  $J$  on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  by

$$J(a\alpha, b\beta) = \frac{1}{\sqrt{3}}(a(\alpha - 2\beta), b(2\alpha - \beta)), \quad (6)$$

for  $\alpha, \beta \in \mathfrak{sl}(2, \mathbb{R})$ . This almost complex structure satisfies

$$J^2 = -\mathrm{Id}, \quad g(X, Y) = \langle X, Y \rangle + \langle JX, JY \rangle, \quad (7)$$

where  $\langle, \rangle$  the product metric of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  associated to the metric given in (1). The last equality implies that  $J$  is compatible with  $g$ . Hence we may say that  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), g, J)$  is an almost Hermitian manifold. Using (7), we get the following explicit expression for  $g$ :

$$g((a\alpha, b\beta), (a\gamma, b\delta)) = \frac{2}{3}\langle (a\alpha, b\beta), (a\gamma, b\delta) \rangle - \frac{1}{3}\langle (a\beta, b\alpha), (a\gamma, b\delta) \rangle,$$

with  $\alpha, \beta, \gamma, \delta \in \mathfrak{sl}(2, \mathbb{R})$ .

We denote by  $\tilde{\nabla}$  the Levi-Civita connection on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  associated with the pseudo-nearly Kähler metric  $g$ . We denote the covariant derivative of  $J$  by  $G$ , which turns out to be skew symmetric. Hence,  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), g, J)$  is a pseudo-nearly Kähler manifold. As such,  $G$  satisfies

$$g(G(X, Y), Z) + g(G(X, Z), Y) = 0, \quad G(X, JY) + JG(X, Y) = 0, \quad (8)$$

where  $X, Y$  and  $Z$  are vector fields on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . These equations imply that

$$g(G(X, Y), JZ) + g(G(X, Z), JY) = 0. \quad (9)$$

Consider now the almost product structure  $P$  on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by

$$P(a\alpha, b\beta) = (a\beta, b\alpha). \quad (10)$$

The tensor  $P$  has the following properties:

$$\begin{aligned} P^2 &= \mathrm{Id}, & g(PX, PY) &= g(X, Y), \\ PJ &= -JP, & g(PX, Y) &= g(X, PY), \\ PG(X, Y) + G(PX, PY) &= 0. \end{aligned} \quad (11)$$

for any pair of vector fields  $X, Y$  on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . The covariant derivative of  $P$  can be expressed in terms of  $P, J$  and  $G$  in the following way:

$$(\tilde{\nabla}_X P)Y = \frac{1}{2}(JG(X, PY) + JPG(X, Y)). \quad (12)$$

Six-dimensional nearly Kähler manifolds carry a distinguished constant, known as the *type*. In particular,  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), g, J)$  has type  $-\frac{2}{3}$ . Namely,  $G$  satisfies the formula

$$\begin{aligned} g(G(X, Y), G(Z, W)) &= -\frac{2}{3}(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)) \\ &\quad + g(JX, Z)g(Y, JW) - g(JX, W)g(Y, JZ). \end{aligned} \quad (13)$$

The curvature tensor  $\tilde{R}$  of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  associated to the Levi-Civita connection  $\tilde{\nabla}$  of the pseudo-nearly Kähler metric  $g$  is given by

$$\begin{aligned} \tilde{R}(U, V)W &= -\frac{5}{6}(g(V, W)U - g(U, W)V) \\ &\quad -\frac{1}{6}(g(JV, W)JU - g(JU, W)JV - 2g(JU, V)JW) \\ &\quad -\frac{2}{3}(g(PV, W)PU - g(PU, W)PV \\ &\quad \quad + g(JPV, W)JPU - g(JPU, W)JPV). \end{aligned} \quad (14)$$

**2.3. The manifold  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  as a pseudo-Riemannian product.** Take the product metric  $\langle, \rangle$  on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  arising from the Lorentzian metric on  $\mathrm{SL}(2, \mathbb{R})$  given in (1). The nearly Kähler metric  $g$  and  $\langle, \rangle$  are related by

$$\langle X, Y \rangle = 2g(X, Y) + g(X, PY), \quad (15)$$

Products of (pseudo-)Riemannian manifolds carry a canonical almost product structure  $Q$  compatible with the product metric, given by

$$Q(X_1, X_2) = (-X_1, X_2).$$

Similar to the relation between the product metric and the nearly Kähler metric,  $P$  and  $Q$  are related by

$$QX = -\frac{1}{\sqrt{3}}(2PJX - JX). \quad (16)$$

Consider the immersion of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  with the product metric  $\langle, \rangle$  into  $M(2, \mathbb{R}) \times M(2, \mathbb{R}) \cong \mathbb{R}_4^8$ . Let  $D$  be the Euclidean connection of  $\mathbb{R}_4^8$ . The Gauss formula splits  $D$  into tangent and normal parts:

$$D_X Y = \nabla_X^E Y + h^E(X, Y),$$

with  $X, Y$  vector fields on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . The connection  $\nabla^E$  is the Levi-Civita connection associated to the product metric  $\langle, \rangle$  and  $h^E$  is the so-called second fundamental form. We have an expression for  $\nabla^E$  in terms of the connection  $\tilde{\nabla}$  associated to  $g, J, P$  and  $G$ :

$$\nabla_X^E Y = \tilde{\nabla}_X Y + \frac{1}{2}(JG(X, PY) + JG(Y, PX)). \quad (17)$$

For  $(a, b) \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  we have

$$h^E(X, Y)_{(a,b)} = \frac{1}{2}\langle X, Y \rangle_{(a,b)} + \frac{1}{2}\langle Y, QX \rangle_{(-a,b)}.$$

Hence,

$$D_X Y = \nabla_X^E Y + \frac{1}{2}\langle X, Y \rangle_{(a,b)} + \frac{1}{2}\langle Y, QX \rangle_{(-a,b)}. \quad (18)$$

**2.4. The isometry group.** The connected component of the identity of the isometry group of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  is

$$\mathrm{Iso}_o(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}),$$

where an element  $\phi_{(a,b,c)}$  acts on a point  $(p, q)$  by  $\phi_{(a,b,c)}(p, q) = (apc^{-1}, bqc^{-1})$ .

The isometries  $\phi_{(a,b,c)}$  preserve  $P$  and  $J$ , in the sense that  $d\phi_{(a,b,c)} \circ J = J \circ d\phi_{(a,b,c)}$  and  $d\phi_{(a,b,c)} \circ P = P \circ d\phi_{(a,b,c)}$ . These isometries are not the only ones that satisfy these properties. Given three matrices  $a, b$  and  $c$  with determinant  $-1$ , the map  $(p, q) \mapsto (apc^{-1}, bqc^{-1})$  is also an isometry that preserves  $J$  and  $P$ .

Denote by  $\mathrm{SL}^\pm(2, \mathbb{R})$  the group of all matrices in  $M(2, \mathbb{R})$  with determinant  $\pm 1$ . We can write any matrix of  $\mathrm{SL}^\pm(2, \mathbb{R})$  as  $i^k a$ , where  $i$  is the matrix given in (3),  $k \in \{0, 1\}$  and  $a \in \mathrm{SL}(2, \mathbb{R})$ . Thus, we have  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) \rtimes \mathbb{Z}_2 \subset \mathrm{Iso}(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}))$ .

Permutations of elements of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  also give rise to isometries of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ :

$$\begin{aligned} \Psi_{0,0}(p, q) &= (p, q), & \Psi_{1,0}(p, q) &= (q, p), \\ \Psi_{0,2\pi/3}(p, q) &= (pq^{-1}, q^{-1}), & \Psi_{1,2\pi/3}(p, q) &= (q^{-1}, pq^{-1}), \\ \Psi_{0,4\pi/3}(p, q) &= (qp^{-1}, p^{-1}), & \Psi_{1,4\pi/3}(p, q) &= (p^{-1}, qp^{-1}). \end{aligned} \quad (19)$$

These isometries are not included in  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Moreover, each one of these is in a different connected component of  $\mathrm{Iso}(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}))$  and satisfies

$$J \circ d\Psi_{\kappa, \tau} = (-1)^\kappa d\Psi_{\kappa, \tau} \circ J, \quad P \circ d\Psi_{\kappa, \tau} = d\Psi_{\kappa, \tau} \circ (\cos \tau P + \sin \tau JP).$$

Later on, we prove that these are *all* the isometries of the nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ .

A key result in the classification of Riemannian homogeneous nearly Kähler manifolds by Butrille [4] is the existence of a unique nearly Kähler structure on  $\mathbb{S}^3 \times \mathbb{S}^3$ . Consequently, the almost complex structure on  $\mathbb{S}^3 \times \mathbb{S}^3$  is unique up to sign.

However, in [18] it is shown that  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  has a unique *left invariant* nearly Kähler structure, which does not necessarily imply that  $J$  is unique up to sign. Therefore, by isometry group of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  we mean the set of all diffeomorphisms preserving the almost Hermitian structure. That is, those isometries  $\mathcal{F}$  of  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), g)$  that preserve  $J$ , i.e.  $\mathcal{F}_*J = \pm J\mathcal{F}_*$ .

**Lemma 3.** *Any almost product structure  $\tilde{P}$  on  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  that satisfies all properties in (11) and (14) is given by*

$$\tilde{P} = \cos \eta P + \sin \eta JP, \quad (20)$$

where  $\eta$  is equal to  $\frac{2\pi}{3}$  or  $\frac{4\pi}{3}$  and  $P$  is given in (10). Conversely, if  $\tilde{P}$  is given by (20), then it satisfies (11) and (14). Moreover, it also satisfies (12).

*Proof.* A similar result was proven for the Riemannian analogue  $\mathbb{S}^3 \times \mathbb{S}^3$  in [13]. We can follow the same proof for  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ .  $\square$

The following lemma is a well known result.

**Lemma 4.** *Let  $\{\alpha_1, \alpha_2, \alpha_3\}$  and  $\{\beta_1, \beta_2, \beta_3\}$  be bases of  $\mathfrak{sl}(2, \mathbb{R})$ . If  $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$  for all  $i, j \in \{1, 2, 3\}$ , then there exists a matrix  $c$  in  $\mathrm{SL}^\pm(2, \mathbb{R})$  such that  $c\alpha_i c^{-1} = \beta_i$ . In other words,  $\mathrm{SL}^\pm(2, \mathbb{R})/\mathbb{Z}_2$  is isomorphic to  $\mathrm{SO}(2, 1)$ .*

With these lemmas we prove the following statement.

**Theorem 1.** *The isometry group of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  is the semi-direct product  $(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) \rtimes (\mathbb{Z}_2 \times S_3)$ , where  $S_3$  is the symmetric group of order 6 generated by  $\{\Psi_{1,0}, \Psi_{1,4\pi/3}\}$*

**Remark 5.** An element  $(a, b, c, \Psi, k)$  acts on a point  $(p, q)$  by

$$(a, b, c, \Psi, k) \cdot (p, q) = \Psi \circ \phi_{i^k(a,b,c)}(p, q).$$

*Proof of Theorem 1.* We already know that the given group is included in  $\mathrm{Iso}(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}))$ . Here we show the oposite inclusion.

Let  $\mathcal{F}$  be an isometry of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Thus, there exist  $\kappa_0 \in \{0, 1\}$  satisfying

$$\mathcal{F}_*J = (-1)^{\kappa_0} J\mathcal{F}_*.$$

As  $\mathcal{F}_*P(\mathcal{F}^{-1})_*$  is an almost product structure satisfying (11) and (14), Lemma 3 implies that

$$\mathcal{F}_*P(\mathcal{F}^{-1})_* = \cos \tau_0 P + \sin \tau_0 JP,$$

for some  $\tau_0 \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ . By taking the composition  $\mathcal{F} \circ \Psi_{\kappa_0, (-1)^{\kappa_0} \tau_0}$  we may assume that  $\mathcal{F}$  preserves  $P$  and  $J$ . Let  $(p_o, q_o) \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  such that  $\mathcal{F}(\mathrm{Id}_2, \mathrm{Id}_2) = (p_o, q_o)$ . Then by taking the composition  $\mathcal{F} \circ \phi_{(p_o^{-1}, q_o^{-1}, \mathrm{Id}_2)}$  we may also assume that  $\mathcal{F}(\mathrm{Id}_2, \mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$ .



Let  $\alpha \in \mathfrak{sl}(2, \mathbb{R})$ . Then we write  $\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 0) = (\beta, \gamma)$ . Since  $\mathcal{F}$  preserves  $P$  we know that  $\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(0, \alpha) = (\gamma, \beta)$ . We compute

$$\begin{aligned} \mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}J(\alpha, 0) &= \frac{1}{\sqrt{3}}\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 2\alpha) \\ &= \frac{1}{\sqrt{3}}\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 0) + \frac{1}{\sqrt{3}}\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(0, 2\alpha) \\ &= \frac{1}{\sqrt{3}}(\beta, \gamma) + \frac{2}{\sqrt{3}}(\gamma, \beta) \\ &= \frac{1}{\sqrt{3}}(\beta + 2\gamma, 2\beta + \gamma). \end{aligned}$$

On the other hand, as  $\mathcal{F}$  preserves  $J$ , we have that  $\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}J(\alpha, 0)$  equals

$$\begin{aligned} J\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 0) &= J(\beta, \gamma) \\ &= \frac{1}{\sqrt{3}}(\beta - 2\gamma, 2\beta - \gamma). \end{aligned}$$

Therefore, we obtain that  $\gamma = 0$ . Moreover, since  $\mathcal{F}$  is an isometry, we deduce that  $\mathcal{F}_*$  maps a set  $\{(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0)\}$  to a set  $\{(\beta_1, 0), (\beta_2, 0), (\beta_3, 0)\}$  such that  $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$ .

Now, using Lemma 4, we may compose  $\mathcal{F}$  with an isometry of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \rtimes \mathbb{Z}_2$  to assume that  $\mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 0) = (\alpha, 0)$  for all  $\alpha \in \mathfrak{sl}(2, \mathbb{R})$ . Since  $\mathcal{F}$  preserves  $P$ , we have that

$$\begin{aligned} \mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, \beta) &= \mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(\alpha, 0) + \mathcal{F}_{*(\text{Id}_2, \text{Id}_2)}(0, \beta) \\ &= (\alpha, 0) + P(\beta, 0) \\ &= (\alpha, \beta). \end{aligned}$$

Since isometries are determined by a point and the differential at that point, the argument above shows that  $\mathcal{F}^{-1}$  is in  $(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})) \rtimes (\mathbb{Z}_2 \times S_3)$ , hence  $\mathcal{F}$  also belongs to this group.  $\square$

### 3. LAGRANGIAN SUBMANIFOLDS OF $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$

Let  $f : M \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  be a non-degenerate pseudo-Riemannian immersion. The Gauss formula relates the Levi-Civita connection on  $M$  and the nearly Kähler connection  $\tilde{\nabla}$  by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (21)$$

where  $X, Y$  are vector fields on  $M$  and  $h$  is a symmetric bilinear normal form called the second fundamental form. On the other hand, the Weingarten formula gives a relation between  $\tilde{\nabla}$  and the normal connection:

$$\tilde{\nabla}_X \xi = -S_\xi X + \nabla_X^\perp \xi,$$

where  $X$  and  $\xi$  are tangent and normal vector fields on  $M$ , respectively. Given  $\xi$  a normal vector field to  $M$ , the tensor  $S_\xi$  is called the shape operator. It is linear at  $\xi$  and  $X$  and symmetric (with respect to  $g$ ), related to  $h$  by  $g(S_\xi X, Y) = g(h(X, Y), \xi)$ .

