Stabilité de quelques problèmes d'évolution

Julie Valein

N° d'ordre : 08/36

THÈSE

présentée à

L'UNIVERSITÉ DE VALENCIENNES ET DU HAINAUT-CAMBRÉSIS

ECOLE DOCTORALE RÉGIONALE SCIENCES POUR L'INGÉNIEUR
LILLE NORD-DE-FRANCE - 072

en vue d'obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ

Spécialité : MATHÉMATIQUES APPLIQUÉES

par

Julie VALEIN

Stabilité de quelques problèmes d'évolution

Soutenue le 20 Novembre 2008 devant le jury composé de

Directeur de thèse :	Serge NICAISE, Université de Valenciennes et du Hainaut-Cambrésis
Rapporteurs :	Kaïs AMMARI, Faculté des Sciences de Monastir
	Marius TUCSNAK, Université Henri Poincaré Nancy 1
Examinateurs :	Félix ALI MEHMETI, Université de Valenciennes et du Hainaut-Cambrésis
	Joachim VON BELOW, Université du Littoral Côte d'Opale
	Olivier GOUBET, Université de Picardie Jules Verne

N° d'ordre : 08/36

THÈSE

présentée à

L'UNIVERSITÉ DE VALENCIENNES ET DU HAINAUT-CAMBRÉSIS

ECOLE DOCTORALE RÉGIONALE SCIENCES POUR L'INGÉNIEUR LILLE NORD-DE-FRANCE - 072

en vue d'obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ

Spécialité : MATHÉMATIQUES APPLIQUÉES

par

Julie VALEIN

Stabilité de quelques problèmes d'évolution

Soutenue le 20 Novembre 2008 devant le jury composé de

Directeur de thèse :	Serge NICAISE, Université de Valenciennes et du Hainaut-Cambrésis
Rapporteurs :	Kaïs AMMARI, Faculté des Sciences de Monastir
	Marius TUCSNAK, Université Henri Poincaré Nancy 1
Examinateurs :	Félix ALI MEHMETI, Université de Valenciennes et du Hainaut-Cambrésis
	Joachim VON BELOW, Université du Littoral Côte d'Opale
	Olivier GOUBET, Université de Picardie Jules Verne

Stabilité de quelques problèmes d'évolution

Dans cette thèse nous étudions la stabilisation de quelques équations d'évolution par rétroaction (feedback). Tout d'abord, nous considérons la stabilisation de l'équation des ondes sur des réseaux 1-d par des feedbacks situés aux noeuds. Dans le premier chapitre, en supposant que le poids du feedback avec retard est plus petit que celui sans retard, nous donnons des conditions spectrales pour obtenir la stabilité forte, exponentielle ou polynomiale en nous ramenant à l'étude d'une inégalité d'observabilité pour le problème conservatif. Dans le second chapitre, nous transférons des inégalités d'observabilité à poids déjà existantes pour un autre problème conservatif en inégalités d'observabilité faibles pour le système dissipé sans retard. Grâce à une inégalité d'interpolation, nous obtenons des taux de décroissance explicites qui dépendent des propriétés géométriques et topologiques du réseau. Nous développons ensuite, dans le chapitre 3, une théorie abstraite pour les équations d'évolution du second ordre avec retard généralisant les résultats du chapitre 1. Nous étudions le cas où le retard dépend du temps pour les équations des ondes et de la chaleur dans le chapitre 4. En émettant certaines hypothèses sur ce retard et en utilisant une fonctionnelle de Lyapunov appropriée, nous prouvons que l'énergie est exponentiellement décroissante et nous donnons explicitement son taux de décroissance. Enfin, nous montrons dans le chapitre 5, qu'une technique de filtrage permet d'obtenir une décroissance quasi-exponentielle de l'équation des ondes discrétisée en espace par différences finies avec un amortissement interne.

Mots-clés : Théorie du contrôle, stabilisation, équations d'évolution, terme de retard, réseaux, semi-discrétisation.

Stability of some evolution problems

In this PhD thesis we study the stabilization of some evolution equations by feedback laws. First we consider the stabilization of the wave equation on 1-d networks with nodal feedbacks. In chapter 1, assuming that the weight of the feedback without delay is smaller than the one with delay, we give spectral conditions to obtain the strong, exponential or polynomial stability, by studying an observability inequality for the conservative system. In chapter 2 we transfer known observability results for another conservative system into a weighted observability estimate for the dissipative one without delay. Thanks to an interpolation inequality, we obtain explicit decay rates which depend on the geometric and topological properties of the network. Then we develop, in chapter 3, an abstract theory for second order evolution equation with delay, which generalizes the results of chapter 1. We study the case where the delay depends on time for the heat and wave equations in chapter 4. Using some assumptions about the delay and an appropriate Lyapunov functional, we prove that the energy is exponentially decreasing and we give explicitely its decay rate. Finally, we show, in chapter 4, that a filtering technique allows to obtain a quasi-exponential decay of a finite difference space discretization of the wave equation by pointwise interior stabilization.

Key words : Control theory, stabilization, evolution equations, delay term, network, semi-discretization.

Spécialité : Mathématiques Appliquées

Laboratoire de Mathématiques et leurs Applications (LAMAV), Université de Valenciennes et du Hainaut-Cambrésis, Le Mont-Houy, 59313 Valenciennes Cedex 9

Remerciements

En premier lieu, je tiens à remercier chaleureusement Serge Nicaise pour sa disponibilité, son dynamisme et sa gentillesse. Travailler avec lui fut un grand plaisir qui, je l'espère, pourra se poursuivre. Il a su me guider avec un enthousiasme constant et communicatif, et m'encourager pendant ces trois années. Il m'a témoigné sa confiance en m'incitant à participer à de nombreuses conférences. Ses grandes qualités scientifiques et humaines ont été indispensables à l'élaboration de cette thèse. Pour tout cela, je ne l'en remercierai jamais assez.

Ma gratitude va à Kaïs Ammari et à Marius Tucsnak d'avoir accepté de rapporter sur mes travaux. Leur lecture attentive et leurs remarques judicieuses ont été précieuses.

Je suis également reconnaissante à Félix Ali Mehmeti, Joachim von Below et Olivier Goubet pour leur participation à mon jury.

J'apprécie l'intérêt que tous ont porté à mon travail. C'est pour moi un grand honneur d'avoir un tel jury.

J'exprime ma profonde et sincère reconnaissance à Enrique Zuazua pour m'avoir accueillie à l'Universidad Autonoma de Madrid. Ces six mois de travail en Espagne m'ont beaucoup appris. J'ai particulièrement apprécié sa grande culture mathématique et sa faculté à exprimer simplement des idées parfois complexes. J'ai été sensible à l'accueil cordial et à la gentillesse des membres du département de mathématiques de cette université.

Un grand merci à Sylvain Ervedoza pour toutes les discussions mathématiques ou autres que nous avons eues à Madrid et qui se poursuivent aujourd'hui.

Les membres du LAMAV m'ont accueillie comme une collègue à part entière et m'ont permis de passer trois années très agréables. Ils m'ont fait confiance en me proposant des enseignements intéressants et enrichissants ; leurs appuis et leurs conseils pédagogiques m'ont aidée dans mes fonctions nouvelles d'enseignante. Je les en remercie vivement, ainsi que Nabila Daifi pour son aide administrative précieuse et sa bonne humeur.

Je veux également faire part de mon amitié à Isabelle, ma collègue de bureau.

Je tiens à remercier les informaticiens du « couloir d'à côté » : Célia, Christophe, Dana, Dimitri, Eric, Fred, Nico, Patrick, pour tous les bons moments de détente passés ensemble.

A toutes les personnes que j'ai rencontrées lors des conférences et avec qui j'ai eu de fructueuses discussions, merci.

A tous mes amis extérieurs à l'université qui m'ont permis de m'évader de mes travaux et de décompresser : Yann, Ludo, Claire, Laure, Ludo, Anais, Caro, Sophie, et à mon professeur de danse Sophie Cirier, un grand merci.

Enfin, je souhaite de tout coeur remercier ma famille : mes parents, ma soeur Sarah et mes grands-parents, pour leur amour et leur soutien sans faille. Je les remercie de m'avoir supportée et encouragée pendant les moments de doute. Je n'aurai jamais pu faire cette thèse sans eux.

Table des matières

Introduction

La Théorie du Contrôle des Equations aux Dérivées Partielles intervient dans différents contextes et de plusieurs manières.

Les problèmes de contrôlabilité, d'observabilité et de stabilité des Equations aux Dérivées Partielles ont fait l'objet, récemment, de nombreux travaux. Dans cette thèse nous nous sommes intéressés à l'étude de la stabilisation de quelques équations d'évolution.

Le problème de contrôlabilité peut se formuler simplement : considérons un système d'évolution décrit par des équations différentielles ordinaires ou par des équations aux dérivées partielles et un intervalle de temps $[0, T]$. Peut-on amener les solutions d'un état initial (au temps $t = 0$) à un état final (au temps $t = T$) en agissant par un contrôle approprié appliqué sur le bord ou dans une partie du domaine dans laquelle l'équation évolue ?

Il y a eu d'intensives recherches sur le sujet depuis les trois dernières décennies. Nous renvoyons par exemple aux livres de J.-L. Lions [70, 71], de Lasiecka et Triggiani [67] et de Fattorini [43], aux articles de Russell [100] et de Zuazua [111, 113]...

Dans un cadre fonctionnel approprié, le problème de contrôlabilité est équivalent à celui d'observabilité qui consiste à analyser si l'énergie totale des solutions peut être évaluée au moyen de mesures partielles sur un sous-ensemble du domaine ou du bord. Pour obtenir des estimées d'observabilité il existe diverses méthodes, comme la technique des multiplicateurs (par exemple [61, 70]), l'analyse microlocale ([15, 25]), les inégalités de Carleman ([28, 44, 46, 69]) ou encore les critères fréquentiels ([26, 77]), les critères spectraux ([73, 95]) ou les inégalités d'Ingham ([53, 57, 62]).

La stabilisation a pour but d'atténuer les vibrations par rétro-action (feedback) ; elle consiste donc à garantir la décroissance de l'énergie des solutions vers 0 de façon plus ou moins rapide par un mécanisme de dissipation.