A totally geodesic submanifold is a submanifold whose geodesics are also geodesics of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ . This is equivalent to a vanishing second fundamental form.

The most fundamental equations of submanifold theory are the Gauss and Codazzi equations, which give expressions for the tangent and normal parts of the curvature tensor, respectively.

Namely,

$$\begin{aligned} \left(\tilde{R}(X, Y)Z\right)^\top &= R(X, Y)Z + S_{h(X, Z)}Y - S_{h(Y, Z)}X, \\ \left(\tilde{R}(X, Y)Z\right)^\perp &= (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \end{aligned}$$

where  $(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$  and  $\tilde{R}, R$  are the curvature tensors of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and  $M$ , respectively.

Assume now that  $M$  is a Lagrangian submanifold. That is, a non-degenerate three-dimensional submanifold such that  $J$  maps tangent spaces of  $M$  into normal spaces, and vice versa.

In [18] we find properties of the tensor  $G$  of a pseudo-nearly Kähler manifold and the second fundamental form of a Lagrangian submanifold:

$$\begin{aligned} g(G(X, Y), Z) &= 0, \\ g(h(X, Y), JZ) &= g(h(X, Z), JY), \\ S_{JX}Y &= Jh(X, Y) \end{aligned} \tag{22}$$

Lagrangian submanifolds are particularly interesting, since their behavior with respect to the almost Hermitian structure, and because of the following property, also found in [18]:

**Proposition 6.** *Any Lagrangian submanifold of a six-dimensional strictly pseudo-nearly Kähler manifold is orientable and minimal.*

The tangent bundle of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  splits into the tangent and normal bundles of a submanifold  $M$ . Moreover, if  $M$  is Lagrangian, we have that  $TSL(2, \mathbb{R}) \times SL(2, \mathbb{R}) = TM \oplus JTM$ . Hence, there exist two endomorphisms  $A, B : TM \rightarrow TM$  such that  $P|_M = A + JB$ , where  $P$  is the almost product structure given in (10). From Equation (11) we deduce that  $A$  and  $B$  are symmetric with respect to  $g$ , commute with each other and  $A^2 + B^2 = \text{Id}$ . Then, the Gauss and Codazzi equations follow from (14) as:

$$\begin{aligned} R(X, Y)Z &= -\frac{5}{6}(g(Y, Z)X - g(X, Z)Y) \\ &\quad - \frac{2}{3}(g(AY, Z)AX - g(AX, Z)AY + g(BY, Z)BX \\ &\quad - g(BX, Z)BY) - S_{h(X, Z)}Y + S_{h(Y, Z)}X, \end{aligned} \tag{23}$$

$$\begin{aligned} (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) &= -\frac{2}{3}(g(AY, Z)JBX - g(AX, Z)JBY \\ &\quad - g(BY, Z)JAX + g(BX, Z)JAY). \end{aligned} \tag{24}$$

A  $\Delta_i$ -orthonormal basis of  $\mathbb{R}_1^3$  is a basis  $\{e_1, e_2, e_3\}$  such that the matrix of inner products is given by  $\Delta_i$ , where

$$\Delta_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given a  $\Delta_i$ -orthonormal basis of  $T_p M$  we can extend this concept to what we call a  $\Delta_i$ -orthonormal frame.

Given a  $\Delta_i$ -orthonormal frame  $\{E_1, E_2, E_3\}$  on a Lagrangian submanifold  $M$  of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ , by (22) we have that  $G(E_i, E_j)$  is a normal vector field. Moreover, by (9) and (13)

we have that  $JG(E_j, E_k) = \alpha E_l$  where  $\alpha$  depends on  $\Delta_i$ , as indicated in the following table.

	$\Delta_1$	$\Delta_2$	$\Delta_3$	
$JG(E_1, E_2)$	$\sqrt{\frac{2}{3}}E_3$	$\sqrt{\frac{2}{3}}E_3$	$\sqrt{\frac{2}{3}}E_3$	(25)
$JG(E_1, E_3)$	$-\sqrt{\frac{2}{3}}E_2$	$-\sqrt{\frac{2}{3}}E_1$	$\sqrt{\frac{2}{3}}E_2$	
$JG(E_2, E_3)$	$-\sqrt{\frac{2}{3}}E_1$	$\sqrt{\frac{2}{3}}E_2$	$\sqrt{\frac{2}{3}}E_1$	

**Lemma 7** ([2]). *Let  $M$  be a Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and  $P$  the almost product structure given in (10). Then there exists a Lagrangian submanifold  $N$  congruent to  $M$  such that the restriction of  $P$  to  $N$  can be written as  $P|_N = A + JB$ , where  $A, B : TN \rightarrow TN$  must have one of the following forms, with respect to a  $\Delta_i$ -orthonormal frame  $\{E_1, E_2, E_3\}$ :*

$$\text{Type I. } A = \begin{pmatrix} \cos 2\theta_1 & 0 & 0 \\ 0 & \cos 2\theta_2 & 0 \\ 0 & 0 & \cos 2\theta_3 \end{pmatrix}, \quad B = \begin{pmatrix} \sin 2\theta_1 & 0 & 0 \\ 0 & \sin 2\theta_2 & 0 \\ 0 & 0 & \sin 2\theta_3 \end{pmatrix},$$

with  $\Delta_i = \Delta_1$  and  $\theta_1 + \theta_2 + \theta_3 = 0$  modulo  $\pi$ ,

$$\text{Type II. } A = \begin{pmatrix} \cos 2\theta_1 & 1 & 0 \\ 0 & \cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \sin 2\theta_1 & -\cot 2\theta_1 & 0 \\ 0 & \sin 2\theta_1 & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with  $\Delta_i = \Delta_2$ ,  $2\theta_1 + \theta_2 = 0$  modulo  $\pi$  and  $\theta_1 \neq 0, \pi/2$ ,

$$\text{Type III. } A = \begin{pmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}, \quad B = \pm \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-4}{3\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

with  $\Delta_i = \Delta_2$ ,

$$\text{Type IV. } A = \begin{pmatrix} \cosh \psi \cos 2\theta_1 & \sinh \psi \sin \theta_2 & 0 \\ -\sinh \psi \sin \theta_2 & \cosh \psi \cos 2\theta_1 & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix},$$

$$B = \begin{pmatrix} \cosh \psi \sin 2\theta_1 & \sinh \psi \cos \theta_2 & 0 \\ -\sinh \psi \cos \theta_2 & \cosh \psi \sin 2\theta_1 & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with  $\Delta_i = \Delta_3$ ,  $2\theta_1 + \theta_2 = 0$  modulo  $\pi$ ,  $\theta_2 \neq 0, \pi$  and  $\psi \neq 0$ .

The functions  $\theta_i$  and  $\psi$  are called the angle functions. Given the extrinsically invariant nature of the type, we say that a Lagrangian submanifold  $M$  is of type I, II, III or IV. Likewise, if  $M$  is of type i we say that  $A$  and  $B$  take type i form on  $M$ .

Suppose  $M$  and  $N$  are congruent Lagrangian submanifolds of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Namely, there exists  $\mathcal{F} \in \text{Iso}(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$  such that  $\mathcal{F}(M) = N$ . The following lemma provides us a way to see how  $P$  projects to  $A$  and  $B$  on  $N$  in terms of  $A$  and  $B$  on  $M$ .

**Lemma 8.** *Let  $M$  be a Lagrangian submanifold of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and  $P$  the almost product structure given in (10). Assume that  $A$  and  $B$  are such that  $P|_M = A + JB$ . Let  $\mathcal{F}$  be an isometry of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Then  $P|_{\mathcal{F}(M)} = \tilde{A} + J\tilde{B}$  with*

$$\begin{aligned}\tilde{A} &= \mathcal{F}_*(\cos \tau A + (-1)^\kappa \sin \tau B)\mathcal{F}_*^{-1} \\ \tilde{B} &= \mathcal{F}_*(-\sin \tau A + (-1)^\kappa \cos \tau B)\mathcal{F}_*^{-1}\end{aligned}$$

where  $\tau$  and  $\kappa$  are such that  $\mathcal{F}_*P = (\cos \tau P + \sin \tau JP)\mathcal{F}_*$  and  $\mathcal{F}_*J = (-1)^\kappa \mathcal{F}_*$ .

Let  $M$  be a Lagrangian submanifold of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  and let  $\{E_1, E_2, E_3\}$  be a  $\Delta_i$ -orthonormal frame such that  $A$  and  $B$  take type I, II, III or IV form. We define the functions  $\omega_{ij}^k$  and  $h_{ij}^k$  by

$$\nabla_{E_i} E_j = \sum_{k=1}^3 \omega_{ij}^k E_k, \quad h(E_i, E_j) = \sum_{k=1}^3 h_{ij}^k J E_k,$$

where  $\nabla$  is the Levi-Civita connection of the submanifold and  $h$  the second fundamental form. The symmetry of  $h$ , the second equation of (22) and the compatibility of the connection with the metric yield many symmetries on  $\omega_{ij}^k$  and  $h_{ij}^k$ , depending on  $\Delta_l$ .

First, for  $\Delta_1$ - and  $\Delta_3$ -orthonormal frames  $\{E_i\}$  we have

$$\delta_k \omega_{ij}^k = -\delta_j \omega_{ik}^j, \quad h_{ij}^k = h_{ji}^k = \delta_j \delta_k h_{ik}^j,$$

where

$$\delta_i = g(E_i, E_i).$$

This implies that  $\omega_{ij}^j = 0$  for all  $i, j = 1, 2, 3$ .

For  $\Delta_2$ -orthonormal frames we obtain

$$\omega_{ij}^k = -\omega_{ik}^{\hat{j}}, \quad h_{ij}^k = h_{ji}^k = h_{ik}^{\hat{j}}$$

where  $\hat{2} = 1$ ,  $\hat{1} = 2$  and  $\hat{3} = 3$ . As before, we have that  $\omega_{33}^3 = 0$ . Also, if  $j = 1$ ,  $k = 2$  or  $j = 2$ ,  $k = 1$  then  $\omega_{ij}^k = 0$ .

Now, the frame  $\{E_1, E_2, E_3\}$  is chosen in terms of  $P$ . Thus, Equation (12) will impose conditions on the functions  $\omega_{ij}^k$  and  $h_{ij}^k$ . We divide between types I, II, III and IV.

**3.1. Lagrangian submanifolds of type I.** Equation (12) for Type I Lagrangian submanifolds yields the following lemma, which was proven in [2].

**Lemma 9.** *Let  $M$  be a Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type I form in Lemma 7 with respect to a  $\Delta_1$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Except for  $h_{12}^3$ , all the components of the second fundamental form are given by the derivatives of the angle functions  $\theta_1, \theta_2$  and  $\theta_3$ :*

$$E_i(\theta_j) = -\delta_i \delta_j h_{jj}^i, \quad (26)$$

where  $\delta_i = g(E_i, E_i)$ . Also

$$h_{ij}^k \cos(\theta_j - \theta_k) = \left(\frac{1}{\sqrt{6}} \delta_k \varepsilon_{ij}^k - \omega_{ij}^k\right) \sin(\theta_j - \theta_k), \quad (27)$$

for  $j \neq k$ .

**3.2. Lagrangian submanifolds of type II.** The covariant derivative of  $P$  in Equation (12) yields the following lemma for type II Lagrangian submanifolds.

**Lemma 10.** *Let  $M$  be a Lagrangian submanifold of the pseudo-nearly Kähler  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  are of type II in Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then  $h(E_1, E_1) = 0$ . Moreover, the derivatives of the angles are given by*

$$\begin{aligned} E_1(\theta_1) &= -h_{11}^1 = 0, & E_2(\theta_1) &= -h_{22}^2, & E_3(\theta_1) &= -h_{12}^3 \\ E_1(\theta_2) &= -h_{33}^2, & E_2(\theta_2) &= -h_{33}^1, & E_3(\theta_2) &= -h_{33}^3. \end{aligned} \quad (28)$$

Furthermore,

$$h_{33}^1 = -2h_{22}^2, \quad h_{33}^2 = -2h_{11}^1 = 0, \quad h_{33}^3 = -2h_{12}^3. \quad (29)$$

*Proof.* Computing Equation (12) with  $X = E_2, Y = E_1$  and looking on the components of  $E_2$  and  $JE_2$  we obtain

$$h_{11}^1 \sin 2\theta_1 = 0, \quad h_{11}^1 \cos 2\theta_1 = 0.$$

Since sine and cosine never vanish at the same time we get that  $h_{11}^1 = 0$ . Same can be done computing Equation (12) with  $X = E_1, Y = E_1$  and  $X = E_3, Y = E_1$  and looking in the directions of  $E_2$  and  $JE_2$ , obtaining  $h_{11}^2 = h_{11}^3 = 0$ . Looking in the direction of  $E_1$  and  $JE_1$  on the same equations, we obtain the derivatives of the function  $\theta_1$ . We derive  $E_i(\theta_2)$  by computing Equation (12) with  $X = E_i$  and  $Y = E_3$ .

We obtain the last statement either from Proposition 6, or from (28) and the fact that  $2\theta_1 + \theta_2 = 0$ .  $\square$

**3.3. Lagrangian submanifolds of type III.** For Lagrangian submanifolds of type III, Equation (12) gives expressions for all functions  $\omega_{ij}^k$ , given in the following lemma.

**Lemma 11.** *Let  $M$  be a Lagrangian submanifold of the pseudo-nearly Kähler  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type III form in Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then we have  $h(E_1, E_1) = 0$  and*

$$\begin{aligned} h_{12}^3 &= \omega_{11}^1 = \omega_{11}^3 = \omega_{33}^2 = 0, \\ \omega_{12}^3 &= \frac{\sqrt{2} + (-1)^{k+1} 3h_{22}^2}{2\sqrt{3}}, \\ \omega_{31}^1 &= \frac{\sqrt{2} + (-1)^{k+1} 12h_{22}^2}{2\sqrt{3}}, \\ \omega_{21}^3 &= -\frac{\sqrt{2} + (-1)^k 6h_{22}^2}{2\sqrt{3}}, \\ \omega_{22}^2 &= \frac{(-1)^k (h_{22}^2 - 3h_{22}^3)}{\sqrt{3}}, \\ \omega_{33}^1 &= \frac{(-1)^{k+1} (4h_{22}^2 - 3h_{22}^3)}{2\sqrt{3}}, \\ \omega_{22}^3 &= \frac{(-1)^{k+1} (9h_{22}^1 - 8h_{22}^2 + 6h_{22}^3)}{6\sqrt{3}}. \end{aligned} \quad (30)$$

where  $(-1)^k$  with  $k \in \{0, 1\}$  is the sign of  $B$  in Lemma 7.