Plus précisément, le problème de stabilisation auquel on s'intéresse revient à déterminer le comportement asymptotique de l'énergie que l'on note $E(t)$ (c'est la norme des solutions dans l'espace d'état), à étudier sa limite afin de déterminer si cette limite est nulle ou pas, et, si cette limite est nulle, à donner une estimation de la vitesse de décroissance de l'énergie vers zéro.

On peut noter des différences entre les problèmes de contrôlabilité et ceux de stabilité. D'une part, dans les premiers le temps varie dans un intervalle fini $[0, T]$, alors que pour les problèmes de stabilisation le temps t tend vers l'infini. D'autre part, dans les problèmes de contrôlabilité, le contrôle peut entrer dans le système librement en boucle ouverte, tandis que pour les problèmes de stabilisation, le contrôle est de la forme feedback ou en boucle fermée.

Malgré cela, les liens entre les problèmes de stabilisation et de contrôlabilité sont étroits et l'on démontre certaines implications entre ces deux problèmes (voir par exemple [71,84, 100]). Cependant, la stabilisation ne peut pas toujours être obtenue comme conséquence de la contrôlabilité, et c'est pour cela que son étude est souvent faite indépendamment et directement.

Il existe plusieurs degrés de stabilité que l'on peut étudier. Le premier degré consiste à analyser simplement la décroissance de l'énergie des solutions vers zéro, i.e. :

$$E(t) \to 0, \quad \text{lorsque } t \to +\infty.$$

C'est ce que l'on appelle la stabilisation forte.

Pour le second, on s'intéresse à la décroissance de l'énergie la plus rapide, c'est-à-dire lorsque celle-ci tend vers 0 de manière exponentielle, i.e. :

$$E(t) \leq Ce^{-\delta t}, \quad \forall t > 0,$$

où C et δ sont des constantes positives avec C qui dépend des données initiales.

Quant au troisième, il étudie des situations intermédiaires, dans lesquelles la décroissance des solutions n'est pas exponentielle, mais du type polynomial par exemple :

$$E(t) \leq \frac{C}{t^{\alpha}}, \quad \forall t > 0,$$

où C et α sont des constantes positives avec C qui dépend des données initiales. Dans ce cas, il faut prendre des données initiales plus régulières, dans le domaine de l'opérateur.

Nos travaux s'orientent dans trois directions : la stabilisation sur des réseaux 1-d, la stabilisation avec un terme de retard, et l'aspect numérique.

L'étude des équations aux dérivées partielles sur des structures en forme de réseau ou de graphe a connu de réelles avancées depuis le début des années 80 ([4,76,83]). Récemment, de nombreux auteurs se sont intéressés aux problèmes de contrôle sur les réseaux unidimensionnels, nous pouvons citer par exemple les livres [38], [63], [64] et [65]. Ces études utilisent des résultats dans plusieurs domaines : séries de Fourier non-harmoniques, approximations Diophantiennes, théorie des graphes, techniques de propagation des ondes. Dans [38], il est démontré des résultats de contrôlabilité en fonction des propriétés d'irrationnalité des rapports des longueurs des cordes du réseau. Dans le cas des réseaux en forme d'étoile ou d'arbre, la contrôlabilité est prouvée en utilisant la formule de D'Alembert, tandis que pour les réseaux généraux (qui peuvent contenir des circuits), les auteurs utilisent le théorème de Beurling-Malliavin. Dans les deux premiers chapitres de cette thèse, nous nous sommes intéressés à la stabilisation de l'équation des ondes sur un réseau de cordes en utilisant deux méthodes différentes que l'on présentera plus tard.

Une recherche très active s'est amorcée récemment sur les problèmes de stabilisation avec effet retard. Les phénomènes de retard (en temps) apparaissent dans de nombreuses applications, par exemple en biologie [51], en mécanique [1] ou encore en automatique [101]. Il est bien connu que le terme avec retard dans le feedback peut être la cause d'instabilité [16–18] : un retard arbitrairement petit dans le feedback peut déstabiliser le système comme le montre [39–41,75,85,98,110]. Mais, a contrario, un terme de retard peut aussi améliorer la performance du système [1,101]. Les problèmes de stabilité de systèmes avec retard revêtent donc une importance non négligeable et les chapitres 1, 3 et 4 de ce travail traitent de ces questions. Pour régler les problèmes d'instabilité qui peuvent survenir dans les systèmes avec retard, nous avons considéré dans ces trois chapitres deux types de dissipation : un premier opérateur de feedback sans retard et un second avec retard. Pour avoir la stabilité de ces systèmes, nous avons supposé que le feedback avec retard est majoré par celui sans retard (dans un sens que nous préciserons plus tard) dans le but de compenser les effets d'instabilité qui peuvent intervenir. Cette idée a été introduite par Xu, Yung et Li [110] et Nicaise et Pignotti [85]. Dans le premier chapitre nous avons considéré l'équation des ondes sur un réseau 1-d avec un terme de retard dans les feedbacks. Dans le chapitre 3 nous avons généralisé cette approche dans le cadre des équations d'évolution abstraites du second ordre avec un terme de retard dans les feedbacks (non bornés). Dans le chapitre 4, nous avons étudié le cas où le retard dépend du temps (et donc n'est plus constant) pour les équations unidimensionnelles de la chaleur et des ondes.

Une autre question intéressante dans le domaine du contrôle et de la stabilisation porte sur l'étude des approximations numériques associées au système considéré. En pratique, le modèle continu est approché par un modèle discret en espace ou en temps (par la méthode des différences finies, des éléments finis ou des volumes finis). Dans de nombreux problèmes de contrôle, le schéma numérique a tendance à mal se comporter et, en général, le contrôle du problème discret ne converge pas vers le contrôle du problème continu. Ce phénomène est dû aux modes étrangers que le schéma numérique introduit à hautes fréquences ; le schéma génère des oscillations à hautes fréquences qui n'existent pas dans le modèle continu. Par conséquent, contrôler un modèle numérique discrétisé ne garantit pas d'obtenir une bonne approximation numérique du contrôle du modèle continu. L'inégalité d'observabilité discrète n'est en fait pas uniforme par rapport au pas de discrétisation, en raison de l'existence des solutions à hautes fréquences dont la vitesse de groupe de propagation est de l'ordre du paramètre de discrétisation. Il se produit le même phénomène dans le cas de la stabilisation avec un taux de décroissance de l'énergie discrétisée non-uniforme par rapport au pas de discrétisation [14, 96, 102, 103]. Ce phénomène a été mis en évidence dans les années 1990 ([49, 50]), et ces auteurs ont proposé des techniques très efficaces afin de restaurer la convergence du contrôle discret, comme la régularisation de Tychonoff [49, 50], l'algorithme bi-grille [47, 79], la méthode des éléments finis mixtes [14,30,31,48,78], la viscosité numérique [96, 102] ou le filtrage des hautes fréquences [56, 112]. Ces méthodes ont pour effet de redresser le spectre discret pour les hautes fréquences. Nous avons étudié, dans le dernier chapitre de cette thèse, la stabilisation de l'équation des ondes discrétisée en espace par différences finies avec un amortissement en un point intérieur et avons prouvé une décroissance quasi-exponentielle en filtrant les hautes fréquences.

Nous allons maintenant rappeler brièvement les principaux outils et méthodes employés. Nous utilisons principalement, dans les chapitres 1, 3 et 5, la méthodologie de Ammari et Tuscnak [11] pour les équations d'évolution du second ordre. Les résultats de stabilité sont basés sur l'obtention d'inégalités d'observabilité pour le problème conservatif associé.

Considérons donc l'équation d'évolution suivante

$$\left\{ \begin{array}{c} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0,\ t > 0 \\ \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1, \end{array} \right. \tag{1}$$

où $A : D(A) \to H$ est un opérateur positif, auto-adjoint avec inverse compact dans un espace de Hilbert H, U est un espace de Hilbert (identifié à son dual) et $B \in \mathcal{L}(U, D(A^{\frac{1}{2}})')$. L'énergie de $\omega(t)$ à l'instant t est définie par

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|_H^2 + \left\| A^{\frac{1}{2}}\omega(t) \right\|_H^2 \right).$$

C'est un système dissipatif, qui vérifie formellement

$$E'(t) = -\|B^*\dot{\omega}(t)\|_U^2 \leq 0. \tag{2}$$

Nous allons expliquer sur cet exemple la méthodologie employée dans les chapitres 1, 3 et 5, sans faire aucune démonstration mais en donnant les outils principaux. La stabilité de cet exemple a déjà été étudiée dans [11] et nos rappels suivent celle-ci.

Notre première préoccupation est l'existence et l'unicité des solutions de (1). Pour cela, nous réécrivons (1) comme un système du premier ordre

$$\left\{ \begin{array}{l} U'(t) = \mathcal{A}U(t),\ t > 0 \\ U(0) = U_0 = (\omega_0,\ \omega_1), \end{array} \right. \tag{3}$$

où $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ et $\mathcal{H}$ un espace de Hilbert. L'outil principal est alors la théorie des semi-groupes et en particulier le Théorème de Lumer-Phillips. Rappelons tout d'abord la définition d'un semi-groupe fortement continu (voir [74, 93] pour plus de détails) :

Définition 0.0.1. *Une famille $S(t)$ $(0 \leq t < \infty)$ d'opérateurs linéaires bornés dans un espace de Banach H est appelée un semi-groupe fortement continu (ou C_0 semi-groupe) si*

i) $S(t_1 + t_2) = S(t_1)S(t_2)$, $\forall t_1, t_2 \geq 0$,

ii) $S(0) = I$,

iii) Pour tout $x \in H$, $S(t)x$ est continu en t sur $[0, \infty)$.

La famille $S(t)$ est appelée C_0 semi-groupe de contractions (ou semi-groupe de contractions) si de plus $\|S(t)\| \leq 1$ pour tout $t \geq 0$.

Pour un tel semi-groupe, on définit un opérateur A de domaine $D(A)$ qui est l'ensemble des points x tel que la limite

$$Ax = \lim_{h \to 0} \frac{S(h)x - x}{h},$$

existe. L'opérateur A est alors appelé le générateur infinitésimal du semi-groupe $S(t)$. Pour un opérateur A donné, si A coïncide avec le générateur infinitésimal de $S(t)$, on dit alors qu'il génère un semi-groupe fortement continu $S(t)$, $t \geq 0$.

Le résultat suivant est connu sous le nom de Théorème de Lumer-Phillips :

Théorème 0.0.1 (Théorème de Lumer-Phillips). *Soit A un opérateur linéaire de domaine $D(A)$ dense dans un espace de Hilbert H. Si*

i) A est dissipatif, i.e. $Re(Ax, x) \geq 0$ pour tout $x \in D(A)$

ii) pour tout $\lambda > 0$, $\lambda I - A$ est surjectif,

alors A génère un C_0 semi-groupe de contractions sur H.

Si un opérateur vérifie les conditions i) et ii), on dit aussi qu'il est m-dissipatif.

Dans notre exemple, nous devons donc vérifier que $\mathcal{A}$ est m-dissipatif pour obtenir l'existence et l'unicité des solutions de (3). De plus, si $U_0 \in \mathcal{H}$, alors $U \in C([0, +\infty), \mathcal{H})$ et si $U_0 \in D(\mathcal{A})$, alors $U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H})$.

Cette méthode sera appliquée dans les trois premiers chapitres. Pour le chapitre 4 (où nous étudions la stabilisation de l'équation des ondes et de la chaleur avec un retard dépendant du temps), l'opérateur $\mathcal{A}(t)$ de (3) dépendra du temps avec un domaine $D(\mathcal{A}(t))$ qui sera lui indépendant du temps. Nous utiliserons alors la technique de normes variables de Kato (voir [59, 60] et le chapitre 4 pour plus de détails).

Une fois l'existence et l'unicité des solutions de notre système (1) prouvées ainsi que la décroissance de l'énergie (qui est la norme des solutions dans $\mathcal{H}$), nous étudions la stabilité forte, c'est-à-dire nous donnons des conditions nécessaires et suffisantes pour que l'énergie tende vers zéro. Pour cela, nous utilisons dans les chapitres 1 et 5 le principe d'invariance de LaSalle et dans le chapitre 3 un résultat de Arendt et Batty [12].

Rappelons ce qu'est le principe d'invariance de LaSalle. Dans ce paragraphe, (Z, d) désigne un espace métrique complet.

Définition 0.0.2. *Un système dynamique sur Z est une famille $\{S(t)\}_{t \geq 0}$ d'applications sur Z telle que $S(t) \in C(Z, Z)$, $\forall t \geq 0$ et vérifiant i)-iii) de la Définition 0.0.1.*

Définition 0.0.3. *Soit $z \in Z$. L'ensemble*

$$\omega(z) = \{y \in Z \, : \, \exists t_n \to \infty, \, S(t_n)z \to y \text{ lorsque } n \to \infty\}$$

est appelé ensemble ω-limite de z.

Remarque 1. *Pour tout $z \in Z$ et $t \geq 0$, $S(t)(\omega(z)) \subset \omega(z)$.*

De plus, si $\bigcup_{t \geq 0} \{S(t)z\}$ est relativement compact dans Z, alors $S(t)(\omega(z)) = \omega(z) \neq \emptyset$.

Définition 0.0.4. *Une fonction $\phi \in C(Z, \mathbb{R})$ est dite fonction de Lyapounov pour $\{S(t)\}_{t\geq 0}$ si on a*

$$\phi(S(t)z) \leq \phi(z),\ \forall z \in Z,\ \forall t \geq 0.$$

Après ces quelques rappels, nous pouvons énoncer le principe d'invariance de LaSalle :

Théorème 0.0.2 (Principe d'invariance de LaSalle). *Soit ϕ une fonction de Lyapounov pour $\{S(t)\}$ et $z \in Z$ tel que $\cup_{t\geq 0}\{S(t)z\}$ soit relativement compact dans Z. Alors :*
(i) $L = \lim_{t\to\infty} \phi(S(t)z)$ existe
(ii) $\phi(y) = L$ pour tout $y \in \omega(z)$.

Dans notre exemple, nous appliquerons le principe d'invariance de LaSalle à l'ensemble relativement compact $\cup_{t\geq 0}S(t)U_0$, où $S(t)$ désigne le C_0 semi-groupe de contractions généré par $\mathcal{A}$ et à la fonctionnelle de Lyapounov $\phi = \|.\|_{\mathcal{H}}$.

Il nous reste alors à examiner la stabilité exponentielle ou polynomiale de notre système dissipé (1). En suivant [11], cette étude est basée sur l'obtention d'une inégalité d'observabilité du problème conservatif associé à (1). Celui-ci est simplement obtenu en prenant comme opérateur de feedback $B = 0$, i.e.

$$\left\{ \begin{array}{l} \ddot{\phi}(t) + A\phi(t) = 0,\ t > 0 \\ \phi(0) = \omega_0,\ \dot{\phi}(0) = \omega_1, \end{array} \right. \tag{4}$$

et pour données initiales les données initiales de ω solution de (1). Nous vérifions alors facilement que l'énergie des solutions de ce système est constante en temps t ; c'est pour cela que nous parlons de système conservatif. Les inégalités d'observabilité sont alors de deux types

$$\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 \leq C\int_0^T \|(B^*\phi)'(t)\|_U^2\,dt \tag{5}$$

et

$$\|\omega_0\|^2_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{m}{2}})} \leq C\int_0^T \|(B^*\phi)'(t)\|_U^2\,dt, \tag{6}$$

amenant deux types de décroissance : exponentielle ou polynomiale. Ces inégalités d'observabilité sont prouvées, dans les chapitres 1, 3 (et 5 pour une version discrète) en développant les solutions du problème conservatif associé en séries de Fourier et en utilisant des critères spectraux associés à l'opérateur $A^{\frac{1}{2}}$ et l'inégalité d'Ingham [57] :

Théorème 0.0.3 (Inégalité d'Ingham). *Soit $(\lambda_n)_{n\in\mathbb{Z}}$ une suite de nombres réels vérifiant la condition suivante dite condition du gap*

$$\exists\gamma > 0,\ \forall n \in \mathbb{Z},\ \lambda_{n+1} - \lambda_n \geq \gamma. \tag{7}$$

Alors pour tout $T > 2\pi/\gamma$, il existe deux constantes positives C_1 et C_2 telles que la fonction $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$ satisfait l'inégalité d'Ingham suivante :

$$C_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_0^T |f(t)|^2\,dt \leq C_2 \sum_{n\in\mathbb{Z}} |a_n|^2.$$

Cependant il arrive dans de très nombreux cas que la condition du gap (7) des valeurs propres de l'opérateur $A^{\frac{1}{2}}$ ne soit pas vérifiée, comme par exemple dans le cas de réseaux de cordes (voir chapitre 1). Nous pouvons alors utiliser l'inégalité d'Ingham généralisée qui est valable si la suite $(\lambda_n)_n$ satisfait (7) par blocs, c'est-à-dire si nous avons

$$\exists M > 0, \exists \gamma > 0, \forall n \in \mathbb{Z}, \lambda_{n+M} - \lambda_n \geq M\gamma > 0 \tag{8}$$

(voir [13,62] et les chapitres 1 et 3 pour plus de détails).

Une autre manière d'obtenir les inégalités d'observabilité (5) et (6) est d'utiliser la formule de D'Alembert (voir [38]).

Une fois ces inégalités d'observabilité prouvées, pour obtenir les résultats de stabilisation, les auteurs de [11] et [6–9] décomposent la solution ω de (1) sous la forme

$$\omega = \phi + \psi, \tag{9}$$

où ϕ est solution du problème conservatif (4) et ψ est solution de

$$\begin{cases} \ddot{\psi}(t) + A\psi(t) = -BB^*\dot{\omega}(t), \, t > 0 \\ \psi(0) = 0, \, \dot{\psi}(0) = 0, \end{cases} \tag{10}$$

avec données initiales nulles et pour laquelle ils montrent un résultat de régularité.

Avec ce résultat de régularité et avec (5) et (2), ils obtiennent l'estimée suivante :

$$E(0) - E(T) \geq CE(T),$$

et donc

$$E(T) \leq \gamma E(0), \quad \text{avec } 0 < \gamma < 1,$$

ce qui amène à la décroissance exponentielle de l'énergie en appliquant cette estimée sur des intervalles de temps $[(m-1)T, \, mT]$ (où $m \in \mathbb{N}$) successivement. Ceci est possible car le système (1) est invariant par translation en temps.

Pour la décroissance polynomiale, une technique similaire est employée en utilisant le Lemme 5.2 de [10]. Une fois encore, le fait que le système soit invariant par translation en temps est essentiel.

Cette méthode a été utilisée au cours des chapitres 1, 2, 3 et également dans une version discrète dans le chapitre 5.

En revanche, nous ne pouvons pas appliquer cette méthode au problème de la stabilisation de l'équation des ondes (ou de la chaleur) avec un terme de retard dépendant du temps dans les feedbacks, puisque le système n'est alors plus invariant par translation en temps. Aussi, dans le chapitre 4, nous avons choisi une autre méthode et l'introduction de fonctionnelles de Lyapounov appropriées.

Notons que les chapitres de cette thèse correspondent à des articles qui ont été publiés ([89]) ou soumis ([87, 88, 90, 106]). Nous avons donc gardé la structure générale de ces articles ; seules les références ont été regroupées dans une bibliographie commune.

Nous allons maintenant présenter les différents chapitres de cette thèse.