*Proof.* From computing Equation (12) with  $X = E_i$ ,  $Y = E_1$ ,  $i = 1, 2, 3$  and looking at the components in the direction of  $JE_2$  it follows  $h(E_1, E_1) = 0$ . Now, if we compute Equation (12) with  $X = E_i$ ,  $Y = E_1$ ,  $X = E_2$ ,  $Y = E_3$  in the direction on  $E_1$  and  $JE_2$  respectively we get  $h_{12}^3 = \omega_{11}^3 = 0$ . The rest of the equations are obtained in the same way.  $\square$

**3.4. Lagrangian submanifolds of type IV.** From Lemma 7 we know that  $2\theta_1 + \theta_2 = 0$  modulo  $\pi$ . Thus, we write  $\theta_2 = k\pi - 2\theta_1$  where  $k = 0, 1$ . Contrarily to type II Lagrangian submanifolds, here it is necessary to distinguish between  $k = 0$  and  $k = 1$ , since both  $\theta_2$  and  $2\theta_2$  appear in the expressions for  $A$  and  $B$ .

From Proposition 6 we may assume that

$$h_{33}^i = h_{22}^i - h_{11}^i$$

for  $i = 1, 2, 3$ . Now, from Equation (12) we obtain the next lemma.

**Lemma 12.** *Given a Lagrangian submanifold of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  of type IV we have that*

$$\begin{aligned} E_1(\theta_1) &= \frac{h_{22}^1 - h_{11}^1}{2}, & E_2(\theta_1) &= \frac{h_{11}^2 - h_{22}^2}{2}, & E_3(\theta_1) &= \frac{h_{22}^3 - h_{11}^3}{2}, \\ E_1(\psi) &= (-1)^k 2h_{11}^2, & E_2(\psi) &= (-1)^{k+1} 2h_{22}^1, & E_3(\psi) &= (-1)^{k+1} 2h_{12}^3. \end{aligned} \quad (31)$$

Moreover, we produce the following expressions for the functions  $\omega_{ij}^k$ .

$$\begin{aligned} \omega_{11}^2 &= \frac{(-1)^k}{2} (h_{11}^1 + h_{22}^1) \coth \psi, \\ \omega_{22}^1 &= \frac{(-1)^{k+1}}{2} (h_{11}^2 + h_{22}^2) \coth \psi, \\ \omega_{32}^1 &= \frac{(-1)^k}{2} (h_{11}^3 + h_{22}^3) \coth \psi - \frac{1}{\sqrt{6}}. \end{aligned} \quad (32)$$

Also,

$$\begin{aligned} \omega_{11}^3 &= \frac{h_{11}^3 \sin 6\theta_1 + (-1)^k h_{12}^3 \sinh \psi}{\cos 6\theta_1 - \cosh \psi}, \\ \omega_{12}^3 &= \frac{h_{12}^3 \sin 6\theta_1 + (-1)^{k+1} h_{11}^3 \sinh \psi}{\cos(6\theta_1) - \cosh \psi} + \frac{1}{\sqrt{6}}, \\ \omega_{22}^3 &= \frac{h_{22}^3 \sin 6\theta_1 + (-1)^{k+1} h_{12}^3 \sinh \psi}{\cos 6\theta_1 - \cosh \psi}, \\ \omega_{21}^3 &= \frac{h_{12}^3 \sin 6\theta_1 + (-1)^k h_{22}^3 \sinh \psi}{\cos 6\theta_1 - \cosh \psi} - \frac{1}{\sqrt{6}}, \\ \omega_{33}^1 &= \frac{(h_{11}^1 - h_{22}^1) \sin 6\theta_1 + (-1)^k (h_{22}^2 - h_{11}^1) \sinh \psi}{\cos 6\theta_1 - \cosh \psi}, \\ \omega_{33}^2 &= \frac{(h_{11}^2 - h_{22}^2) \sin 6\theta_1 + (-1)^{k+1} (h_{22}^1 - h_{11}^2) \sinh \psi}{\cos 6\theta_1 - \cosh \psi}. \end{aligned} \quad (33)$$

*Proof.* From computing  $(\nabla_{E_1} P)E_1$  and looking at the components in the direction of  $E_2$  and  $JE_2$  we get the equation

$$\begin{pmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ -\sin 2\theta_1 & \cos 2\theta_1 \end{pmatrix} \begin{pmatrix} \cosh \psi (E_1(\psi) - (-1)^k 2h_{11}^2) \\ -\sinh \psi (2E_1(\theta_1) + h_{11}^1 - h_{22}^1) \end{pmatrix} = 0.$$

From  $(\nabla_{E_2}P)E_2$  and  $(\nabla_{E_3}P)E_3$  we derive the rest of the equations in Equation (31) in a similar way.

Having Equation (31) we look at the rest of the components of  $(\nabla_{E_i}P)E_i$  and we furnish the expressions in (32).

By computing  $(\nabla_{E_i}P)E_j$  with  $i \neq j$  we come to linear equations which yield (33).  $\square$

#### 4. EXTRINSICALLY HOMOGENEOUS LAGRANGIAN SUBMANIFOLDS

In this section we first prove that for each case of Lemma 7 there is a unique frame  $\{E_i\}_i$  with respect to which  $P$  takes that particular shape. Consequently, the associated angle functions,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant. Afterwards, we describe the examples given in Theorem 2 and provide a classification for each type of Lagrangian submanifold.

**4.1. The uniqueness of the frames.** We consider each case of Lemma 7 separately.

**4.1.1. Lagrangian submanifolds of type I.** It is straightforward to check that for type I Lagrangian submanifolds the frame  $\{E_1, E_2, E_3\}$  is unique if and only if the functions  $\theta_i$  are all different modulo  $\pi$ . Later on, we will see that if two of them are equal, the submanifold is totally geodesic. Hence, now we focus on the case where all angles are different modulo  $\pi$ .

**Proposition 13.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $\{E_1, E_2, E_3\}$  is the unique  $\Delta_1$ -orthonormal frame such that  $A$  and  $B$  take type I form in Lemma 7. Then the functions  $\theta_i$ ,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant.*

*Proof.* We have to show that  $\theta_{ip} = \theta_{iq}$  for any two points  $p$  and  $q$  in  $M$ . By hypothesis there is a Lie subgroup  $H$  of  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  such that  $H$  acts transitively on  $M$ . Therefore, there exists an isometry  $\phi \in H$  such that  $\phi(p) = q$ . We have that

$$P_p E_{ip} = \cos 2\theta_{ip} E_{ip} + \sin 2\theta_{ip} J_p E_{ip}$$

In subsection 2.4 we saw that isometries in  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  preserve  $P$  and  $J$ . Thus, we apply  $\phi$  to both sides:

$$\begin{aligned} P_q \phi_* E_{ip} &= \phi_* P_p E_{ip} = \phi_*(\cos 2\theta_{ip} E_{ip} + \sin 2\theta_{ip} J_p E_{ip}) \\ &= \cos 2\theta_{ip} \phi_* E_{ip} + \sin 2\theta_{ip} \phi_* J_p E_{ip} \\ &= \cos 2\theta_{ip} \phi_* E_{ip} + \sin 2\theta_{ip} J_q \phi_* E_{ip} \end{aligned}$$

Since  $\{E_i\}_i$  is the unique frame with respect to which  $A$  and  $B$  are diagonal, we have  $\phi_* E_{ip} = E_{iq}$  and  $\theta_{iq} = \theta_{ip}$ .

It follows from  $\phi(M) = M$  that  $\phi$  preserves  $\nabla$  and  $h$ . Thus using a similar argument we get that  $\omega_{ij}^k$  and  $h_{ij}^k$  are constant.  $\square$

**4.1.2. Lagrangian submanifolds of type II.**

**Proposition 14.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take the type II form in Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . If  $\theta_1 \neq \theta_2$  modulo  $\pi$  then the frame is unique up to signs. If instead  $\theta_1 = \theta_2$  there is a unique frame, up to signs, such that  $h_{22}^1 = g(h(E_2, E_2), J E_2) = 0$ . In both cases, the functions  $\theta_i$ ,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant.*

*Proof.* The last statement follows from the uniqueness (even if it is up to sign) of the frame as in the proof of Proposition 13.

Suppose that  $\theta_1 \neq \theta_2$  and that  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$  is a frame on  $M$  such that

$$\begin{aligned} P\tilde{E}_1 &= \cos 2\tilde{\theta}_1 \tilde{E}_1 + \sin 2\tilde{\theta}_1 J\tilde{E}_1, \\ P\tilde{E}_2 &= \tilde{E}_1 + \cos 2\tilde{\theta}_1 \tilde{E}_2 - \cot 2\tilde{\theta}_1 J\tilde{E}_1 + \sin 2\tilde{\theta}_1 J\tilde{E}_2 \\ P\tilde{E}_3 &= \cos 2\tilde{\theta}_2 \tilde{E}_3 + \sin 2\tilde{\theta}_2 J\tilde{E}_3. \end{aligned}$$

Hence, at any point of  $M$  the eigenvalues of  $A$  and  $B$  are  $\{\cos 2\theta_1, \cos 2\theta_2\}$  and  $\{\sin 2\theta_1, \sin 2\theta_2\}$ , respectively. Moreover, the associated eigenvectors are the same. The eigenspace associated to  $\cos 2\theta_1$  and  $\sin 2\theta_1$  is lightlike and the one associated to  $\cos 2\theta_2$  and  $\sin 2\theta_2$  is spacelike, therefore  $\tilde{\theta}_1 = \theta_1$  and  $\tilde{\theta}_2 = \theta_2$  modulo  $\pi$ . As the eigenvectors  $E_3$  and  $\tilde{E}_3$  are both unit length, we derive that  $\tilde{E}_3 = \pm E_3$ . Similarly, we get  $\tilde{E}_1 = cE_1$  for  $c \in \mathbb{R}$  and therefore  $\tilde{E}_2 = c^{-1}E_2$ . Computing  $A\tilde{E}_2$  we produce

$$A\tilde{E}_2 = \frac{1}{c}(E_1 + \cos 2\theta_1 E_2) = \frac{1}{c^2}\tilde{E}_1 + \cos 2\theta_1 \tilde{E}_2$$

thus  $c^2 = 1$ . Since  $\{\tilde{E}_i\}$  also has to satisfy the relations in (25) we obtain  $\tilde{E}_3 = E_3$ .

Suppose now that  $\theta_2 = \theta_1$  modulo  $\pi$ . This means that the eigenspace associated to  $\cos 2\theta_1$  is two-dimensional. Therefore any linear isometry that preserves the eigenspace preserves the form of  $A$  and  $B$ . Let  $T$  be the linear isometry defined by  $TE_i = \tilde{E}_i$ . After some computations we obtain that  $T$  has the form

$$T = \begin{pmatrix} \varepsilon & -\varepsilon \frac{t^2}{2} & t \\ 0 & \varepsilon & 0 \\ 0 & -\varepsilon t & 1 \end{pmatrix},$$

for  $\varepsilon = \pm 1$  and some  $t \in \mathbb{R}$ .

Computing  $h(\tilde{E}_2, \tilde{E}_2)$  and using Lemma 10 yields  $\tilde{h}_{22}^1 = h_{22}^1 - 2th_{22}^3$  and  $\tilde{h}_{22}^3 = h_{22}^3$ . Suppose that  $h_{22}^3 = 0$ , then the Gauss equation (23) with  $X = E_2, Y = E_3, Z = E_3$  implies  $1/6 = 3/2$ , a contradiction. Then we can choose  $t = h_{22}^1/(2h_{22}^3)$ , thus  $\tilde{h}_{22}^1 = 0$ . In the same way, we can obtain that it is the unique (up to sign) frame with this condition. As before, we conclude that  $\omega_{ij}^k$  and  $h_{ij}^k$  are constant for this frame.  $\square$

#### 4.1.3. Lagrangian submanifolds of type III.

**Proposition 15.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type III form in Lemma 7 with respect to a  $\Delta_2$ -orthogonal frame  $\{E_1, E_2, E_3\}$ . Then the frame is unique and the functions  $\theta_i$ ,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant.*

*Proof.* Suppose that  $\{\tilde{E}_i\}_i$  is another  $\Delta_2$ -orthonormal frame with respect to which  $A$  and  $B$  take type III form in Lemma 7. We denote by  $T$  the linear isometry given by  $TE_i = \tilde{E}_i$ . We write

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

First notice that  $E_1$  spans the unique eigenspace of  $A$  and  $B$ , thus  $t_{21} = t_{31} = 0$ . Computing  $g(TE_1, TE_2)$  we furnish  $t_{22}t_{11} = 1$ . In the same way, computing  $g(TE_1, TE_3)$  and  $g(TE_3, TE_3)$  we come to  $t_{23} = 0$  and  $t_{33} = \varepsilon = \pm 1$ . Computing  $ATE_2 = -\frac{1}{2}TE_2 + TE_3$  we get that  $t_{13} = t_{32}$  and  $t_{11} = \varepsilon$ . Computing  $g(E_2, E_3)$  and  $g(E_2, E_2)$  we obtain  $t_{13} = 0$  and  $t_{12} = 0$ . By asking  $JG(TE_1, TE_2) = \sqrt{\frac{2}{3}}TE_3$  we see that  $\varepsilon = 1$ .

As in the previous propositions, the last statement follows from the uniqueness of the frame.  $\square$



4.1.4. *Lagrangian submanifolds of type IV.*

**Proposition 16.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type IV form in Lemma 7 with respect to  $\Delta_3$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then the frame is unique and the functions  $\theta_i$ ,  $\psi$ ,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant.*

*Proof.* In order to simplify the proof we write  $A$  and  $B$  as

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \cos 2\theta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma & \delta & 0 \\ -\delta & \gamma & 0 \\ 0 & 0 & \sin 2\theta_2 \end{pmatrix},$$

with respect to  $\{E_i\}_i$ .

Suppose there exists a  $\Delta_3$ -orthonormal frame  $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$  and functions  $\tilde{\alpha}$ ,  $\tilde{\beta}$ ,  $\tilde{\gamma}$ ,  $\tilde{\delta}$  and  $\tilde{\theta}_2$  such that  $A$  and  $B$  take the form

$$A = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} & 0 \\ -\tilde{\beta} & \tilde{\alpha} & 0 \\ 0 & 0 & \cos 2\tilde{\theta}_2 \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{\gamma} & \tilde{\delta} & 0 \\ -\tilde{\delta} & \tilde{\gamma} & 0 \\ 0 & 0 & \sin 2\tilde{\theta}_2 \end{pmatrix},$$

with respect to  $\{\tilde{E}_i\}_i$ .