Dans le premier chapitre, nous étudions la stabilisation de l'équation des ondes sur un réseau unidimensionnel de N branches e_j avec un terme de retard dans les feedbacks situés aux noeuds. Plus précisément, le déplacement u_j le long de la corde e_j vérifie l'équation des ondes sur cette corde e_j de longueur l_j :

$$\frac{\partial^2 u_j}{\partial t^2}(x,\, t) - \frac{\partial^2 u_j}{\partial x^2}(x,\, t) = 0, \quad 0 < x < l_j,\, t > 0, \quad \forall j \in \{1, ..., N\}. \tag{11}$$

Nous supposons la continuité en tous les noeuds intérieurs, i.e. en tous les sommets de $\mathcal{V}_{int}$:

$$u_j(v,\, t) = u_l(v,\, t) = u(v,t) \quad \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \tag{12}$$

où ici $\mathcal{E}_v$ est l'ensemble des branches qui relient le sommet v. Nous fixons une partition des noeuds extérieurs :

$$\mathcal{V}_{ext} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{V}_{ext}^c,$$

où nous imposons les conditions de Dirichlet aux noeuds de $\mathcal{D}$

$$u_{j_v}(v,t) = 0 \quad \forall v \in \mathcal{D}, t > 0, \tag{13}$$

où j_v est le seul élément de $\mathcal{E}_v$ pour $v \in \mathcal{V}_{ext}$, les conditions de Neumann aux noeuds de $\mathcal{N}$

$$\frac{\partial u_{j_v}}{\partial n_{j_v}}(v,t) = 0 \quad \forall v \in \mathcal{N}, t > 0,$$

où $\partial u_{j_v}/\partial n_{j_v}(v,.)$ désigne la dérivée normale extérieure de u_j au sommet v. De plus nous considérons un sous-ensemble $\mathcal{V}_{int}^c$ de l'ensemble des noeuds intérieurs $\mathcal{V}_{int}$. Pour tous les noeuds intérieurs qui ne sont pas dissipés la loi de Kirchhoff est vérifiée :

$$\sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\, t) = 0 \quad \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c, t > 0. \tag{14}$$

Enfin pour tous les noeuds dissipés, c'est-à-dire pour tous les sommets de

$$\mathcal{V}_c = \mathcal{V}_{int}^c \cup \mathcal{V}_{ext}^c,$$

nous imposons deux types de dissipation, une sans retard et avec un poids $\alpha_1^{(v)} \geq 0$, et une avec retard $\tau_v > 0$ et avec un poids $\alpha_2^{(v)} \geq 0$:

$$\sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\, t) = -\left(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\, t) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\, t - \tau_v)\right) \quad \forall v \in \mathcal{V}_c, t > 0. \tag{15}$$

Notons que $\alpha_1^{(v)}$, $\alpha_2^{(v)}$ et τ_v dépendent du noeud v dissipé. Enfin les données initiales sont

$$u(t=0) = u^{(0)}, \; \frac{\partial u}{\partial t}(t=0) = u^{(1)}, \tag{16}$$

ainsi que

$$\frac{\partial u}{\partial t}(v, t-\tau_v) = f_v^0(t-\tau_v) \quad \forall v \in \mathcal{V}_c,\, 0 < t < \tau_v,$$

à cause du retard dans le système.

En réécrivant ce système en un système du premier ordre (3) et en appliquant le théorème de Lumer-Phillips comme expliqué dans le début de l'introduction, nous prouvons que ce système est bien posé (c'est-à-dire qu'il admet une solution unique qui dépend continûment des données initiales) sous la condition

$$\alpha_2^{(v)} \leq \alpha_1^{(v)}, \quad \forall v \in \mathcal{V}_c. \tag{17}$$

Nous devons cependant restreindre cette hypothèse à

$$\alpha_2^{(v)} < \alpha_1^{(v)}, \quad \forall v \in \mathcal{V}_c \tag{18}$$

pour obtenir la décroissance stricte de l'énergie

$$E(t) := \frac{1}{2}\sum_{j=1}^{N} \int_0^{l_j} \left(\left(\frac{\partial u_j}{\partial t}\right)^2 + \left(\frac{\partial u_j}{\partial x}\right)^2\right) dx + \sum_{v \in \mathcal{V}_c} \frac{\xi^{(v)}}{2} \left(\int_0^1 \left(\frac{\partial u}{\partial t}(v, t-\tau_v\rho)\right)^2 d\rho\right),$$

où $\xi^{(v)}$ est une constante positive satisfaisant (qui existe par (18))

$$\tau_v \alpha_2^{(v)} < \xi^{(v)} < \tau_v(2\alpha_1^{(v)} - \alpha_2^{(v)}),\, \forall v \in \mathcal{V}_c. \tag{19}$$

En utilisant le principe d'invariance de LaSalle et sous la condition (18), nous donnons alors une condition nécessaire et suffisante pour que l'énergie tende vers zéro quand $t \to \infty$, c'est-à-dire pour avoir la stabilité forte. Cette condition est une condition spectrale associée au problème conservatif (4) (c'est-à-dire pour $\alpha_2^{(v)} = \alpha_1^{(v)} = 0$). Par exemple, dans le cas le plus simple où, si nous notons $(\lambda_k^2)_k$ et $(\varphi_k)_k$ les valeurs propres et vecteurs propres du système conservatif associé, le gap simple (7) est vérifié et les valeurs propres sont simples, cette condition nécessaire et suffisante de stabilité forte s'écrit

$$\forall k \in \mathbb{N}, \sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 > 0.$$

Nous avons des conditions similaires dans le cas où les valeurs propres sont multiples et où le gap généralisé (8) est vérifié.

Dans le cas où (18) n'est pas vérifié, nous avons prouvé sur un exemple que le système est instable en exhibant une suite de retards et une suite de points où il y a dissipation pour lesquelles les solutions du système dissipé ont une énergie constante.

L'étude de la stabilité exponentielle et polynomiale est basée, comme expliqué précédemment, sur l'obtention d'inégalités d'observabilité pour le problème conservatif en développant les solutions en séries de Fourier et en utilisant l'inégalité d'Ingham. Nous donnons une condition nécessaire et suffisante, de type spectral, pour obtenir l'inégalité d'observabilité correspondante à (5), c'est-à-dire

$$\left\|u^{(0)}\right\|_{V}^{2}+\left\|u^{(1)}\right\|_{L^{2}(\mathcal{R})}^{2} \leq C \sum_{v \in \mathcal{V}_{c}} \int_{0}^{T}\left(\frac{\partial \phi}{\partial t}(v, t)\right)^{2} d t, \tag{20}$$

où ϕ est la solution correspondant à (4) et C, T sont des constantes strictement positives. Dans le cas où le gap simple (7) est vérifié et les valeurs propres sont simples, cette condition nécessaire et suffisante devient :

$$\exists \alpha>0, \forall k \in \mathbb{N}, \sum_{v \in \mathcal{V}_{c}}\left|\varphi_{k}(v)\right|^{2} \geq \alpha. \tag{21}$$

De la même manière la condition spectrale plus faible

$$\exists m \in \mathbb{N}, \exists \alpha>0, \forall k \in \mathbb{N}, \sum_{v \in \mathcal{V}_{c}}\left|\varphi_{k}(v)\right|^{2} \geq \frac{\alpha}{k^{2 m}} \tag{22}$$

est équivalente à l'inégalité d'observabilité correspondante à (6), c'est-à-dire

$$\sum_{k \geq 1} \frac{1}{k^{2 m}}\left(a_{k}^{2} \lambda_{k}^{2}+b_{k}^{2}\right) \leq C \sum_{v \in \mathcal{V}_{c}} \int_{0}^{T}\left(\frac{\partial \phi}{\partial t}(v, t)\right)^{2} d t, \tag{23}$$

où $u^{(0)}=\sum_{k \geq 1} a_{k} \varphi_{k}$, $u^{(1)}=\sum_{k \geq 1} b_{k} \varphi_{k}$, ϕ est la solution correspondant à (4), C, T sont des constantes strictement positives et $m \in \mathbb{N}^{*}$.

Pour obtenir les résultats de stabilité exponentielle et polynomiale il suffit alors de démontrer un résultat de régularité pour le système correspondant à (10) et d'utiliser, comme expliqué précedemment, la technique de [11]. Ce résultat de régularité pour le système correspondant à (10) est assez technique et utilise notamment des constructions locales.

Ainsi la décroissance exponentielle de l'énergie provient de l'inégalité d'observabilité (20), et donc de (21) dans le cas où le gap simple (7) est vérifié et les valeurs propres sont simples.

L'inégalité d'observabilité (23) (et donc (22) dans le cas où le gap simple (7) est vérifié et les valeurs propres sont simples) implique la décroissance polynomiale de l'énergie en utilisant en plus un lemme d'interpolation.

Tous ces résultats sont illustrés d'exemples concrets. Nous remarquons que le type de stabilité dépend des conditions aux bords et des propriétés d'irrationnalité des rapports des longueurs du réseau. Pour des réseaux complexes, comportant par exemple des boucles, nous utilisons un résultat de [20, 81] qui donne explicitement le spectre de l'opérateur de Laplace via des propriétés algébriques du réseau dans le cas où les branches sont de mêmes longueurs.

Dans le premier chapitre, les conditions suffisantes (et nécessaires) pour obtenir la stabilité sont des conditions spectrales ; elles requièrent le calcul explicite du spectre de l'opérateur de Laplace sur le réseau, ce qui n'est pas évident pour des réseaux quelconques. De plus, notre analyse est limitée à des inégalités d'observabilité fortes (amenant à des résultats de décroissance exponentielle ou polynomiale), qui ont lieu pour une classe restreinte de réseau. Dans le chapitre 2, nous répondons à ces limites pour des réseaux arbitraires dans lesquels nous pouvons démontrer des inégalités d'observabilité plus faibles amenant à des taux de décroissance plus lents.

Dans le chapitre 2, nous étudions la stabilisation de l'équation des ondes sur des réseaux 1-d en ne dissipant qu'en un seul noeud extérieur (que l'on suppose être v_1) et nous considérons le même système que dans le chapitre 1, c'est-à-dire nous considérons (11)-(16) mais sans retard (i.e. $\alpha_2^{(v_1)} = 0$ et $\alpha_1^{(v_1)} = \alpha$) et avec

$$\mathcal{N} = \emptyset, \quad \mathcal{V}_c = \{v_1\}, \quad \mathcal{V}_{int}^c = \emptyset.$$

Le but de ce chapitre est de développer une méthode systématique pour obtenir des taux de décroissance sur des réseaux quelconques et de donner un résultat général permettant de transformer un résultat d'observabilité pour un système conservatif en stabilisation pour le système dissipé. Contrairement au chapitre précédent, nous ne prenons pas pour le système conservatif le système correspondant à (4) (i.e. le système (11)-(16) avec $\alpha_1^{(v_1)} = \alpha = 0$ et $\alpha_2^{(v_1)} = 0$), mais le système (11)-(14) avec la condition de Dirichlet au noeud v_1 :

$$\psi_1(0, t) = 0 \tag{24}$$

à la place de la condition de Neumann.

L'étude de l'observabilité de ce problème est motivée par les problèmes de contrôle et a déjà été réalisée ([33–38]).

Le but est donc de montrer le lien entre les résultats d'observabilité existants pour la solution ψ de (11)-(14) et (24) et la stabilisation du système dissipé. Plus précisement, notons $(\tilde{\lambda}_n^2)_{n\geq 1}$ la suite des valeurs propres correspondant au problème (11)-(14) et (24) et $(\varphi_n^D)_{n\geq 1}$ les vecteurs propres correspondants à $(\tilde{\lambda}_n^2)$ formant une base orthonormale de $L^2(\mathcal{R})$. Sous des conditions sur la topologie du réseau et les longueurs des branches, Dager et Zuazua [33–38] ont prouvé des inégalités d'observabilité à poids pour (11)-(14) et (24) de la forme

$$E_*^D(\psi, 0) := \sum_{n\geq 1} c_n^2 (\tilde{\lambda}_n^2 \psi_{0,n}^2 + \psi_{1,n}^2) \leq C \int_0^T \left| \frac{\partial \psi_1}{\partial x}(0, t) \right|^2 dt, \tag{25}$$

pour une constante positive C, où $\psi_{0,n}$, $\psi_{1,n}$ sont les coefficients de Fourier des données initiales de ψ dans la base $(\varphi_n^D)_n$ et avec des poids positifs $(c_n^2)_{n\geq 1}$ dépendants des propriétés du réseau.

Cette inégalité d'observabilité à poids est prouvée de différentes manières : par la formule de D'Alembert (ce qui évite de calculer le spectre) pour les arbres ou par le développement en séries de Fourier des solutions et le théorème de Beurling-Malliavin pour les réseaux plus complexes, comprenant des boucles par exemple.

La clé de ce chapitre réside dans le fait que l'on peut obtenir, de manière systématique, l'inégalité d'observabilité à poids pour la solution du problème dissipé directement par

(25). Ceci nous évite de refaire toute l'analyse (en particulier en théorie des nombres), assez subtile, déjà effectuée sur les poids c_n^2 dans [38].

Pour cela, nous décomposons la solution du système dissipé u comme la somme de w, une solution de (11)-(14) et (24) avec donnée initiale appropriée $(u^{(0)} - u_1^{(0)}(0)\varphi,\, u^{(1)})$ (où φ est une fonction régulière donnée telle que $\varphi_1(0) = 1$), et d'un reste. Appliquant (25) à w, nous obtenons l'estimée d'observabilité à poids pour u solution du problème dissipé

$$E_*^D(w,\, 0) + u_1^{(0)}(0)^2 \leq C_T \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, t)\right)^2 dt, \tag{26}$$

où $E_*^D(w,\, 0)$ est défini par (25), avec des poids $(c_n^2)_n$ dépendant du réseau. Si les poids c_n^2 sont non nuls pour tout $n \in \mathbb{N}^*$, l'énergie du système dissipatif tend vers 0 quand $t \to \infty$. Cependant, en général, les poids tendent vers 0 quand $n \to \infty$, la quantité est plus faible que la norme dans l'espace d'énergie et le taux de décroissance n'est pas exponentiel.

Il est important de souligner que (26) est vrai sous les mêmes hypothèses sur le réseau que pour (25) pour le problème de Dirichlet (11)-(14) et (24). Nous n'avons donc pas besoin d'hypothèse supplémentaire sur le réseau.

Pour obtenir les propriétés de décroissance à partir de (26), nous regardons cette inégalité comme une estimée d'observabilité faible dans laquelle l'énergie observée $E_-(0)$ est égale, pour parler rapidemment, à $E_*^D(w,\, 0) + u_1^{(0)}(0)^2$. En pratique nous prenons souvent, si nécessaire, des points situés sur l'enveloppe convexe de c_n^2 au lieu des poids c_n^2 eux-mêmes dans la définition de E_*^D. L'énergie observée E_- est plus faible que la norme dans l'espace d'énergie des données initiales qui est nécessaire pour prouver la décroissance exponentielle, et par conséquent nous obtenons des taux de décroissance plus faibles (logarithmique par exemple). Pour obtenir des taux de décroissance explicites à partir de l'inégalité d'observabilité faible nous utilisons une inégalité d'interpolation qui est une variante de celle de Bégout et Soria [19] et une généralisation de l'inégalité d'Hölder. Pour cela nous avons besoin de supposer plus de régularité des données initiales. Pour être plus précis nous considérons les données initiales $(u^{(0)},\, u^{(1)}) \in X_s := [D(\mathcal{A}),\, D(\mathcal{A}^0)]_{1-s}$ pour $0 < s < 1/2$. Nous en déduisons une inégalité d'interpolation de la forme

$$1 \leq \Phi_s\left(\frac{E_-(0)}{CE_u(0)}\right) \frac{\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}{C'E_u(0)},$$

où Φ_s est une fonction croissante qui dépend de s et de l'énergie E_- (et donc des poids c_n^2).

L'inégalité d'interpolation précédente entraîne

$$E_-(0) \geq CE_u(0)\Phi_s^{-1}\left(\frac{E_u(0)}{C'\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}\right).$$

Avec la dérivée de l'énergie et (26), nous obtenons

$$E_u(0) - E_u(T) \geq CE_u(0)\Phi_s^{-1}\left(\frac{E_u(0)}{C'\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}\right),$$

ce qui implique, par la technique de Ammari et Tucsnak [11]

$$\forall t > 0,\, E_u(t) \leq C\Phi_s\left(\frac{1}{t+1}\right)\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2. \tag{27}$$

Evidemment, le taux de décroissance dans (27) dépend du comportement de la fonction Φ_s près de 0. Donc, dans le but de déterminer le taux de décroissance explicite nous avons besoin d'une description précise de la fonction Φ_s, qui dépend de s et des énergies E et E_- et donc, des poids $(c_n^2)_n$ de (26). Ces poids dépendent de la topologie du réseau et des propriétés de théorie des nombres des longueurs des cordes.

Cette approche nous permet d'obtenir de manière systématique des taux de décroissance pour l'énergie des solutions régulières du système comme une conséquence des propriétés d'observabilité d'un système conservatif.

Dans les deux premiers chapitres, nous avons démontré des résultats de stabilité sur des réseaux en utilisant des inégalités d'observabilité pour des systèmes conservatifs (différents dans ces deux chapitres). Le premier chapitre traite en plus de stabilisation de système avec retard. En mettant en parallèle ce premier chapitre et le travail de Nicaise et Pignotti [85] sur les problèmes avec retard pour l'équation des ondes sur des domaines de $\mathbb{R}^n$, $n \geq 1$, nous avons remarqué que les méthodes développées pour démontrer l'existence de solution et trouver les taux de décroissance présentent des similarités. Cette observation nous a amenés à considérer un système abstrait et général qui contient une grande classe de problèmes avec des feedbacks avec retard, permettant de retrouver ces résultats et d'en obtenir d'autres, dans le même esprit que l'article de Ammari et Tucsnak [11].

Dans le troisième chapitre, nous considérons ainsi le système (1) mais avec un retard dans le feedback, c'est-à-dire

$$\left\{\begin{array}{c} \ddot{\omega}(t) + A\omega(t) + B_1B_1^*\dot{\omega}(t) + B_2B_2^*\dot{\omega}(t-\tau) = 0,\, t > 0 \\ \omega(0) = \omega_0,\, \dot{\omega}(0) = \omega_1, \\ B_2^*\dot{\omega}(t-\tau) = f^0(t-\tau),\, 0 < t < \tau, \end{array}\right. \tag{28}$$

où $\tau > 0$ est le retard, $A : D(A) \longrightarrow H$ est un opérateur positif, auto-adjoint avec inverse compact dans un espace de Hilbert H, U_1, U_2 sont des espaces de Hilbert (identifiés à leur dual) et $B_i \in \mathcal{L}(U, D(A^{1/2})')$, $i = 1, 2$. Notons $V = D(A^{1/2})$.

La première question naturelle à se poser est l'existence et l'unicité des solutions. Pour cela, nous réécrivons ce système comme un système du premier ordre (3) et nous appliquons le théorème de Lumer-Phillips (comme expliqué dans le début de l'introduction), en supposant la condition suivante

$$\exists 0 < \alpha \leq 1,\, \forall u \in V,\, \left\|B_2^*u\right\|_{U_2}^2 \leq \alpha \left\|B_1^*u\right\|_{U_1}^2. \tag{29}$$

Cette condition est cohérente avec (17) en prenant $B_i^*\varphi = (\sqrt{\alpha_i^{(v)}}\varphi(v))_{v\in\mathcal{V}_c}$ pour $\varphi \in D(A^{1/2})$. Comme dans le premier chapitre, nous devons restreindre notre hypothèse à

$$\exists 0 < \alpha < 1,\, \forall u \in V,\, \left\|B_2^*u\right\|_{U_2}^2 \leq \alpha \left\|B_1^*u\right\|_{U_1}^2, \tag{30}$$

pour obtenir la décroissance stricte de l'énergie

$$E(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega\right\|_H^2 + \|\dot{\omega}\|_H^2 + \tau\xi\int_0^1 \|B_2^*\dot{\omega}(t-\tau\rho)\|_{U_2}^2\, d\rho\right), \tag{31}$$

où ξ est une constante positive satisfaisant

$$1 < \xi < \frac{2}{\alpha} - 1, \tag{32}$$

qui existe puisque $0 < \alpha < 1$. De plus, cette hypothèse semble réaliste car, sans cette hypothèse, il existe des cas où des instabilités peuvent apparaitre, comme nous l'avons vu dans le chapitre 1 (voir aussi [85, 110]).

Sous cette condition et en utilisant un résultat de [12] (voir aussi [105]) nous obtenons une condition nécessaire et suffisante pour la stabilité forte du système, qui est

$$\text{Pour tout vecteur propre non nul } \varphi \in D(A) \text{ de } A,\ B_1^*\varphi \neq 0.$$

Remarquons que cette dernière condition est indépendante du retard et par conséquent sous la condition (30), notre système est fortement stable si et seulement si le même système sans retard est fortement stable.