Since  $\cos 2\theta_2$  and  $\sin 2\theta_2$  are the only eigenvalues of  $A$  and  $B$ ,  $\tilde{\theta}_2 = \theta_2$  modulo  $\pi$  and  $\tilde{E}_3 = \varepsilon E_3$  with  $\varepsilon = \pm 1$ . We denote by  $T$  the linear isometry defined by  $TE_i = \tilde{E}_i$ . Given that  $\{\tilde{E}_i\}$  is an  $\Delta_3$ -orthonormal frame, we may assume that  $\tilde{E}_1$  and  $\tilde{E}_2$  do not have components in the direction of  $E_3$ . Hence, we can write  $T$  as

$$T = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

with  $t \in \mathbb{R}$ . Requiring  $ATE_1 = (\tilde{\alpha}TE_1 - \tilde{\beta}TE_2)$  and  $ATE_2 = (\tilde{\beta}TE_1 + \tilde{\alpha}TE_2)$  we obtain

$$\begin{aligned} \alpha \cosh t + \beta \sinh t &= \tilde{\alpha} \cosh t - \tilde{\beta} \sinh t, \\ -\beta \cosh t + \alpha \sinh t &= -\tilde{\beta} \cosh t + \tilde{\alpha} \sinh t, \\ \beta \cosh t + \alpha \sinh t &= \tilde{\beta} \cosh t + \tilde{\alpha} \sinh t, \\ \alpha \cosh t - \beta \sinh t &= \tilde{\alpha} \cosh t + \tilde{\beta} \sinh t. \end{aligned}$$

Combining these equations we get

$$\begin{aligned} (\alpha - \tilde{\alpha}) \cosh t &= 0 \\ (\beta - \tilde{\beta}) \cosh t &= 0 \end{aligned}$$

therefore  $\tilde{\alpha} = \alpha$  and  $\tilde{\beta} = \beta$ . We may use the same argument to deduce that  $\tilde{\delta} = \delta$  and  $\tilde{\gamma} = \gamma$ . We compute again  $ATE_1 = (\tilde{\alpha}TE_1 - \tilde{\beta}TE_2)$  and we derive  $t = 0$ . From  $JG(\tilde{E}_1, \tilde{E}_2) = \sqrt{\frac{2}{3}}\tilde{E}_3$  it follows that  $\varepsilon = 1$ .

In a similar way as in the proofs of propositions 13-15 we obtain that  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  are constant. Computing  $\frac{\alpha}{\beta}$  and  $\frac{\alpha}{\tilde{\beta}}$  we obtain that  $\psi$ ,  $\theta_1$  and  $\theta_2$  are constant as well. Finally using the uniqueness of the frame we get that  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant.  $\square$

**4.2. Extrinsically homogeneous Lagrangian submanifolds of type I.** The following proposition proven in [2] gives us a characterization of totally geodesic Lagrangian submanifolds of type I:

**Proposition 17.** *Let  $M$  be a Lagrangian submanifold of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  of type I in Lemma 7. If two angles are equal modulo  $\pi$ , then  $M$  is totally geodesic.*

Moreover, in the same paper all the totally geodesic Lagrangian submanifolds are classified up to congruence by the following theorem, which we rewrite to fit better in this article.

**Theorem 18.** *Let  $f : (M, g) \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  be a totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Then  $f(M)$  is congruent to an open subset of the following extrinsically homogeneous Lagrangian embeddings:*

$(M, g)$	$f$	$H$
$(\mathrm{SL}(2, \mathbb{R}), \frac{2}{3}\langle, \rangle)$	$u \mapsto (u, u)$	$\{(u, u, \mathrm{Id}_2) : u \in \mathrm{SL}(2, \mathbb{R})\}$
$(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}^+)$	$u \mapsto (u, iui)$	$\{(u, iui, \mathrm{Id}_2) : u \in \mathrm{SL}(2, \mathbb{R})\}$
$(\mathrm{SL}(2, \mathbb{R}), g_{\kappa, \tau}^-)$	$u \mapsto (u, -kuk)$	$\{(u, -kuk, \mathrm{Id}_2) : u \in \mathrm{SL}(2, \mathbb{R})\}$

where  $H$  acts transitively on  $M$  with null isotropy, and  $g_{\kappa, \tau}^+, g_{\kappa, \tau}^-$  are Berger-like metrics on  $\mathrm{SL}(2, \mathbb{R})$  stretched in spacelike and timelike directions, respectively.

This theorem implies that any Lagrangian submanifold of type I with two equal angle functions modulo  $\pi$  is extrinsically homogeneous. Therefore, to complete the classification of extrinsically homogeneous Lagrangian submanifolds we assume that the submanifold is not totally geodesic and that all angle functions are different modulo  $\pi$ .

**Proposition 19.** *Let  $M$  be a non-totally geodesic extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  of type I. Let  $\theta_i, i = 1, 2, 3$  be the angle functions associated to the  $\Delta_1$ -orthonormal frame with respect of which  $A$  and  $B$  are diagonal. Then  $(\theta_1, \theta_2, \theta_3)$  is a permutation of  $(0, \pi/3, 2\pi/3)$  and the manifold  $M$  has constant sectional curvature. Moreover, the sectional curvature is either equal to 0 or to  $-\frac{3}{8}$ .*

*Proof.* Lagrangian submanifolds of type I are essentially an analogue of Lagrangian submanifolds of  $\mathbb{S}^3 \times \mathbb{S}^3$ . In [3] the authors proved for  $\mathbb{S}^3 \times \mathbb{S}^3$  that the angle functions of non-totally geodesic Lagrangian submanifolds are constant and a permutation of  $(0, \frac{\pi}{3}, \frac{2\pi}{3})$ . The same argument works for  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ .

By Lemma 8 we may assume that  $(\theta_1, \theta_2, \theta_3) = (0, \pi/3, 2\pi/3)$ . From Lemma 9 we know that all the functions  $h_{ij}^k$  are equal to zero, except for  $h_{12}^3$ , which from Proposition 13 we know is constant. Then the Codazzi equation (24) with  $X = E_1, Y = E_2, Z = E_2$  yields that  $h_{12}^3$  is either equal to  $\frac{1}{2\sqrt{2}}$  or to  $-\frac{1}{\sqrt{2}}$ . Both cases imply that the sectional curvature is constant. In the former case the sectional curvature is equal to  $-\frac{3}{8}$  and in the latter case the sectional curvature is equal to 0.  $\square$

**Example 20.** Let  $f : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  be the isometric immersion given by  $u \mapsto (iui^{-1}, juju^{-1})$  and let  $\{X_1, X_2, X_3\}$  be the frame on  $\mathrm{SL}(2, \mathbb{R})$  given in (4). We may compute

$$\begin{aligned} f_*(X_1) &= (0, juju^{-1}(-2uii^{-1})), \\ f_*(X_2) &= (iuii^{-1}(-2uju^{-1}), 0), \\ f_*(X_3) &= (iuii^{-1}(-2uku^{-1}), juju^{-1}(-2uku^{-1})). \end{aligned}$$

It follows from the definition of  $J$  in (6) that

$$\begin{aligned} Jf_*(X_1) &= \frac{1}{\sqrt{3}}(\mathbf{i}u\mathbf{i}u^{-1}(4u\mathbf{i}u^{-1}), \mathbf{j}u\mathbf{j}u^{-1}(2u\mathbf{i}u^{-1})), \\ Jf_*(X_2) &= \frac{1}{\sqrt{3}}(\mathbf{i}u\mathbf{i}u^{-1}(-2u\mathbf{j}u^{-1}), \mathbf{j}u\mathbf{j}u^{-1}(-4u\mathbf{j}u^{-1})), \\ Jf_*(X_3) &= \frac{1}{\sqrt{3}}(\mathbf{i}u\mathbf{i}u^{-1}(2u\mathbf{k}u^{-1}), \mathbf{j}u\mathbf{j}u^{-1}(-2u\mathbf{k}u^{-1})). \end{aligned}$$

We can easily check that  $f$  is a Lagrangian immersion by computing  $g(Jf_*(X_i), f_*(X_j)) = 0$  for  $i, j = 1, 2, 3$ . Moreover, we have

$$\begin{aligned} Pf_*(X_1) &= (\mathbf{i}u\mathbf{i}u^{-1}(-2u\mathbf{i}u^{-1}), 0) = -\frac{1}{2}f_*(X_1) - \frac{\sqrt{3}}{2}Jf_*(X_1), \\ Pf_*(X_2) &= (0, \mathbf{j}u\mathbf{j}u^{-1}(-2u\mathbf{j}u^{-1})) = -\frac{1}{2}f_*(X_2) + \frac{\sqrt{3}}{2}Jf_*(X_2), \\ Pf_*(X_3) &= (\mathbf{i}u\mathbf{i}u^{-1}(-2u\mathbf{k}u^{-1}), \mathbf{j}u\mathbf{j}u^{-1}(-2u\mathbf{k}u^{-1})) = f_*(X_3). \end{aligned}$$

Let  $H$  be the subgroup of  $\text{Iso}_o(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))$  given by  $\{(\mathbf{i}u\mathbf{i}, \mathbf{j}u\mathbf{j}, u) : u \in \text{SL}(2, \mathbb{R})\} \cong \text{SL}(2, \mathbb{R})$ . Then  $f(\text{SL}(2, \mathbb{R})) = H \cdot (\text{Id}_2, \text{Id}_2)$ . Notice that  $H$  acts on  $f(\text{SL}(2, \mathbb{R}))$  with isotropy  $\mathbb{Z}_2$ . Hence, the embedding  $\text{PSL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) : [u] \mapsto (\mathbf{i}u\mathbf{i}u^{-1}, \mathbf{j}u\mathbf{j}u^{-1})$  is congruent to  $f$ .

**Proposition 21.** *Any extrinsically homogeneous non-totally geodesic Lagrangian submanifold of the pseudo-nearly Kähler  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  of type I with  $h_{12}^3 = \frac{1}{2\sqrt{2}}$  is congruent to an open subset of the image of  $\text{PSL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) : [u] \mapsto (\mathbf{i}u\mathbf{i}u^{-1}, \mathbf{j}u\mathbf{j}u^{-1})$ .*

*Proof.* Let  $f: M \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  be a non-totally geodesic extrinsically homogeneous Lagrangian immersion of type I. Let  $\{E_1, E_2, E_3\}$  be the frame on  $M$  such that  $JG(E_1, E_2) = \sqrt{\frac{2}{3}}E_3$  with angle functions given by  $(\theta_1, \theta_2, \theta_3) = (0, \pi/3, 2\pi/3)$ . Moreover, assume that  $h_{12}^3 = g(h(E_1, E_2), E_3) = \frac{1}{2\sqrt{2}}$ . We have

$$PE_1 = E_1, \quad PE_2 = -\frac{1}{2}E_2 + \frac{\sqrt{3}}{2}JE_2, \quad PE_3 = -\frac{1}{2}E_3 - \frac{\sqrt{3}}{2}JE_3.$$

Hence, according to Equation (16), we have

$$QE_1 = \sqrt{3}JE_1, \quad QE_2 = -E_2, \quad QE_3 = E_3. \quad (34)$$

From Proposition 19 it follows that  $M$  has constant sectional curvature  $-\frac{3}{8}$ . Thus  $M$  is locally isometric to  $(\text{SL}(2, \mathbb{R}), \frac{8}{3}\langle, \rangle)$  (see [17]), where  $\langle, \rangle$  is the metric given in (1). Then we may identify

$$E_1 = \sqrt{\frac{3}{8}}X_3, \quad E_2 = \sqrt{\frac{3}{8}}X_2, \quad E_3 = \sqrt{\frac{3}{8}}X_1.$$

where  $\{X_1, X_2, X_3\}$  is the frame on  $\text{SL}(2, \mathbb{R})$  given in (4). Now we write the immersion  $f(u) = (p(u), q(u))$  and  $f_*(E_i)u = (D_{E_i}f)_u = (p(u)\alpha_i(u), q(u)\beta_i(u))$  where  $\alpha_i(u), \beta_i(u) \in \mathfrak{sl}(2, \mathbb{R})$ . By Equation (34) we have  $\alpha_1 = \beta_1$ ,  $\beta_2 = 0$  and  $\alpha_3 = 0$ . We know from (17) that  $\nabla_{E_1}^E E_1 = \nabla_{E_2}^E E_2 =$

$\nabla_{E_3}^E E_3 = 0$  and

$$\begin{aligned}
\nabla_{E_1}^E E_2 &= \nabla_{E_1}^E E_3 = 0, \\
\nabla_{E_2}^E E_1 &= -\sqrt{\frac{3}{2}}E_3 = -\sqrt{\frac{3}{2}}(0, q\beta_3), \\
\nabla_{E_2}^E E_3 &= -\frac{1}{2}\sqrt{\frac{3}{2}}(E_1 + QE_1) = -\sqrt{\frac{3}{2}}(0, q\alpha_1), \\
\nabla_{E_3}^E E_1 &= \sqrt{\frac{3}{2}}E_2 = \sqrt{\frac{3}{2}}(p\alpha_2, 0), \\
\nabla_{E_3}^E E_2 &= \frac{1}{2}\sqrt{\frac{3}{2}}(E_1 - QE_1) = \sqrt{\frac{3}{2}}(p\alpha_1, 0).
\end{aligned} \tag{35}$$

Throughout this proof, we will denote by  $\langle \cdot, \cdot \rangle_\times$  the product metric associated to the metric  $\langle \cdot, \cdot \rangle$  on  $SL(2, \mathbb{R})$  given in (1). By Equation 15  $E_1, E_2, E_3$  are orthogonal with respect to the product metric  $\langle \cdot, \cdot \rangle_\times$  and their lengths are

$$\langle E_1, E_1 \rangle_\times = -3, \quad \langle E_2, E_2 \rangle_\times = \langle E_3, E_3 \rangle_\times = \frac{3}{2}.$$

This implies that

$$\langle \alpha_1, \alpha_1 \rangle = -\frac{3}{2}, \quad \langle \alpha_2, \alpha_2 \rangle = \langle \beta_3, \beta_3 \rangle = \frac{3}{2} \tag{36}$$

On the one hand, Equation (18) yields

$$D_{E_i} D_{E_j} f = \nabla_{E_i}^E E_j + \frac{1}{2} \langle E_i, E_j \rangle (p, q) + \frac{1}{2} \langle E_i, QE_j \rangle (-p, q),$$

and on the other hand, by Equation (5), we furnsih

$$\begin{aligned}
D_{E_i} D_{E_j} f &= (p\alpha_i \alpha_j + pE_i(\alpha_j), q\beta_i \beta_j + qE_i(\beta_j)) \\
&= (p(\alpha_i \times \alpha_j) + \langle \alpha_i, \alpha_j \rangle p + pE_i(\alpha_j), q(\beta_i \times \beta_j) + \langle \beta_i, \beta_j \rangle q + qE_i(\beta_j)).
\end{aligned}$$

Therefore

$$\nabla_{E_i}^E E_j = (p\alpha_i \times \alpha_j + pE_i(\alpha_j), q\beta_i \times \beta_j + qE_i(\beta_j)),$$

where  $E_i(\alpha) = d\alpha(E_i)$  thinking of  $\alpha$  as a map from  $SL(2, \mathbb{R})$  into  $\mathfrak{sl}(2, \mathbb{R})$ . Hence, using (35) we obtain

$$\alpha_1 \times \alpha_2 = -\sqrt{\frac{3}{2}}\beta_3$$

and also

$$\begin{aligned}
E_1(\alpha_1) &= 0, & E_2(\alpha_1) &= -\sqrt{\frac{3}{2}}\beta_3, & E_3(\alpha_1) &= \sqrt{\frac{3}{2}}\alpha_2, \\
E_1(\alpha_2) &= \sqrt{\frac{3}{2}}\beta_3, & E_2(\alpha_2) &= 0, & E_3(\alpha_2) &= \sqrt{\frac{3}{2}}\alpha_1, \\
E_1(\beta_3) &= -\sqrt{\frac{3}{2}}\alpha_2, & E_2(\beta_3) &= -\sqrt{\frac{3}{2}}\alpha_1, & E_3(\beta_3) &= 0.
\end{aligned}$$

In terms of the vector fields  $X_i$  this translates into the following differential equations:

$$\begin{aligned}
X_1(\alpha_1) &= 2\alpha_2, & X_2(\alpha_1) &= -2\beta_3, & X_3(\alpha_1) &= 0, \\
X_1(\alpha_2) &= 2\alpha_1, & X_2(\alpha_2) &= 0, & X_3(\alpha_2) &= 2\beta_3, \\
X_1(\beta_3) &= 0, & X_2(\beta_3) &= -2\alpha_1, & X_3(\beta_3) &= -2\alpha_2.
\end{aligned} \tag{37}$$

From (36) and Lemma 4 we know that there exists  $c \in SL^\pm(2, \mathbb{R})$  such that

$$\alpha_1(\text{Id}_2) = -\sqrt{\frac{3}{2}}c\mathbf{k}c^{-1}, \quad \alpha_2(\text{Id}_2) = -\sqrt{\frac{3}{2}}c\mathbf{j}c^{-1}, \quad \beta_3(\text{Id}_2) = -\sqrt{\frac{3}{2}}c\mathbf{i}c^{-1}. \tag{38}$$

Therefore, as the solution of the system (37) with initial conditions (38) is unique, we have that

$$\alpha_1(u) = -\sqrt{\frac{3}{2}}cu\mathbf{k}u^{-1}c^{-1}, \quad \alpha_2(u) = -\sqrt{\frac{3}{2}}u\mathbf{j}u^{-1}c^{-1}, \quad \beta_3(u) = -\sqrt{\frac{3}{2}}cu\mathbf{i}u^{-1}c^{-1}.$$

We can check easily that they satisfy the equations in (37). By the homogeneity of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  we can take initial conditions  $f(\mathrm{Id}_2) = (\mathrm{Id}_2, \mathrm{Id}_2)$ . Applying the isometry of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by  $(p, q) \mapsto (cpc^{-1}, cq c^{-1})$  we may assume that

$$\alpha_1(u) = -\sqrt{\frac{3}{2}}\mathbf{u}\mathbf{k}u^{-1}, \quad \alpha_2(u) = -\sqrt{\frac{3}{2}}\mathbf{u}\mathbf{j}u^{-1}, \quad \alpha_3(u) = -\sqrt{\frac{3}{2}}\mathbf{c}\mathbf{u}\mathbf{i}u^{-1}.$$

Then, the immersion  $f = (p, q)$  given by  $p = \mathbf{i}\mathbf{u}\mathbf{i}u^{-1}$ ,  $q = \mathbf{j}\mathbf{u}\mathbf{j}u^{-1}$  is the unique solution of the differential equation  $D_{E_i}p = p\alpha_i$ ,  $D_{E_i}q = q\beta_i$ , with  $i = 1, 2, 3$ .  $\square$

**Example 22.** Let  $H$  be the Lie subgroup of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by

$$H = \{(e^{vi}, e^{wj}, e^{uk}) : v, w \in \mathbb{R}, u \in [0, 2\pi)\} \cong \mathbb{R}^2 \times \mathbb{S}^1.$$

The subgroup  $H$  acts transitively on the submanifold  $f : \mathbb{R}_1^3/\mathbb{Z} \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by

$$f(u, v, w) = (e^{vi}e^{-uk}, e^{wj}e^{-uk}).$$

Moreover, the isotropy of  $H$  is trivial.

The derivatives of  $f$  are given by

$$f_u = (-e^{vi}e^{-uk}\mathbf{k}, -e^{wj}e^{-uk}\mathbf{k}), \quad f_v = (e^{vi}e^{-uk}e^{uk}\mathbf{i}e^{-uk}, 0), \quad f_w = (0, e^{wj}e^{-uk}e^{uk}\mathbf{j}e^{-uk}).$$

Applying the almost complex structure yields the following expressions:

$$\begin{aligned} Jf_u &= \frac{1}{\sqrt{3}}(e^{vi}e^{-uk}\mathbf{k}, -e^{wj}e^{-uk}\mathbf{k}), \\ Jf_v &= \frac{1}{\sqrt{3}}(e^{vi}e^{-uk}e^{uk}\mathbf{i}e^{-uk}, 2e^{vi}e^{-uk}e^{uk}\mathbf{i}e^{-uk}), \\ Jf_w &= \frac{1}{\sqrt{3}}(-2e^{wj}e^{-uk}e^{uk}\mathbf{j}e^{-uk}, -e^{wj}e^{-uk}e^{uk}\mathbf{j}e^{-uk}). \end{aligned}$$

We can easily check that

$$g(Jf_u, f_v) = g(Jf_u, f_w) = g(Jf_v, f_w) = 0,$$

which shows that this submanifold is Lagrangian. After applying the tensor  $P$  we obtain

$$\begin{aligned} Pf_u &= (-e^{vi}e^{-uk}\mathbf{k}, -e^{wj}e^{-uk}\mathbf{k}) = f_u, \\ Pf_v &= (0, e^{wj}e^{-uk}e^{uk}\mathbf{i}e^{-uk}) = -\frac{1}{2}f_v + \frac{\sqrt{3}}{2}Jf_v, \\ Pf_w &= (e^{wj}e^{-uk}e^{uk}\mathbf{j}e^{-uk}, 0) = -\frac{1}{2}f_w - \frac{\sqrt{3}}{2}Jf_w. \end{aligned}$$

Thus,  $f$  is a flat, extrinsically homogeneous Lagrangian submanifold of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  of type I with constant angles  $(\theta_1, \theta_2, \theta_3) = (0, \frac{\pi}{3}, \frac{2\pi}{3})$ .

**Proposition 23.** *Any extrinsically homogeneous non-totally geodesic Lagrangian submanifold of type I of the nearly Kähler  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  with  $h_{12}^3 = -\frac{1}{\sqrt{2}}$  is congruent to an open subset of the image of the Lagrangian embedding  $f : \mathbb{R}_1^3/\mathbb{Z} \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by*

$$f(u, v, w) = (e^{vi}e^{-uk}, e^{wj}e^{-uk}).$$

*Proof.* Notice first that all the coefficients of the connection and second fundamental form vanish except for  $h_{12}^3 = g(h(E_1, E_2), JE_3) = -\frac{1}{\sqrt{2}}$ . We also know that the angle functions are given by  $(2\theta_1, 2\theta_2, 2\theta_3) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$ . So we can find a local frame such that  $JG(E_1, E_3) = \sqrt{\frac{2}{3}}E_3$  and

$$PE_1 = E_1, \quad PE_2 = -\frac{1}{2}E_2 + \frac{\sqrt{3}}{2}JE_2, \quad PE_3 = -\frac{1}{2}E_3 - \frac{\sqrt{3}}{2}JE_3.$$

From the relation between  $Q$  and  $P$  in (16) it follows that

$$QE_1 = \sqrt{3}JE_1, \quad QE_2 = -E_2, \quad QE_3 = E_3. \quad (39)$$

Using that  $h_{ii}^j = 0$ ,  $h_{12}^3 = -\frac{1}{\sqrt{2}}$  and Equation (27), we deduce that  $[E_i, E_j] = 0$  for  $i, j = 1, 2, 3$ . Then we write  $E_1 = f_u$ ,  $E_2 = f_v$ ,  $E_3 = f_w$  for  $u, v, w$  local coordinates. Thus, Equation (39) implies that

$$p_w = 0, \quad q_v = 0, \quad q_u = p_u. \quad (40)$$

Moreover, we have  $\nabla_{f_u}^E f_u = \nabla_{f_v}^E f_v = \nabla_{f_w}^E f_w = 0$  and

$$\begin{aligned} \nabla_{f_u}^E f_v &= \nabla_{f_v}^E f_u = -\frac{1}{2}\sqrt{\frac{3}{2}}f_w - \frac{3}{2\sqrt{2}}Jf_w, \\ \nabla_{f_u}^E f_w &= \nabla_{f_w}^E f_u = \frac{1}{2}\sqrt{\frac{3}{2}}f_v - \frac{3}{2\sqrt{2}}Jf_v, \\ \nabla_{f_v}^E f_w &= \nabla_{f_w}^E f_v = 0. \end{aligned}$$

From the relation between the Euclidean metric with the nearly Kähler metric in (15) we know that  $f_u, f_v, f_w$  are also orthogonal with respect to the induced Euclidean product metric. Furthermore, their inner products are given by

$$\langle f_u, f_u \rangle = -3, \quad \langle f_v, f_v \rangle = \langle f_w, f_w \rangle = \frac{3}{2}.$$

Furthermore,

$$\begin{aligned} \langle f_u, Qf_u \rangle &= 0, \quad \langle f_u, Qf_v \rangle = 0, \quad \langle f_u, Qf_w \rangle = 0, \\ \langle f_v, Qf_w \rangle &= 0, \quad \langle f_v, Qf_v \rangle = -\frac{3}{2}, \quad \langle f_w, Qf_w \rangle = \frac{3}{2}. \end{aligned}$$

Thus, using (40) we get that  $p_u, p_v$  and  $q_w$  are orthogonal and

$$\langle p_u, p_u \rangle = -\frac{3}{2}, \quad \langle p_v, p_v \rangle = \langle p_w, p_w \rangle = \frac{3}{2}. \quad (41)$$

From the expression for the Euclidean connection  $D$  of  $\mathbb{R}_4^8$  given in Equation (18) we obtain

$$\begin{aligned} f_{uu} &= -\frac{3}{2}f, & f_{uv} &= -\frac{1}{2}\sqrt{\frac{3}{2}}f_w - \frac{3}{2\sqrt{2}}Jf_w, \\ f_{uw} &= \frac{1}{2}\sqrt{\frac{3}{2}}f_v - \frac{3}{2\sqrt{2}}Jf_v, & f_{vv} &= \frac{3}{4}f - \frac{3}{4}Qf, \\ f_{vw} &= 0, & f_{ww} &= \frac{3}{4}f + \frac{3}{4}Qf. \end{aligned}$$

Hence we produce differential equations for  $p$  and  $q$ :

$$\begin{aligned} p_{uu} &= -\frac{3}{2}p, \quad p_{vv} = \frac{3}{2}p, \quad p_{uw} = \sqrt{\frac{3}{2}}pq^{-1}q_w, \\ q_{uu} &= -\frac{3}{2}q, \quad q_{ww} = \frac{3}{2}q, \quad q_{uw} = -\sqrt{\frac{3}{2}}qp^{-1}p_v. \end{aligned} \quad (42)$$

By applying an isometry of the type  $(p, q) \mapsto (ap, bq)$  we may assume that  $p(0) = \text{Id}_2$ ,  $q(0) = \text{Id}_2$ . Now, because of Equation (41), there exists a matrix  $c \in \text{SL}^\pm(2, \mathbb{R})$  such that

$$p_u(0) = \sqrt{\frac{3}{2}}c\mathbf{k}c^{-1}, \quad p_v(0) = \sqrt{\frac{3}{2}}c\mathbf{i}c^{-1}, \quad q_u(0) = \sqrt{\frac{3}{2}}c\mathbf{k}c^{-1}, \quad q_w(0) = \sqrt{\frac{3}{2}}c\mathbf{j}c^{-1}.$$

Applying the isometry  $(p, q) \mapsto (cpc^{-1}, cq c^{-1})$  we obtain that any solution of the system (42) is congruent to the map  $f(u, v, w) = (e^{\sqrt{\frac{3}{2}}vi} e^{-\sqrt{\frac{3}{2}}uk}, e^{\sqrt{\frac{3}{2}}wj} e^{-\sqrt{\frac{3}{2}}uk})$ . Then, changing the coordinates by  $u \rightarrow \sqrt{\frac{2}{3}}u$ ,  $v \rightarrow \sqrt{\frac{2}{3}}v$  and  $w \rightarrow \sqrt{\frac{2}{3}}w$ , we get the map  $(u, v, w) \mapsto (e^{vi} e^{-uk}, e^{wj} e^{-uk})$ .  $\square$

### 4.3. Extrinsically homogeneous Lagrangian submanifolds of type II.

**Proposition 24.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type II form in Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_3, E_3\}$ . Then  $\theta_1 = \theta_2 = \pi/3$  modulo  $\pi$ . Also, all the components of the second fundamental form and of the connection are equal to zero except for  $h_{22}^3$ ,  $\omega_{12}^3$ ,  $\omega_{21}^3$ ,  $\omega_{31}^1$  and  $\omega_{22}^3$ . There are only two possibilities for the value of these constants:*

- (1)  $h_{22}^3 = -\frac{\sqrt{2}}{3}$ ,  $\omega_{12}^3 = -\sqrt{\frac{3}{2}}$ ,  $\omega_{21}^3 = -\sqrt{\frac{3}{2}}$ ,  $\omega_{31}^1 = 0$ ,  $\omega_{33}^2 = 0$ ,
- (2)  $h_{22}^3 = \frac{2\sqrt{2}}{3}$ ,  $\omega_{12}^3 = \sqrt{\frac{3}{2}}$ ,  $\omega_{21}^3 = \sqrt{\frac{3}{2}}$ ,  $\omega_{31}^1 = \sqrt{\frac{3}{2}}$ .

Moreover, in both cases the sectional curvature is constant and equal to  $-\frac{3}{2}$ .

*Proof.* As indicated in Proposition 14, we have to distinguish between two cases: when  $\theta_1 = \theta_2$  and when  $\theta_1 \neq \theta_2$ .

Suppose first that  $\theta_1 \neq \theta_2$ . By Proposition 14 both angles and the functions  $h_{ij}^k, \omega_{ij}^k$  are constant. It follows from computing the Codazzi equation (24) with  $X = E_3, Y = E_2$  and  $Z = E_3$  that  $\sin(2(\theta_1 - \theta_2)) = 0$ , hence  $\theta_1$  and  $\theta_2$  are equal modulo  $\pi/2$ . Recall that for type II submanifolds  $\sin 2\theta_1$  is different from zero,  $2\theta_1 + \theta_2 = 0$  modulo  $\pi$  and the angles are different modulo  $\pi$ . Therefore  $\theta_1 = \frac{\pi}{6}, \theta_2 = \frac{2\pi}{3}$  or  $\theta_1 = \frac{5\pi}{6}, \theta_2 = \frac{\pi}{3}$ . From Lemma 8 we know that these two cases are congruent via the isometry  $\Psi_{0,1}$  given in (19). It follows from Equation (12) that

$$\omega_{11}^1 = \omega_{11}^3 = \omega_{33}^2 = 0, \quad \omega_{21}^3 = -\frac{1}{\sqrt{6}}, \quad \omega_{12}^3 = \frac{1}{\sqrt{6}}, \quad \omega_{31}^1 = \frac{\sqrt{3}}{2}h_{22}^3 + \frac{1}{\sqrt{6}}.$$

Computing the Codazzi equation (24) with  $X = E_3, Y = E_2, Z = E_3$  and  $X = E_1, Y = E_2, Z = E_2$  we obtain

$$\frac{4 - 3h_{22}^3(3h_{22}^3 + \sqrt{2})}{3\sqrt{3}} = 0, \quad 8\sqrt{3} - 3\sqrt{6}h_{22}^3 = 0,$$

which is a contradiction.