Dans la troisième étape, sous la condition (30) et une hypothèse de borne venant de [11] entre la résolvante de A et les opérateurs B_1 et B_2 :

si $\beta > 0$ est fixé et $C_\beta = \{\lambda \in \mathbb{C}\,|\Re\lambda = \beta\}$, la fonction

$$\lambda \in C_\beta \to H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1}B \in \mathcal{L}(U) \text{ est bornée,} \tag{33}$$

où $B = (B_1, B_2) \in \mathcal{L}(U, V')$ avec $U = U_1 \times U_2$, nous prouvons que la décroissance exponentielle du système (28) provient de l'estimée d'observabilité (5) pour le problème conservatif (4), en utilisant la décomposition (9) et la méthode de [11] comme expliqué dans le début de l'introduction. Une fois de plus, cette estimée d'observabilité est indépendante du terme avec retard $B_2B_2^*\dot{\omega}(t-\tau)$ et par conséquent, sous les conditions (30) et (33), la décroissance exponentielle de (1) implique la décroissance exponentielle de (28). Malgré tout nous donnons la dépendance de la décroissance par rapport au retard, en particulier nous montrons que si le retard augmente le taux de décroissance diminue.

Une analyse similaire pour la décroissance polynomiale est effectuée en prenant l'estimée d'observabilité plus faible (6) et en utilisant le lemme technique 5.2 de [11]. Une fois de plus, nous montrons que si le retard augmente le taux de décroissance diminue.

Pour pouvoir appliquer plus facilement ces résultats, nous donnons la preuve de ces estimées d'observabilité (5) et (6) en écrivant les solutions en séries de Fourier, en utilisant l'inégalité d'Ingham (classique ou généralisée) et une réduction à des conditions entre les vecteurs propres de A et l'opérateur de feedback B_1^*. La condition nécessaire et suffisante pour obtenir (5), dans le cas où le gap simple (7) est vérifié et les valeurs propres sont simples, est exactement (21) dans le cas des réseaux (chapitre 1).

Nous terminons ce chapitre en illustrant ces résultats d'exemples dans lesquels notre cadre abstrait s'applique. A notre connaissance, ces exemples, à l'exception du premier qui reprend le chapitre 1, sont nouveaux, puisque en particulier notre cadre abstrait permet de considérer des opérateurs de feedback B_1 et B_2 différents, à condition qu'ils vérifient (30). Nous étudions par exemple l'équation des poutres d'Euler-Bernoulli avec des termes de damping internes et localisés en un point, l'équation des ondes avec des termes de damping répartis (1-d et multi-d) ou encore l'équation des ondes sur un réseau avec des termes de damping répartis.

Dans le chapitre 4, nous étudions la stabilisation des équations de la chaleur et des ondes avec un terme de retard qui dépend du temps (et non plus constant comme auparavant). Les systèmes n'étant plus invariants par translation en temps, les techniques utilisées jusqu'alors (c'est-à-dire se ramener à une inégalité d'observabilité) ne sont plus applicables. Nous allons donc utiliser une méthode différente en introduisant des fonctionnelles de Lyapounov appropriées. Plus précisément nous considérons l'équation des ondes suivante

$$\left\{\begin{array}{cc} u_{tt}(x,t) - au_{xx}(x,t) = 0, & 0 < x < \pi,\, t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = -\alpha_1 u_t(\pi,t) - \alpha_2 u_t(\pi, t-\tau(t)), & t > 0, \\ u(x,0) = u^0(x),\, u_t(x,0) = u^1(x), & 0 < x < \pi, \\ u_t(\pi, t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{array}\right. \tag{34}$$

avec un paramètre constant $a > 0$ et où α_1, α_2 sont des nombres réels positifs, où le retard $\tau(t)$ est une fonction du temps qui satisfait

$$\forall t > 0,\, \dot{\tau}(t) \leq d < 1, \tag{35}$$

$$\exists M > 0,\, \forall t > 0,\, 0 < \tau_0 \leq \tau(t) \leq M, \tag{36}$$

et

$$\forall T > 0,\, \tau \in W^{2,\infty}([0,T]). \tag{37}$$

Comme pour les chapitres précédents, nous nous intéressons tout d'abord à l'existence et l'unicité des solutions de ce système et pour cela nous réécrivons notre système en un système du premier ordre (3) et nous supposons que α_1 et α_2 vérifient

$$\alpha_2^2 \leq (1-d)\alpha_1^2. \tag{38}$$

Cependant à cause du retard dépendant du temps, nous n'utiliserons pas le théorème de Lumer-Phillips car l'opérateur $\mathcal{A}$ de (3) dépend du temps. Mais comme le domaine de l'opérateur $\mathcal{A}(t)$ est constant en temps, nous employons la technique de normes variables de Kato [59, 60] en prenant pour produit scalaire de $\mathcal{H} = \{\phi \in H^1(0,\pi) : \phi(0) = 0\} \times L^2(0,\pi) \times L^2(0,1)$ un produit scalaire qui dépend du temps

$$\left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{\omega} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^\pi (au_x\tilde{u}_x + \omega\tilde{\omega})dx + q\tau(t)\int_0^1 z(\rho)\tilde{z}(\rho)d\rho,$$

où q est une constante positive choisie telle que

$$\Psi_q = \frac{1}{2}\begin{pmatrix} q - 2a\alpha_1 & -a\mu_1\alpha_2 \\ -a\alpha_2 & -q(1-d) \end{pmatrix}$$

soit négative (ce qui est garanti par (35) et (38)), et avec norme associée $\|.\|_t$.

Comme pour les chapitres 1 et 3, nous restreignons l'hypothèse (38) en

$$\alpha_2^2 < (1-d)\alpha_1^2. \tag{39}$$

pour obtenir la décroissance stricte de l'énergie $E(t)$, qui correspond à

$$E(t) = \frac{1}{2}\left\|(u,\, u_t,\, u_t(\pi,\, t-\tau(t)\rho))^T\right\|_t^2, \tag{40}$$

où q est une constante positive choisie telle que Ψ_q soit définie négative (ce qui est garanti par (35) et (39)). Nous remarquons que dans le cas où le retard est constant en temps, i.e. $\tau(t) = \tau$ pour tout $t > 0$ (et donc $d = 0$), nous retrouvons les résultats des chapitres 1 et 3 et de [85] pour la même énergie. En effet, l'énergie est alors strictement décroissante sous la condition (18), ce qui correspond à (39) pour $d = 0$.

Sous les hypothèses (35), (36) et (39), nous prouvons la stabilité exponentielle de l'équation des ondes (34) en utilisant la fonctionnelle de Lyapounov suivante

$$\mathcal{E}(t) = E(t) + \gamma\left(2\int_0^\pi x u_t u_x dx + \mathcal{E}_2(t)\right), \tag{41}$$

où $\gamma > 0$ est un paramètre fixé suffisamment petit, E est l'énergie standard definie par (40) avec q une constante positive fixée telle que Ψ_q soit définie négative et où $\mathcal{E}_2$ est définie par

$$\mathcal{E}_2(t) = q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u_t^2(\pi,\, s)ds = q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u_t^2(\pi,\, t-\tau(t)\rho)d\rho, \tag{42}$$

avec $\delta > 0$.

La fonctionnelle de Lyapounov $E(t) + 2\gamma\int_0^\pi x u_t u_x dx$ est standard dans les problèmes avec conditions au bord avec mémoire (voir par exemple [86]). Nous avons ajouté deux termes à l'énergie standard $E(t)$ pour prendre en compte la dépendance de τ par rapport à t. De plus nous remarquons que les énergies E et $\mathcal{E}$ sont équivalentes.

De cette manière, sous les conditions (35), (36) et (39), nous prouvons que ce système est exponentiellement stable et nous donnons également le taux exact de décroissance de façon explicite. Il dépend du retard $\tau(t)$ (et plus particulièrement de M et d), de α_1, α_2, de a et de δ. Nous remarquons que nous pouvons choisir δ tel que la décroissance soit aussi rapide que possible pour une fonction τ fixée. Cependant notre taux de décroissance va diminuer si le maximum M de τ augmente, ce qui est cohérent avec l'étude du chapitre 3.

Nous effectuons également dans ce chapitre 4 la même étude pour l'équation de la chaleur.

Dans le cinquième et dernier chapitre nous nous intéressons à la stabilisation de l'équation des ondes discrétisée en espace par différences finies avec un amortissement en un point intérieur. Le principal problème pour l'étude de la stabilisation des approximations numériques d'un système est que le taux de décroissance n'est pas uniforme par rapport au pas de discrétisation. Il faut donc utiliser une méthode pour redresser le spectre discret pour les hautes fréquences.

Plus précisément, nous considérons l'équation des ondes sur un intervalle de longueur 1 avec un amortissement en $\xi \in (0, 1)$

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & 0 < x < 1,\ t > 0, \\ y(0,t) = 0,\ y_x(1,t) = 0 & t > 0, \\ y(\xi_-, t) = y(\xi_+, t) & t > 0, \\ y_x(\xi_-, t) - y_x(\xi_+, t) = -\alpha y_t(\xi, t) & t > 0, \\ y(t=0) = y^{(0)},\ y_t(t=0) = y^{(1)} & 0 < x < 1, \end{cases} \tag{43}$$

où $(y^{(0)}, y^{(1)}) \in V \times L^2(0,1)$, $V = \{y \in H^1(0,1)\,;\, y(0) = 0\}$ et α est une constante positive. Ce système (43) est bien posé dans l'espace d'énergie $V \times L^2(0,1)$ (voir [6] ou le début de l'introduction).

L'énergie de la solution du système (43) est donnée par

$$E(t) = \frac{1}{2}\int_0^1 (|y_t(x,t)|^2 + |y_x(x,t)|^2)dx$$

et vérifie la loi de dissipation suivante

$$\frac{dE(t)}{dt} = -\alpha\,|y_t(\xi,t)|^2. \tag{44}$$

De plus, nous savons que (voir [6]) $\lim\limits_{t\to\infty} E(t) = 0$ pour toute donnée initiale dans $V \times L^2(0,1)$ si et seulement si

$$\xi \neq \frac{2p}{2q+1},\ \forall p, q \in \mathbb{N} \tag{45}$$

et, comme nous l'avons signalé précédemment, la décroissance exponentielle de la solution de (43) est équivalente à une estimée d'observabilité pour le système conservatif associé (4). Dans ce cas, l'estimée d'observabilité est vérifiée si et seulement si ξ est un nombre rationnel qui admet une décomposition en fraction irréductible de la forme

$$\xi = \frac{p}{q}, \text{ où } p \text{ est impair},$$

et par conséquent sous cette condition, le système (43) est exponentiellement stable dans l'espace d'énergie. Nous supposons donc que ξ est fixé et vérifie ces conditions.