Suppose now that  $\theta_1 = \theta_2$  modulo  $\pi$ . Using that  $2\theta_1 + \theta_2 = 0$  modulo  $\pi$  we deduce  $\theta_1 = \theta_2 = \pi/3$  or  $2\pi/3$ . By Lemma 8 we know that these cases are congruent via the isometry  $\Psi_{0,1}$  given in (19). Thus, we only consider the cases where  $\theta_1 = \theta_2 = \pi/3$ .

By Lemma 10 we have  $h_{22}^2 = h_{12}^3 = 0$ . Moreover, by Proposition 14 we may assume that  $h_{22}^1 = 0$  and that all the functions  $h_{ij}^k, \omega_{ij}^k$  are constant. Hence, from Equation (12) we obtain

$$\omega_{11}^1 = \omega_{11}^3 = \omega_{22}^2 = \omega_{33}^2 = 0.$$

We also get

$$\omega_{21}^3 = \sqrt{3}h_{22}^3 - \frac{1}{\sqrt{6}}, \quad \omega_{31}^1 = \frac{\sqrt{3}}{2}h_{22}^3 + \frac{1}{\sqrt{6}}.$$

Thus computing the Codazzi equation with  $X = E_3, Y = E_2$  and  $Z = E_2$  yields  $\omega_{33}^1 = 0$ . Moreover, we obtain

$$-9(h_{22}^3)^2 + 3\sqrt{2}h_{22}^3 + 4 = 0.$$

Hence  $h_{22}^3 = -\frac{\sqrt{2}}{3}$  or  $h_{22}^3 = \frac{2\sqrt{2}}{3}$ .

If  $h_{22}^3 = -\frac{\sqrt{2}}{3}$  then the Codazzi equation with  $X = E_1, Y = E_2$  and  $Z = E_2$  implies  $\omega_{12}^3 = -\sqrt{\frac{3}{2}}$ . Also we obtain  $\omega_{22}^3 = 0$ .

If instead  $h_{22}^3 = \frac{2\sqrt{2}}{3}$  we obtain  $\omega_{12}^3 = \sqrt{\frac{3}{2}}$ . A straightforward computation shows that both cases have constant sectional curvature equal to  $-\frac{3}{2}$ .  $\square$

Now we exhibit two examples of extrinsically homogeneous Lagrangian submanifolds of type II in Lemma 7.

**Example 25.** Let  $\mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$  be the Bianchi group of type V with group law given by

$$(t, u) \cdot (s, v) = (t + s, \varphi_0(s)u + v)$$

where  $\varphi_0: \mathbb{R} \rightarrow \mathrm{Aut}(\mathbb{R}^2)$  is given by

$$\varphi_0(s) = \begin{pmatrix} e^{-2s} & 0 \\ 0 & e^{-2s} \end{pmatrix}.$$

Let  $\hat{g}$  be the right invariant metric such that its components with respect to the frame of right invariant vector fields  $\left\{ \frac{\partial}{\partial t}, e^{-2t} \frac{\partial}{\partial u_1}, e^{-2t} \frac{\partial}{\partial u_2} \right\}$  are given by

$$\begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

In fact, this Lorentzian manifold is simply connected, geodesically complete and it has constant sectional curvature equal to  $-\frac{3}{2}$ . Hence, by the pseudo-Riemannian analogue of the Killing-Hopf theorem (see [17]), it is isometric to  $\tilde{H}_1^3(-\frac{3}{2})$ , the universal cover of the anti-de Sitter space.

Now let  $\mathfrak{h}$  be the Lie subalgebra of  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$  spanned by  $\{e_1, e_2, e_3\}$  where

$$\begin{aligned} e_1 &= (\mathbf{i}, -\mathbf{i}, -\mathbf{i}), \\ e_2 &= \left( \frac{9}{4}(\mathbf{j} - \mathbf{k}), \frac{1}{2}(\mathbf{j} + \mathbf{k}), 0 \right), \\ e_3 &= \left( \frac{9}{4}(-\mathbf{j} + \mathbf{k}), 0, \frac{1}{2}(\mathbf{j} + \mathbf{k}) \right), \end{aligned}$$

where  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$  are given in (3). The Lie algebra  $\mathfrak{h}$  is a Bianchi Lie algebra of type V with brackets

$$[e_1, e_2] = -2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = 0.$$

Therefore  $\mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$  is the universal cover of  $\exp(\mathfrak{h}) \subset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ . Moreover, the map

$$(w, u, v) \mapsto \exp(we_1 + ue^w w \operatorname{csch}(w)e_2 + ve^w w \operatorname{csch}(w)e_3)$$

defined at  $w = 0$  as  $\exp(ue_2 + ve_3)$ , is a group isomorphism, thus  $\exp \mathfrak{h} \cong \mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2$ . One can check that the immersion  $\iota: \mathbb{R} \ltimes_{\varphi_0} \mathbb{R}^2 \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  given by

$$\iota(w, u, v) = \exp(we_1 + ue^w w \operatorname{csch}(w)e_2 + ve^w w \operatorname{csch}(w)e_3) \cdot (\mathrm{Id}_2, \mathrm{Id}_2) \quad (43)$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame given by

$$E_1 = -e^{-2w} \iota_u - e^{-2w} \iota_v, \quad E_2 = \frac{1}{3}e^{-2w} \iota_u - \frac{1}{3}e^{-2w} \iota_v, \quad E_3 = \sqrt{\frac{3}{8}} \iota_w,$$

is a  $\Delta_2$ -orthonormal frame with respect to which  $A$  and  $B$  take type II form in Lemma 7, with angle functions  $\theta_1 = \theta_2 = \frac{\pi}{3}$ .



**Example 26.** Let  $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$  be the Bianchi group of type III with group law

$$(t, u) \cdot (s, v) = (t + s, \varphi_1(s)u + v)$$

where  $\varphi_1: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$  is given by

$$\varphi_1(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let  $g_\lambda$  be the right invariant metric on  $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$  such that its components with respect to the frame of right invariant vector fields  $\left\{ \frac{\partial}{\partial t}, e^{2t} \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right\}$  are given by

$$\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{2(\lambda-1)}{3} \end{pmatrix},$$

where  $\lambda$  is an arbitrary real number. As in Example 25, this Lorentzian manifold is simply connected, geodesically complete and it has constant sectional curvature equal to  $-\frac{3}{2}$ . Therefore it is isometric to  $\tilde{H}_1^3(-\frac{3}{2})$ .

Let  $\mathfrak{h}$  be the Lie subalgebra of  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$  spanned by  $\{e_1, e_2, e_3\}$  where

$$\begin{aligned} e_1 &= (\mathbf{i}, 0, 0), \\ e_2 &= \left(\frac{1}{2}(\mathbf{j} + \mathbf{k}), 0, 0\right), \\ e_3 &= \left(0, -\frac{\lambda+7}{6}\mathbf{j} + \frac{11-\lambda}{6}\mathbf{k}, -\frac{\lambda+9}{6}\mathbf{j} + \frac{9-\lambda}{6}\mathbf{k}\right). \end{aligned}$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are given in (3). The Lie algebra  $\mathfrak{h}$  is a Bianchi Lie algebra of type III with brackets

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

Therefore  $\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2$  is the universal cover of  $\exp(\mathfrak{h})$ . The map  $\phi_\lambda: \mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2 \rightarrow \exp(\mathfrak{h})$  given by

$$\phi_\lambda(u, v, w) = \exp\left(we_1 + \frac{uwe^{-w}}{\sinh w}e_2 + ve_3\right)$$

with  $\frac{we^{-w}}{\sinh w}$  extended to 1 when  $w = 0$ , is a surjective homomorphism with

$$H_\lambda = \ker(\phi_\lambda) \cong \begin{cases} \mathbb{Z} & \text{when } \lambda = \frac{2n^2}{n^2-m^2}, m > n > 0 \text{ integers,} \\ \{0\} & \text{otherwise.} \end{cases}$$

One can check that the map  $f_\lambda: (\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)/H_\lambda \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  given by

$$f_\lambda(u, v, w) = \phi_\lambda(u, v, w) \cdot (\text{Id}_2, \text{Id}_2),$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame  $\{E_1, E_2, E_3\}$  given by

$$E_1 = e^{2w}(f_\lambda)_u, \quad E_2 = \frac{1-\lambda}{3}e^{2w}(f_\lambda)_u + (f_\lambda)_v, \quad E_3 = \sqrt{\frac{3}{2}}(f_\lambda)_w$$

is a  $\Delta_2$ -orthonormal frame with respect to which  $A$  and  $B$  take type II form in Lemma 7 with  $\theta_1 = \theta_2 = \frac{\pi}{3}$ .

**Remark 27.** For any pair  $\lambda_1, \lambda_2$  the subgroups  $\phi_{\lambda_i}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$  of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  are non-conjugate. That is, there does not exist an automorphism of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  preserving the isotropy subgroup  $\Delta\text{SL}(2, \mathbb{R})$  that maps  $\phi_{\lambda_1}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$  into  $\phi_{\lambda_2}(\mathbb{R} \ltimes_{\varphi_1} \mathbb{R}^2)$ . This can be easily seen since conjugations by elements of  $\text{SL}^\pm(2, \mathbb{R})$  preserve the indefinite inner product of  $\mathfrak{sl}(2, \mathbb{R})$  given in (1).

**Proposition 28.** *Let  $f: M \rightarrow SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type II form in Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then  $M$  is congruent to an open subset of either the image of the immersion in Example 25, or the image of the immersion in Example 26.*

*Proof.* Because of Proposition 24 we may assume that  $\theta_1 = \theta_2 = \frac{2}{3}\pi$ . Moreover, we divide in two cases.

Suppose that  $\omega_{ij}^k$  and  $h_{ij}^k$  satisfy (1) in Proposition 24. We take the frame  $\{\rho E_1, \rho E_2, E_3\}$  where  $\rho$  is a non-vanishing smooth function and solution of

$$E_1(\rho) = E_2(\rho) = 0, \quad E_3(\rho) = \sqrt{\frac{3}{2}}\rho. \quad (44)$$

It is easy to check that  $\rho$  indeed exists and that

$$[\rho E_1, \rho E_2] = [\rho E_1, E_3] = [\rho E_2, E_3] = 0.$$

Hence, there exist local coordinates  $u, v, w$  such that  $\rho E_1 = f_u$ ,  $\rho E_2 = f_v$  and  $E_3 = f_w$  and hence  $\rho(w) = e^{\sqrt{\frac{3}{2}}w}$ . It follows from the relation between  $Q$  and  $P$  given in (16) that

$$Qf_u = -f_u, \quad Qf_v = -\frac{2}{3}f_u - f_v + \frac{2}{\sqrt{3}}Jf_u, \quad Qf_w = -f_w.$$

Writing  $f = (p, q)$  yields

$$q_u = q_w = 0, \quad q_v = \frac{2}{3}qp^{-1}p_u. \quad (45)$$

Using the relation between the nearly Kähler connection  $\tilde{\nabla}$  and the product connection  $\nabla^E$  given in (17) we obtain

$$\begin{aligned} \nabla_{f_u}^E f_u &= 0, & \nabla_{f_u}^E f_v &= -\sqrt{\frac{3}{2}}\rho^2 f_w, & \nabla_{f_u}^E f_w &= \sqrt{\frac{3}{2}}f_u, \\ \nabla_{f_v}^E f_u &= -\sqrt{\frac{3}{2}}\rho^2 f_w, & \nabla_{f_v}^E f_v &= -\sqrt{\frac{2}{3}}\rho^2 f_w, & \nabla_{f_v}^E f_w &= \frac{1}{\sqrt{6}}f_u + \sqrt{\frac{3}{2}}f_v - \frac{1}{\sqrt{2}}Jf_u, \\ \nabla_{f_w}^E f_u &= \sqrt{\frac{3}{2}}f_u, & \nabla_{f_w}^E f_w &= 0, & \nabla_{f_w}^E f_v &= \frac{1}{\sqrt{6}}f_u + \sqrt{\frac{3}{2}}f_v - \frac{1}{\sqrt{2}}Jf_u. \end{aligned} \quad (46)$$

Now we compute

$$\begin{aligned} \langle f_u, f_u \rangle &= 0, & \langle f_v, f_v \rangle &= \rho^2, & \langle f_w, f_w \rangle &= \frac{3}{2}, \\ \langle f_u, f_v \rangle &= \frac{3}{2}\rho^2, & \langle f_u, f_w \rangle &= 0, & \langle f_v, f_w \rangle &= 0, \end{aligned} \quad (47)$$

and

$$\begin{aligned} \langle f_u, Qf_u \rangle &= 0, & \langle f_u, Qf_w \rangle &= 0, & \langle f_v, Qf_w \rangle &= 0, \\ \langle f_u, Qf_v \rangle &= -\frac{3}{2}\rho^2, & \langle f_v, Qf_v \rangle &= -\rho^2, & \langle f_w, Qf_w \rangle &= -\frac{3}{2}, \end{aligned} \quad (48)$$

where  $\langle, \rangle$  is the product metric associated to the metric on  $SL(2, \mathbb{R})$  given in (1). In particular, we have

$$\begin{aligned} \langle p_u, p_u \rangle &= 0, & \langle p_v, p_v \rangle &= \rho^2, & \langle p_w, p_w \rangle &= \frac{3}{2} \\ \langle p_u, p_v \rangle &= \frac{3}{2}\rho^2, & \langle p_u, p_w \rangle &= 0, & \langle p_v, p_w \rangle &= 0. \end{aligned} \quad (49)$$

Here,  $\langle, \rangle$  is the metric on  $SL(2, \mathbb{R})$  given in (1).

To compute the second derivatives of  $f$ , we use the expression for the Euclidean connection of  $\mathbb{R}_4^8$  in (18). Plugging (46), (47) and (48) into (18) we obtain

$$\begin{aligned} f_{uu} &= 0, & f_{uv} &= -\sqrt{\frac{3}{2}}\rho^2 f_w + \frac{3}{4}\rho^2 f - \frac{3}{4}\rho^2 Qf, \\ f_{uw} &= \sqrt{\frac{3}{2}}f_u, & f_{vv} &= -\sqrt{\frac{2}{3}}\rho^2 f_w + \frac{1}{2}\rho^2 f - \frac{1}{2}\rho^2 Qf, \\ f_{vw} &= \frac{1}{\sqrt{6}}f_u + \sqrt{\frac{3}{2}}f_v - \frac{1}{\sqrt{2}}Jf_u, & f_{ww} &= \frac{3}{4}f - \frac{3}{4}Qf. \end{aligned}$$

Looking at each component of  $f$  we obtain differential equations for  $p$  and  $q$ :

$$\begin{aligned} p_{uu} &= 0, & p_{uv} &= -\sqrt{\frac{3}{2}}\rho^2 p_w + \frac{3}{2}\rho^2 p, & p_{uw} &= \sqrt{\frac{3}{2}}p_u, \\ p_{vw} &= \sqrt{\frac{3}{2}}p_v, & p_{vv} &= -\sqrt{\frac{2}{3}}\rho^2 p_w + \rho^2 p, & p_{ww} &= \frac{3}{2}p, & q_{vv} &= 0. \end{aligned} \tag{50}$$

The other derivatives of  $q$  are zero because of (45).

Applying an isometry of the type  $(p, q) \mapsto (ap, bq)$ , we may assume initial conditions  $(p(0), q(0)) = (\text{Id}_2, \text{Id}_2)$ . Then from (49) and (45) it follows that there exists  $c \in \text{SL}^\pm(2, \mathbb{R})$  such that

$$\begin{aligned} p_u(0) &= c \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} c^{-1}, & p_v(0) &= c \begin{pmatrix} 0 & \frac{1}{3} \\ 3 & 0 \end{pmatrix} c^{-1}, \\ p_w(0) &= \sqrt{\frac{3}{2}}c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c^{-1}, & q_v(0) &= c \begin{pmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} c^{-1}. \end{aligned}$$

Applying the isometry  $(p, q) \mapsto (cpc^{-1}, cq c^{-1})$  we obtain that any solution of (50) is congruent to an open subset of the immersion  $f = (p, q)$  where

$$p(u, v, w) = \begin{pmatrix} e^{\sqrt{\frac{3}{2}}w} & e^{\sqrt{\frac{3}{2}}w} (u + \frac{v}{3}) \\ 3e^{\sqrt{\frac{3}{2}}w} v & e^{\sqrt{\frac{3}{2}}w} (v^2 + 3uv) + e^{-\sqrt{\frac{3}{2}}w} \end{pmatrix}, \quad q(v) = \begin{pmatrix} 1 & \frac{2v}{3} \\ 0 & 1 \end{pmatrix}.$$

Finally, taking the change of coordinates  $w \rightarrow 2\sqrt{\frac{2}{3}}w$ ,  $u \rightarrow -\frac{1}{2}(u+v)$  and  $v \rightarrow \frac{3}{2}(u-v)$  we get the immersion in (43).

Now suppose that  $\omega_{ij}^k$  and  $h_{ij}^k$  satisfy (2) in Proposition 24. In this case,  $\omega_{22}^3$  is constant. We define the constant  $\lambda$  as  $\omega_{22}^3 = \sqrt{\frac{2}{3}}(1 - \lambda)$ .

Take the frame  $\left\{ \rho E_1, -\frac{1}{\sqrt{6}}\omega_{22}^3 E_1 + E_2, E_3 \right\}$ , where  $\rho$  is a non-vanishing smooth function and solution of

$$E_1(\rho) = E_2(\rho) = 0, \quad E_3(\rho) = -\sqrt{6}E_3.$$

Using Proposition 24 we can easily check that this is a coordinate frame. We call this frame  $\{f_u, f_v, f_w\}$ . First we notice that  $\rho = e^{-\sqrt{6}w}$ . We obtain from Equation (16) that

$$Qf_u = -f_u, \quad Qf_v = -\frac{2}{3\rho}f_u - f_v + \frac{2}{\sqrt{3}\rho}Jf_u, \quad Qf_w = -f_w,$$

We deduce

$$q_u = q_w = 0, \quad q_v = \frac{2}{3\rho}qp^{-1}p_u. \tag{51}$$

Equation (17) gives us the following expressions

$$\begin{aligned} \nabla_{f_u}^E f_u &= 0, & \nabla_{f_u}^E f_v &= \sqrt{\frac{3}{2}}\rho f_w, & \nabla_{f_v}^E f_v &= -\sqrt{\frac{2}{3}}f_w + \sqrt{2}Jf_w, \\ \nabla_{f_u}^E f_w &= -\sqrt{\frac{3}{2}}f_u, & \nabla_{f_w}^E f_w &= 0, & \nabla_{f_v}^E f_w &= \frac{2\lambda-1}{\sqrt{6\rho}}f_u - \sqrt{\frac{3}{2}}f_v + \frac{1}{\sqrt{2\rho}}Jf_u. \end{aligned}$$

From Equation (15) it follows

$$\begin{aligned} \langle f_u, f_u \rangle &= 0, & \langle f_u, f_v \rangle &= \frac{3\rho}{2}, & \langle f_u, f_w \rangle &= 0, \\ \langle f_v, f_v \rangle &= \lambda, & \langle f_v, f_w \rangle &= 0, & \langle f_w, f_w \rangle &= \frac{3}{2}. \end{aligned}$$

and

$$\begin{aligned} \langle f_u, Qf_u \rangle &= 0, & \langle f_u, Qf_v \rangle &= -\frac{3\rho}{2}, & \langle f_u, Qf_w \rangle &= 0, \\ \langle f_v, Qf_v \rangle &= -\lambda, & \langle f_v, Qf_w \rangle &= 0, & \langle f_w, Qf_w \rangle &= -\frac{3}{2}. \end{aligned}$$

In particular, we have

$$\begin{aligned} \langle p_u, p_u \rangle &= 0, & \langle p_v, p_v \rangle &= \lambda, & \langle p_w, p_w \rangle &= \frac{3}{2}, \\ \langle p_u, p_v \rangle &= \frac{3}{2}\rho, & \langle p_u, p_w \rangle &= 0, & \langle p_v, p_w \rangle &= 0. \end{aligned} \tag{52}$$

We may use Equation (18) to compute

$$\begin{aligned} f_{uu} &= 0, & f_{vv} &= -\sqrt{\frac{2}{3}}f_w + \sqrt{2}Jf_w + \frac{1}{2}\lambda f - \frac{1}{2}\lambda Qf, & f_{ww} &= \frac{3}{4}f - \frac{3}{4}Qf, \\ f_{uw} &= -\sqrt{\frac{3}{2}}f_u, & f_{vw} &= \frac{2\lambda-1}{\sqrt{6\rho}}f_u - \sqrt{\frac{3}{2}}f_v + \frac{1}{\sqrt{2\rho}}Jf_u, & f_{ww} &= \sqrt{\frac{3}{2}}\rho f_w + \frac{3\rho}{4}f - \frac{3\rho}{4}Qf. \end{aligned}$$

Hence we obtain

$$\begin{aligned} p_{uu} &= 0, & p_{uv} &= \sqrt{\frac{3}{2}}\rho p_w + \frac{3}{2}\rho p, & p_{uw} &= -\sqrt{\frac{3}{2}}p_u, \\ p_{vv} &= \lambda p, & p_{vw} &= \frac{1}{\rho}\sqrt{\frac{2}{3}}\lambda p_u - \sqrt{\frac{3}{2}}p_v, & p_{ww} &= \frac{3}{2}p, & q_{vv} &= 2\sqrt{\frac{2}{3}}qp^{-1}p_w. \end{aligned} \tag{53}$$

Applying an isometry of the type  $(p, q) \mapsto (ap, bq)$  we may assume that  $p(0) = \mathrm{Id}_2$  and  $q(0) = \mathrm{Id}_2$ . From (52) and (51) it follows that there exists a matrix  $c \in \mathrm{SL}^\pm(2\mathbb{R})$  such that

$$\begin{aligned} p_u(0) &= c \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} c^{-1}, & p_v(0) &= c \begin{pmatrix} 0 & \frac{\lambda}{3} \\ 3 & 0 \end{pmatrix} c^{-1}, \\ p_w(0) &= \sqrt{\frac{3}{2}}c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c^{-1}, & q_v(0) &= c \begin{pmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} c^{-1}. \end{aligned}$$

Applying the isometry  $(p, q) \mapsto (cpc^{-1}, cq c^{-1})$  of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  we obtain that any solution of (53) is congruent to the solution with  $c = \mathrm{Id}_2$ . After the change of coordinates  $w \rightarrow \sqrt{\frac{3}{2}}w$ ,  $u \rightarrow ue^{-\sqrt{\frac{3}{2}}w}$ , we obtain that such solution is the immersion  $f_\lambda$  in Example 26.  $\square$

#### 4.4. Extrinsically homogeneous Lagrangian submanifolds of type III.

**Example 29.** Let  $\mathbb{R} \times_{\varphi_2} \mathbb{R}^2$  be the Bianchi group of type VI with the group law

$$(t, u) \cdot (s, v) = (t + s, \varphi_2(s)u + v)$$

where  $\varphi_2: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$  is given by

$$\varphi_2(s) = \begin{pmatrix} e^{-2s} & 0 \\ 0 & e^s \end{pmatrix}.$$

Let  $\tilde{g}$  be the right invariant metric such that its components with respect to the frame of right invariant vector fields  $\left\{ \frac{\partial}{\partial t}, e^{-2t} \frac{\partial}{\partial u_1}, e^t \frac{\partial}{\partial u_2} \right\}$  are given by

$$\begin{pmatrix} 0 & 2 & \frac{7}{3}\sqrt{\frac{2}{3}} \\ 2 & 0 & 8\sqrt{\frac{2}{3}} \\ \frac{7}{3}\sqrt{\frac{2}{3}} & 8\sqrt{\frac{2}{3}} & \frac{128}{9} \end{pmatrix}.$$

Now let  $\mathfrak{h}$  be the Lie subalgebra of  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$  spanned by  $\{e_1, e_2, e_3\}$  where

$$\begin{aligned} e_1 &= \left( \frac{1}{18} (27 + 2\sqrt{6}) \mathbf{i} - \left( 2\sqrt{\frac{2}{3}} + \frac{3}{4} \right) \mathbf{j} + \left( \frac{8}{3}\sqrt{\frac{2}{3}} + \frac{3}{4} \right) \mathbf{k}, \right. \\ &\quad - \left( 1 + \frac{17}{12\sqrt{6}} \right) \mathbf{i} + \frac{1}{96} (48 - 17\sqrt{6}) \mathbf{j} - \left( \frac{1}{2} + \frac{85}{48\sqrt{6}} \right) \mathbf{k}, \\ &\quad \left. - \frac{1}{2} \mathbf{i} + \frac{1}{4} (1 - 3\sqrt{6}) \mathbf{j} + \frac{1}{4} (3\sqrt{6} - 1) \mathbf{k} \right), \\ e_2 &= \left( 0, \sqrt{\frac{2}{3}} \mathbf{i} + \frac{1}{2} \sqrt{\frac{3}{2}} \mathbf{j} + \frac{5}{2\sqrt{6}} \mathbf{k}, 0 \right), \\ e_3 &= \left( \frac{8}{9} (2 + 3\sqrt{6}) \mathbf{i} - \frac{2}{3} (7 + 2\sqrt{6}) \mathbf{j} + \frac{2}{9} (37 + 6\sqrt{6}) \mathbf{k}, 0, -6\mathbf{j} + 6\mathbf{k} \right). \end{aligned}$$

The Lie algebra  $\mathfrak{h}$  is a Bianchi Lie algebra of type VI with brackets

$$[e_1, e_2] = -2e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0.$$

Therefore  $\mathbb{R} \times_{\varphi_2} \mathbb{R}^2$  is the universal cover of  $\exp(\mathfrak{h})$ . Moreover, the map defined as

$$(v, u, w) \mapsto \exp\left( ve_1 + \frac{2ue^{2v}v}{e^{2v}-1}e_2 + \frac{vw}{e^v-1}e_3 \right)$$

and as  $\exp(2ue_2 + we_3)$  when  $v = 0$ , is a group isomorphism. Thus  $\exp(\mathfrak{h}) \cong \mathbb{R} \times_{\varphi_2} \mathbb{R}^2$ . One can check that the immersion  $j: \mathbb{R} \times_{\varphi_2} \mathbb{R}^2 \rightarrow \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  given by

$$j(u, v, w) = \exp\left( ve_1 + \frac{2ue^{2v}v}{e^{2v}-1}e_2 + \frac{vw}{e^v-1}e_3 \right) \cdot (\text{Id}_2, \text{Id}_2)$$

is a Lagrangian immersion, whose image is extrinsically homogeneous. The frame given by

$$E_1 = -\sqrt{\frac{3}{2}}e^{-2v}J_u, \quad E_2 = -\frac{1}{\sqrt{6}}J_v, \quad E_3 = \frac{7e^{-2v}}{4\sqrt{6}}J_u + \sqrt{6}J_v - \frac{3}{4}e^vJ_w.$$

is a  $\Delta_2$ -orthonormal frame with respect to which  $A$  and  $B$  take type III form from Lemma 7.

**Proposition 30.** *Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type III form from Lemma 7 with respect to a  $\Delta_2$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then  $M$  is congruent to an open subset of the submanifold given in Example 29.*

*Proof.* By Lemma 8 we may apply the isometry  $\Psi_{0,1}$  in (19) and assume that the sign of  $B$  is  $-1$ . Proposition 15 implies that the components of the second fundamental form and of the connection associated to  $E_1, E_2$  and  $E_3$  are all constant. From the Codazzi equation (24) with  $X = E_1$ ,

$Y = E_2$  and  $Z = E_2$  we have  $h_{22}^2 = \frac{2\sqrt{2}}{3}$ . Then, computing the Codazzi equation with  $X = E_3$ ,  $Y = E_2$  and  $Z = E_2$  we obtain

$$h_{22}^1 = -\frac{13}{18\sqrt{2}}, \quad h_{22}^3 = \frac{5\sqrt{2}}{9}.$$

We define the frame  $\left\{ \rho E_1, E_2, \rho^{-1/2} \left( \frac{7}{12\sqrt{6}} E_1 + \sqrt{6} E_2 + \frac{1}{\sqrt{6}} E_3 \right) \right\}$  where  $\rho$  is a non-vanishing smooth function that satisfies

$$E_1(\rho) = 0, \quad E_2(\rho) = \sqrt{\frac{2}{3}}\rho, \quad E_3(\rho) = -2\sqrt{6}\rho.$$

From (30) we can check that  $\rho$  indeed exists and that the defined frame is a coordinate frame. We write said frame as  $\{f_u, f_v, f_w\}$ . Hence  $\rho = e^{\sqrt{\frac{2}{3}}v}$ . Equation (16) yields

$$\begin{aligned} Qf_u &= f_u, \\ Qf_v &= -\frac{23}{18\rho}f_u - 3f_v + 2\sqrt{\frac{2\rho}{3}}f_w - \frac{7}{6\sqrt{3}\rho}Jf_u - 4\sqrt{3}Jf_v + 2\sqrt{2\rho}Jf_w, \\ Qf_w &= -\frac{7\rho^{-3/2}}{\sqrt{6}}f_u - 4\sqrt{\frac{6}{\rho}}f_v + 5f_w - \frac{5\rho^{-3/2}}{3\sqrt{2}}Jf_u - 12\sqrt{\frac{2}{\rho}}Jf_v + 4\sqrt{3}Jf_w. \end{aligned}$$

If we denote  $f = (p, q)$  then the first equation implies that  $p_u = 0$ . From the second and third equations we obtain

$$p_v = pq^{-1} \left( -\frac{5q_u}{6\rho} - 4q_v + 2\sqrt{\frac{2}{3}}\sqrt{\rho}q_w \right), \quad p_w = pq^{-1} \left( 4q_w - \frac{13q_u + 72\rho q_v}{3\sqrt{6}\rho^{3/2}} \right). \quad (54)$$