Dans ce chapitre nous nous intéressons à la discrétisation en espace par différences finies de (43). Soit $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$ et considérons la subdivision de $(0, 1)$ donnée par

$$0 = x_0 < x_1 = h < \ldots < x_{j-1} < x_j = jh < x_{j+1} < \ldots < x_N < x_{N+1} = 1,$$

i.e. $x_j = jh$ pour tout $j = 0, ..., N+1$. Comme ξ n'est pas nécessairement égal à jh pour tout j, nous fixons $j_N \in \mathbb{N} \cap (0,\, N+1)$ tel que $x_{j_N} \to \xi$ quand $N \to \infty$. La semi-discrétisation en espace de (43) est la suivante : pour l'équation des ondes, nous obtenons

$$y_j'' - \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = 0, \quad t > 0,\, j = 1, ..., N,\, j \neq j_N, \tag{46}$$

les conditions au bord (Dirichlet et Neumann) deviennent

$$y_0 = 0,\, y_{N+1} - y_N = 0, \quad t > 0, \tag{47}$$

l'approximation naturelle de la condition de transmission est

$$\frac{y_{j_N+1} - 2y_{j_N} + y_{j_N-1}}{h} = \alpha y_{j_N}', \quad t > 0, \tag{48}$$

et les données initiales du problème discrétisé sont

$$y_j(t=0) = y_j^{(0)},\, y_j'(t=0) = y_j^{(1)}, \quad j = 1, ..., N. \tag{49}$$

Ici $y_j(t)$ est une approximation de $y(x_j,\, t)$, y étant la solution de (43), à condition que les conditions initiales $(y_j^{(0)},\, y_j^{(1)})$, $j = 0, ..., N+1$ soient des approximations des données initiales dans (43). Notons $y_h = (y_j)_j$, $y_h^{(0)} = (y_j^{(0)})_j$ et $y_h^{(1)} = (y_j^{(1)})_j$. La stabilisation du même type de problème mais avec un amortissement au bord (en $x = 1$) a été effectuée dans [103] (ou avec un amortissement interne localisé dans [102]) en ajoutant une viscosité numérique.

Nous introduisons l'énergie de ce problème discrétisé par

$$E_h(t) = \frac{h}{2} \sum_{j=0,\, j\neq j_N}^{N} \left|y_j'(t)\right|^2 + \frac{h}{2} \sum_{j=0}^{N} \left|\frac{y_{j+1}(t) - y_j(t)}{h}\right|^2, \tag{50}$$

qui est une discrétisation de l'énergie continue E, qui est décroissante et qui vérifie

$$E_h'(t) = -\alpha (y_{j_N}'(t))^2.$$

En utilisant le principe d'invariance de LaSalle, nous montrons que l'énergie discrétisée E_h tend vers 0 quand $t \to \infty$ si et seulement si

$$j_N h \neq \frac{(2-h)l}{2k+1},\, \forall k = 0, ..., N-1,\, l \in \mathbb{N}.$$

Nous remarquons que cette condition est une version discrète de (45) pour le modèle continu.

Ensuite, nous montrons, comme pour ce type de problème (voir [14, 96, 102, 103] par exemple), que la décroissance exponentielle de l'énergie discrète de (46)-(49) n'est pas uniforme par rapport au pas de discrétisation, en prenant $j_N = N\frac{p}{q}$, où p est impair et N

est un multiple de q. Ceci est dû à l'existence des modes étrangers que le schéma numérique introduit à haute fréquence et qui n'apparaissent pas dans le modèle continu.

Pour surmonter cet obstacle, nous filtrons les hautes fréquences et en conséquence nous introduisons la classe $C_h(\gamma)$ des solutions du problème discrétisé générées par les vecteurs propres du problème conservatif discrétisé

$$\left\{\begin{array}{ll} u_j'' - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 0 & t>0,\ j=1,\dots,N \\ u_0 = 0,\ u_{N+1} - u_N = 0 & t>0 \\ u_j(t=0) = y_j^{(0)},\ u_j'(t=0) = y_j^{(1)} & j=1,\dots,N, \end{array}\right. \tag{51}$$

associés aux valeurs propres telles que $\lambda h^2 \leq \gamma$. Ainsi la classe $C_h(\gamma)$ est définie par

$$C_h(\gamma) := \left\{ \sum_{\lambda_{k,h} \leq \frac{\gamma}{h^2}} a_k \varphi^{k,h} \text{ avec } a_k \in \mathbb{R} \right\}.$$

Nous supposons dans la suite que $j_N = \left[\frac{p(2N+1)}{2q}\right] \in \mathbb{N}$ et donc $x_{j_N} \to \xi = \frac{p}{q}$. Nous pouvons alors prouver une inégalité d'observabilité discrète uniforme (en h) pour les solutions u_h de (51) dans la classe $C_h(\gamma)$

$$(T-2)E_{j_N}(u_h, 0) \leq C \int_0^T \left|u_{j_N}'(t)\right|^2 dt,$$

pour un certain γ, puisqu'en filtrant nous pouvons utiliser l'inégalité d'Ingham. En effet, le gap (7) des valeurs propres entrant dans le développement de Fourier des solutions de (51) dans la classe $C_h(\gamma)$ est vérifié.

Grâce à cette inégalité d'observabilité uniforme, nous pouvons prouver la décroissance quasi-exponentielle de l'énergie E_h de (46)-(49), en effectuant la décomposition $y_h = u_h + w_h$ et en prouvant certaines estimées par la technique des multiplicateurs. De plus, sans filtrer, la décroissance quasi-exponentielle n'est pas uniforme par rapport à h, ce qui montre l'avantage de cette technique. Nous parlons de décroissance quasi-exponentielle car nous pouvons majorer $E_h(t)$ par $Ke^{-\omega t}E_h(y_h, 0)$ (qui correspond à la décroissance exponentielle) plus un terme résiduel qui tend vers 0 quand $h \to 0$, en prenant des données initiales suffisamment régulières. Par conséquent cette estimée tend vers la stabilité exponentielle $E(y, t) \leq Ke^{-\omega t}E(y, 0)$ quand $h \to 0$; c'est dans ce sens que cette estimée est quasi-optimale. De plus, le membre de droite de notre estimée de décroissance quasi-exponentielle se comporte comme une fonction exponentielle décroissante pour t entre 0 et $-\frac{\ln h}{\omega} + c$ et est constante (proportionnellement à h) pour t suffisamment grand.

Afin d'illustrer ces résultats, nous avons effectué des tests numériques en considérant un schéma numérique complètement discrétisé qui est implicite. Ces tests numériques montrent que sans filtrer nous n'avons pas de décroissance exponentielle (nous prenons un “grand” vecteur propre) mais que filtrer (en prenant un “petit” vecteur propre) permet de rétablir une décroissance exponentielle.

Chapitre 1

Stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks

1.1 Introduction/Notations

Time delay effects arise in many practical problems, see for instance [1, 51, 101] for biological, electrical engineering, or mechanical applications. Furthermore it is well known that they can induce some instabilities [39, 40, 42, 85, 110], or on the contrary improve the performance of the system [1, 101].

Recently, control problems on 1-d networks are paying attention of many authors, see [38, 65] and the references cited there. We here investigate the effect of time delay in boundary and/or transmission stabilization of the wave equation in $1-d$ networks. To our knowledge, the analysis of this effect to $1-d$ networks is not yet done.

Before going on, let us recall some definitions and notation about $1-d$ networks used in the whole chapter. We refer to [2, 3, 20–23, 82, 91] for more details.

Definition 1.1.1. *A $1-d$ network $\mathcal{R}$ is a connected set of $\mathbb{R}^n$, $n \geq 1$ defined by*

$$\mathcal{R} = \bigcup_{j=1}^{N} e_j$$

where e_j is a curve that we identify with the interval $(0,\, l_j)$, $l_j > 0$, and such that for $k \neq j$, $\overline{e_j} \cap \overline{e_k}$ is either empty or a common extremity called a vertex or a node (here $\overline{e_j}$ means the closure of e_j).

For a function $u\, :\, \mathcal{R} \longrightarrow \mathbb{R}$, we set $u_j = u_{|e_j}$ the restriction of u to the edge e_j.

We denote by $\mathcal{E} = \{e_j\,;\, 1 \leq j \leq N\}$ the set of edges of $\mathcal{R}$ and by $\mathcal{V}$ the set of vertices of $\mathcal{R}$. For a fixed vertex v, let

$$\mathcal{E}_v \;=\; \{j \in \{1, ..., N\}\,;\, v \in \overline{e_j}\}$$

be the set of edges having v as vertex. If card $(\mathcal{E}_v) = 1$, v is an exterior node, while if card $(\mathcal{E}_v) \geq 2$, v is an interior node. We set $\mathcal{V}_{ext}$ the set of exterior nodes and $\mathcal{V}_{int}$ the set of interior nodes. For $v \in \mathcal{V}_{ext}$, the single element of $\mathcal{E}_v$ is denoted by j_v.

We now fix a partition of $\mathcal{V}_{ext}$:

$$\mathcal{V}_{ext} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{V}^c_{ext}.$$

Clearly we will impose Dirichlet boundary condition at the nodes of $\mathcal{D}$, Neumann boundary condition at the nodes of $\mathcal{N}$ and finally a feedback boundary condition at the nodes of $\mathcal{V}^c_{ext}$. We further fix a subset $\mathcal{V}^c_{int}$ of $\mathcal{V}_{int}$, where a feedback transmission condition will be imposed. For shortness, we denote by $\mathcal{V}_c$ the set of controlled nodes, namely

$$\mathcal{V}_c = \mathcal{V}^c_{int} \cup \mathcal{V}^c_{ext}.$$

We also suppose that $\mathcal{D} \neq \emptyset$; so that the H^1 semi-norm becomes a norm.