The relation between the connection  $\nabla^E$  associated to the product metric and the nearly Kähler connection  $\tilde{\nabla}$  given in Equation (17) yields

$$\begin{aligned} \nabla_{f_u}^E f_u &= 0, \\ \nabla_{f_u}^E f_v &= -\frac{11}{4\sqrt{6}}f_u - 3\sqrt{6}\rho f_v + 3\rho^{3/2}f_w + \frac{1}{\sqrt{2}}Jf_u, \\ \nabla_{f_u}^E f_w &= -\frac{13}{4\sqrt{\rho}}f_u - 18\sqrt{\rho}f_v + 3\sqrt{6}\rho f_w + \sqrt{\frac{3}{\rho}}Jf_u, \\ \nabla_{f_v}^E f_v &= \frac{49}{48\sqrt{6}\rho}f_u + \frac{29}{2\sqrt{6}}f_v - \frac{7\sqrt{\rho}}{4}f_w - \frac{17}{9\sqrt{2}\rho}Jf_u - 5\sqrt{2}Jf_v + 2\sqrt{3\rho}Jf_w, \\ \nabla_{f_v}^E f_w &= \frac{91}{96\rho^{3/2}}f_u + \frac{53}{4\sqrt{\rho}}f_v - \frac{43}{4\sqrt{6}}f_w - \frac{25}{6\sqrt{3}\rho^{3/2}}Jf_u - 7\sqrt{\frac{3}{\rho}}Jf_v + \frac{9}{\sqrt{2}}Jf_w, \\ \nabla_{f_w}^E f_w &= \frac{19}{3\sqrt{6}\rho^2}f_u + \frac{12\sqrt{6}}{\rho}f_v - \frac{10}{\sqrt{\rho}}f_w - \frac{17}{3\sqrt{2}\rho^2}Jf_u - \frac{12\sqrt{2}}{\rho}Jf_v + 6\sqrt{\frac{3}{\rho}}Jf_w. \end{aligned}$$

From the relation between the product metric and the nearly Kähler metric in Equation (15) it follows

$$\begin{aligned} \langle f_u, f_u \rangle &= 0, & \langle f_v, f_v \rangle &= 0, & \langle f_w, f_w \rangle &= \frac{4}{\rho}, \\ \langle f_u, f_v \rangle &= \frac{3}{2}\rho, & \langle f_u, f_w \rangle &= 3\sqrt{\frac{3\rho}{2}}, & \langle f_v, f_w \rangle &= \frac{5}{8}\sqrt{\frac{3}{2\rho}}. \end{aligned} \quad (55)$$

and

$$\begin{aligned} \langle f_u, Qf_u \rangle &= 0, & \langle f_v, Qf_v \rangle &= -\frac{4}{3}, & \langle f_w, Qf_w \rangle &= -\frac{4}{\rho}, \\ \langle f_u, Qf_v \rangle &= \frac{3}{2}\rho, & \langle f_u, Qf_w \rangle &= 3\sqrt{\frac{3\rho}{2}}, & \langle f_v, Qf_w \rangle &= -\frac{49}{8\sqrt{6}\rho}. \end{aligned} \quad (56)$$

In particular we have

$$\begin{aligned} \langle q_u, q_u \rangle &= 0, & \langle q_v, q_v \rangle &= -\frac{2}{3}, & \langle q_w, q_w \rangle &= 0, \\ \langle q_u, q_v \rangle &= \frac{3}{2}\rho, & \langle q_u, q_w \rangle &= 3\sqrt{\frac{3\rho}{2}}, & \langle q_v, q_w \rangle &= -\frac{17}{8\sqrt{6\rho}}. \end{aligned} \quad (57)$$

Now we compute

$$\begin{aligned} f_{uu} &= 0, \\ f_{uv} &= -\frac{11}{4\sqrt{6}}f_u - 3\sqrt{6}\rho f_v + 3\rho^{3/2}f_w + \frac{1}{\sqrt{2}}Jf_u + \frac{3\rho}{4}f + \frac{3\rho}{4}Qf, \\ f_{uw} &= -\frac{13}{4\sqrt{\rho}}f_u - 18\sqrt{\rho}f_v + 3\sqrt{6}\rho f_w + \sqrt{\frac{3}{\rho}}Jf_u + \frac{3}{2}\sqrt{\frac{3}{2}}\sqrt{\rho}f + \frac{3}{2}\sqrt{\frac{3\rho}{2}}Qf, \\ f_{vv} &= \frac{49}{48\sqrt{6\rho}}f_u + \frac{29}{2\sqrt{6}}f_v - \frac{7\sqrt{\rho}}{4}f_w - \frac{17}{9\sqrt{2\rho}}Jf_u - 5\sqrt{2}Jf_v + 2\sqrt{3\rho}Jf_w - \frac{2}{3}Qf, \\ f_{vw} &= \frac{91}{96\rho^{3/2}}f_u + \frac{53}{4\sqrt{\rho}}f_v - \frac{43}{4\sqrt{6}}f_w - \frac{25}{6\sqrt{3\rho^{3/2}}}Jf_u - 7\sqrt{\frac{3}{\rho}}Jf_v + \frac{9}{\sqrt{2}}Jf_w + \frac{5}{16}\sqrt{\frac{3}{2\rho}}f - \frac{49}{16\sqrt{6\rho}}Qf, \\ f_{ww} &= \frac{19}{3\sqrt{6\rho^2}}f_u + \frac{12\sqrt{6}}{\rho}f_v - \frac{10}{\sqrt{\rho}}f_w - \frac{17}{3\sqrt{2\rho^2}}Jf_u - \frac{12\sqrt{2}}{\rho}Jf_v + 6\sqrt{\frac{3}{\rho}}Jf_w + \frac{2}{\rho}f - \frac{2}{\rho}Qf. \end{aligned}$$

Hence  $q$  satisfies

$$\begin{aligned} q_{uu} &= 0, \\ q_{ww} &= 0, \\ q_{uv} &= \frac{1}{8} \left( 12\rho \left( q - 2\sqrt{6}q_v + 2\sqrt{\rho}q_w \right) - 5\sqrt{6}q_u \right), \\ q_{uw} &= \frac{1}{4\sqrt{\rho}} \left( 6\rho \left( \sqrt{6}q - 12q_v + 2\sqrt{6\rho}q_w \right) - 17q_u \right), \\ q_{vv} &= \frac{1}{864\rho} \left( 323\sqrt{6}q_u - 72\rho \left( 8q - 17\sqrt{6}q_v + 13\sqrt{\rho}q_w \right) \right), \\ q_{vw} &= \frac{1}{288\rho^{3/2}} \left( 289q_u - 6\rho \left( 17\sqrt{6}q - 204q_v + 30\sqrt{6\rho}q_w \right) \right), \end{aligned} \quad (58)$$

and  $p$  satisfies

$$\begin{aligned} p_{vv} &= \frac{2}{3}p - \frac{1}{18\sqrt{6\rho}} \left( 19pq^{-1}q_u + 72\rho pq^{-1}q_v \right), \\ p_{vw} &= \frac{1}{6\rho^{3/2}} \left( 4\rho \left( \sqrt{6}p - 6pq^{-1}q_v \right) - 7pq^{-1}q_u \right), \\ p_{ww} &= \frac{1}{9\rho^2} \left( 36\rho \left( p - 2\sqrt{6}pq^{-1}q_v + \sqrt{\rho}pq^{-1}q_w \right) - 17\sqrt{6}pq^{-1}q_u \right). \end{aligned} \quad (59)$$

After applying an isometry of the type  $(p, q) \mapsto (ap, bq)$ , we may assume that  $p(0) = \text{Id}_2$  and  $q(0) = \text{Id}_2$ . From (54) and (57) it follows that there exists a matrix  $c \in \text{SL}^\pm(2, \mathbb{R})$  such that

$$\begin{aligned} q_u(0) &= c \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{pmatrix} c^{-1}, \\ q_w(0) &= c \begin{pmatrix} 0 & 0 \\ 3\sqrt{\frac{3}{2}} & 0 \end{pmatrix} c^{-1}, \end{aligned}$$

$$\begin{aligned}
q_v(0) &= c \begin{pmatrix} \frac{1}{72}(-17 + 6\sqrt{6}) & -\frac{17}{36} \\ \frac{1}{144}(233 - 12\sqrt{6}) & \frac{1}{72}(17 - 6\sqrt{6}) \end{pmatrix} c^{-1}, \\
p_v(0) &= c \begin{pmatrix} \frac{1}{9} - \sqrt{\frac{2}{3}} & \frac{2}{9} \\ \sqrt{\frac{2}{3}} - \frac{1}{18} & \sqrt{\frac{2}{3}} - \frac{1}{9} \end{pmatrix} c^{-1}, \\
p_w(0) &= c \begin{pmatrix} -2 + \frac{2}{9}\sqrt{6} & \frac{4}{3}\sqrt{\frac{2}{3}} \\ 2 - \frac{1}{3}\sqrt{\frac{2}{3}} & 2 - \frac{2}{9}\sqrt{6} \end{pmatrix} c^{-1}.
\end{aligned}$$

By applying the isometry  $(p, q) \mapsto (cpc^{-1}, cqc^{-1})$  we obtain that any solution of the system of differential equations given in (58) and (59) is congruent to the solution with  $c = \text{Id}_2$ . This solution is the map given in Example 29 after the change of coordinates  $v \rightarrow \frac{v}{\sqrt{6}}$ ,  $u \rightarrow \sqrt{\frac{3}{2}}u$ ,  $w \rightarrow \frac{1}{4}\sqrt{\frac{3}{2}}w$ .  $\square$

#### 4.5. Extrinsically homogeneous Lagrangian submanifolds of type IV.

**Proposition 31.** *There are no extrinsically homogeneous Lagrangian submanifolds of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  of type IV in Lemma 7.*

*Proof.* Let  $M$  be an extrinsically homogeneous Lagrangian submanifold of the pseudo-nearly Kähler  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ . Suppose that  $A$  and  $B$  take type IV form with respect to a  $\Delta_3$ -orthonormal frame  $\{E_1, E_2, E_3\}$ . Then by Proposition 16 the functions  $\psi$ ,  $\theta_1$ ,  $\theta_2$ ,  $h_{ij}^k$  and  $\omega_{ij}^k$  are constant. Thus by Lemma 12 the functions  $h_{ij}^k$  are all zero except for  $h_{11}^3$ . Recall that we can write  $\theta_2 = -2\theta_1 + k\pi$ , with  $k = 0, 1$ . Computing the Codazzi equation (24) with  $X = E_3$ ,  $Y = E_1$ ,  $Z = E_1$  and  $X = E_1$ ,  $Y = E_2$ ,  $Z = E_2$  yields

$$(h_{11}^3)^2 = \frac{1}{3}(\cosh \psi - \cos 6\theta_1) \cosh \psi, \quad (60)$$

and

$$\sqrt{\frac{2}{3}}h_{11}^3 = \frac{4(-1)^k \sinh(\psi) (-\cos 6\theta_1 \cosh \psi + 3(h_{11}^3)^2 + \cosh^2(\psi))}{3(\cos 6\theta_1 - \cosh \psi)}. \quad (61)$$

Plugging (60) into (61) gives

$$h_{11}^3 = 2\sqrt{\frac{2}{3}}(-1)^{k+1} \sinh 2\psi.$$

Comparing both values of  $(h_{11}^3)^2$  we derive  $\cos 6\theta_1 = 9 \cosh \psi - 8 \cosh 3\psi$ . Then writing the Codazzi equation (24) with  $X = E_1$ ,  $Y = E_3$  and  $Z = E_2$  we obtain  $\sinh 2\psi = 0$ . This is a contradiction since by Lemma 7,  $\psi$  is different from zero.  $\square$

## 5. PROOF OF THE MAIN THEOREM

*Proof of Theorem 2.* By Lemma 7 we separate the argument into four cases.

In [2], the authors proved that any totally geodesic submanifold is of type I and is congruent to one of the first three examples given in Theorem 2. By Proposition 19, Proposition 21 and Proposition 23, any non-totally geodesic extrinsically homogeneous Lagrangian submanifold of type I is congruent to either Example 22 or to Example 20.

Proposition 28 implies that any extrinsically homogeneous Lagrangian submanifold of type II is congruent to an open subset of either the image of the immersion in Example 25 or the image of the immersion in Example 26.

Proposition 30 states that any extrinsically homogeneous Lagrangian submanifold of type III is congruent to the one given in Example 29.



Proposition 31 shows that there are no extrinsically homogeneous Lagrangian submanifolds of type IV.

Except for the first, sixth and seventh examples in Theorem 2, all the submanifolds are not isometric and therefore not congruent. Hence, it only remains to distinguish between the aforementioned cases.

First, the first submanifold in Theorem 2 is the first example of Theorem 18, therefore the only one of these three that is totally geodesic. The sixth submanifold is the immersion  $\iota$  given in Example 25 and the seventh one is the family of immersions  $f_\lambda$  given in Example 26.

Suppose that  $\iota$  is congruent to  $f_\lambda$  for some  $\lambda$ . That means, there exists a isometry  $\mathcal{F}$  of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  that maps one into the other. Suppose that  $\mathcal{F} \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathbb{Z}_2$ . These isometries preserve  $P$  and  $J$ , hence  $A$  and  $B$  have the same shape with respect to  $\{E_i\}_i$  and with respect to  $\{\mathcal{F}_*E_i\}_i$ . Hence,

$$-\frac{\sqrt{2}}{3} = g(h(E_2, E_2), JE_3) = g(\mathcal{F}_*h(E_2, E_2), \mathcal{F}_*JE_3) = g(h(\mathcal{F}_*E_2, \mathcal{F}_*E_2), J\mathcal{F}_*E_3) = \frac{2\sqrt{2}}{3},$$

which is a contradiction.

In Theorem 1 we showed that the isometry group of  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$  is a semidirect product of  $S_3$  with  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times \mathbb{Z}_2$ . Therefore, to complete the proof we can assume that  $\mathcal{F} \in S_3$ , i.e.  $\mathcal{F} = \Psi_{\kappa, \tau}$  for some  $\kappa \in \{0, 1\}$ ,  $\tau \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ . Moreover, we can assume that  $\tau \neq 0$ , since otherwise  $\mathcal{F}$  preserves  $P$ , and therefore we may use the same argument as before, up to sign. From Lemma 8 it follows that  $P$  restricted to  $\mathcal{F}(M)$  takes the shape  $\tilde{A} + J\tilde{B}$ , where

$$\begin{aligned}\tilde{A} &= \cos \tau A + (-1)^\kappa \sin \tau B, \\ \tilde{B} &= -\sin \tau A + (-1)^\kappa \cos \tau B.\end{aligned}$$

For  $\tau \neq 0$  these matrices have a different form than  $A$  and  $B$ , as it can be seen in Lemma 4 of [2]. Therefore, there does not exist such a  $\Delta_2$ -orthonormal frame  $\{E_i\}_i$  such that  $A$  and  $B$  take type II form in Lemma 7 on  $\mathcal{F}(M)$ , which is a contradiction.

Similar arguments can be used to distinguish between  $f_{\lambda_1}$  and  $f_{\lambda_2}$  for  $\lambda_1 \neq \lambda_2$ , by considering the function  $\omega_{33}^2 = \sqrt{\frac{2}{3}}(1 - \lambda)$  instead of  $h_{22}^3$ .  $\square$

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