We can now formulate our initial/boundary value problem :

$$\left\{\begin{array}{ll}
\frac{\partial^2 u_j}{\partial t^2}(x, t) - \frac{\partial^2 u_j}{\partial x^2}(x, t) = 0 & 0 < x < l_j, t > 0, \\
 & \quad \forall j \in \{1, ..., N\}, \\
u_j(v, t) = u_l(v, t) = u(v,t) & \forall j, l \in \mathcal{E}_v, v \in \mathcal{V}_{int}, t > 0, \\
\sum\limits_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v, t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v, t) + \alpha_2^{(v)} \frac{\partial u}{\partial t}(v, t - \tau_v)) & \forall v \in \mathcal{V}_c, t > 0, \\
\sum\limits_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v, t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}^c_{int}, t > 0, \\
u_{j_v}(v,t) = 0 & \forall v \in \mathcal{D}, t > 0, \\
\frac{\partial u_{j_v}}{\partial n_{j_v}}(v,t) = 0 & \forall v \in \mathcal{N}, t > 0, \\
u(t = 0) = u^{(0)}, \frac{\partial u}{\partial t}(t = 0) = u^{(1)}, & \\
\frac{\partial u}{\partial t}(v, t - \tau_v) = f_v^0(t - \tau_v) & \forall v \in \mathcal{V}_c, 0 < t < \tau_v,
\end{array}\right. \tag{1.1}$$

where $\alpha_1^{(v)}, \alpha_2^{(v)} \geq 0$ are fixed nonnegative real numbers, the delay $\tau_v > 0$ is also supposed to be fixed and $\frac{\partial u_j}{\partial n_j}(v, \cdot)$ means the outward normal (space) derivative of u_j at the vertex v.

Note that u_j represents the displacement of the string e_j.

Remark that the condition $\frac{\partial u}{\partial t}(v, t - \tau_v) = f_v^0(t - \tau_v)$ for $v \in \mathcal{V}_c$, $0 < t < \tau_v$ denotes an initial value in the past, but is necessary due to the delay equation.

In the absence of delay, i.e. $\alpha_2^{(v)} = 0$ for all $v \in \mathcal{V}_c$, the above problem has been considered by some authors in some particular situations, for instance Ammari and Tucsnak [11], Ammari, Henrot and Tucsnak [6], Ammari and Jellouli [7, 8], Ammari, Jellouli and Khenissi [9] and Xu, Liu and Liu [109]. In these papers, some sufficient conditions are given in order to guarantee some stabilities of the system. On the contrary, if $\alpha_1^{(v)} = 0$ that is if we have only the delay part in the boundary/transmission condition, system (1.1) may become unstable. See, for instance Datko, Lagnese and Polis [42] for the example of a string. Therefore it is interesting to seek for stabilization results in general 1-d networks when the parameters $\alpha_1^{(v)}$ and $\alpha_2^{(v)}$ are both nonzero. In the special case of one string and a feedback law at one extremity, this problem has been studied by Xu, Yung and Li [110], where the authors use a spectral analysis. For the wave equation in higher dimensional space domain, we refer to [85].

In accordance with [85, 110], assuming that

$$\alpha_2^{(v)} \leq \alpha_1^{(v)}, \forall v \in \mathcal{V}_c,$$

we show the decay of an appropriate energy. We further give a necessary and sufficient condition for the decay to zero of the energy. If the above condition does not hold, we conjecture that the energy does not decay. We do not investigate this problem in its full generality but study it in a particular case.

Now if

$$\alpha_2^{(v)} < \alpha_1^{(v)}, \forall v \in \mathcal{V}_c,$$

we first give a sufficient condition for the exponential decay of the energy. We secondly find a sufficient condition for the polynomial decay of the energy.

Our method is based on the use of observability estimates of the problem without damping. Here we have chosen to obtain these observability estimates by a frequency domain method. The use of other techniques like the D'Alembert representation formula [7,38] may avoid the use of the frequency domain method but give quite often non optimal decay rates for the energy. Note finally that the observability estimate is independent of the delay term.

The chapter is organized as follows. After the recall of some definitions and notation, we show in the second section that our problem is well posed. Then in section 1.3, we prove the decay of an appropriate energy and give a necessary and sufficient condition which guarantees the decay to 0 of the energy. Section 1.4 is devoted to the proof of a regularity result and an a priori estimate used for the stability results. In section 1.5 we give a sufficient condition for the exponential stability of our system. Similarly section 1.6 is concerned with a sufficient condition for the polynomial stability of our system. Finally we end up with some illustrative examples in section 1.7.

In the whole chapter the notation $a \lesssim b$ means that there exists a positive constant C independent of a and b such that $a \leq C\ b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

1.2 Well posedness of the problem

We aim to show that problem (1.1) is well-posed. For that purpose, we use semi-group theory and an idea from [85].

For future uses, we introduce the spatial operator associated with the system similar to (1.1) but without damping. Introduce

$$L^2(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R}; u_j \in L^2(0, l_j), \forall j = 1, \cdots, N\},$$

which is a Hilbert space for the natural inner product. Its associated norm will be denoted by $\|\cdot\|_{L^2(\mathcal{R})}$. Let further V be the Hilbert space

$$V := \{\phi \in \prod_{j=1}^{N} H^1(0,\, l_j) \,:\, \phi_j(v) = \phi_k(v)\, \forall j,\, k \in \mathcal{E}_v,\, \forall v \in \mathcal{V}_{int}\, ;\, \phi_{j_v}(v) = 0\, \forall v \in \mathcal{D}\},$$

equipped with the inner product

$$< \phi,\, \tilde{\phi} >_V = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial \phi_j}{\partial x} \frac{\partial \tilde{\phi}_j}{\partial x} dx.$$

For shortness for $u \in L^1(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R}; u_j \in L^1(0, l_j), \forall j = 1, \cdots, N\}$, we often write

$$\int_{\mathcal{R}} u = \sum_{j=1}^{N} \int_0^{l_j} u_j(x)\, dx.$$

Now we introduce the operator A from $L^2(\mathcal{R})$ into itself by

$$\begin{aligned} D(A) \quad &:= \quad \{u \in V \cap \prod_{j=1}^{N} H^2(0,\, l_j) \,:\, \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = 0,\, \forall v \in \mathcal{V}_{int}\, ; \\ &\qquad \frac{\partial u_{j_v}}{\partial n_{j_v}}(v) = 0,\, \forall v \in \mathcal{N} \cup \mathcal{V}_{ext}^c\}, \\ (Au)_j \quad &= \quad -\frac{\partial^2 u_j}{\partial x^2} \quad \forall j = 1, \cdots, N,\, \forall u \in D(A). \end{aligned}$$

This operator is a positive selfadjoint operator since it is the Friedrichs extension of the triple $(L^2(\mathcal{R}), V, a)$, where the bilinear form a is defined by

$$a(u, v) = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial u_j}{\partial x} \frac{\partial v_j}{\partial x}\, dx,\ \forall u, v \in V.$$

Let further set $X = V \cap \prod_{j=1}^{N} H^2(0, l_j)$, which is a Hilbert space with the inner product

$$(u, v)_X = (u, v)_{L^2(\mathcal{R})} + (\Delta u, \Delta v)_{L^2(\mathcal{R})},\ \forall u, v \in X,$$

where we have set

$$(\Delta u)_j = \frac{\partial^2 u_j}{\partial x^2} \quad \forall j = 1, \cdots, N,\ u \in X.$$

Now we come back to our system (1.1) and transform it as follows. For all $v \in \mathcal{V}_c$ let us introduce the auxiliary variable $z_v(\rho, t) = \frac{\partial u}{\partial t}(v,\, t - \tau_v \rho)$ for $\rho \in (0,\, 1)$ and $t > 0$. In this manner, we eliminate the delay term in (1.1) and problem (1.1) is equivalent to

$$\begin{cases} \frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) = 0 & 0 < x < l_j, \quad t > 0, \, \forall j \in \{1, ..., N\}, \\ \tau_v \frac{\partial z_v}{\partial t}(\rho,\,t) + \frac{\partial z_v}{\partial \rho}(\rho,\,t) = 0 & 0 < \rho < 1, \, t > 0, \, \forall v \in \mathcal{V}_c, \\ u_j(v,\,t) = u_l(v,\,t) = u(v,t) & \forall j, \, l \in \mathcal{E}_v, \, v \in \mathcal{V}_{int}, t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = -(\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\,t) + \alpha_2^{(v)} z_v(1,\,t)) & \forall v \in \mathcal{V}_c, t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c, t > 0, \\ u_{j_v}(v,t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial u_{j_v}}{\partial n_{j_v}}(v,t) = 0 & \forall v \in \mathcal{N}, t > 0, \\ z_v(0,\,t) = \frac{\partial u}{\partial t}(v,\,t) & \forall v \in \mathcal{V}_c, \, t > 0, \\ u(t=0) = u^{(0)}, \, \frac{\partial u}{\partial t}(t=0) = u^{(1)}, & \\ z_v(\rho,\,0) = f_v^0(-\tau_v \rho) & \forall v \in \mathcal{V}_c, \, 0 < \rho < 1. \end{cases} \tag{1.2}$$

Note that z_v satisfies a transport equation in the t, ρ variables, with an initial datum at $t = 0$ and $\rho = 0$.

If we introduce $z = (z_v)_{v \in \mathcal{V}_c}$ and

$$U := \left(u, \, \frac{\partial u}{\partial t}, \, z\right)^\top,$$

then U satisfies

$$U' = \left(\frac{\partial u}{\partial t}, \, \frac{\partial^2 u}{\partial t^2}, \, \frac{\partial z}{\partial t}\right)^\top = \left(\frac{\partial u}{\partial t}, \, \Delta u, \, -\left(\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho}\right)_{v\in\mathcal{V}_c}\right)^\top.$$

Consequently the problem (1.2) may be rewritten as the first order evolution equation

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u^{(0)}, \, u^{(1)}, \, (f^0(-\tau_v \cdot))_v)^\top = U_0, \end{cases} \tag{1.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ w \\ z \end{pmatrix} := \begin{pmatrix} w \\ \Delta u \\ -(\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho})_v \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) := \{&(u, \, w, \, z) \in (V \cap \prod_{j=1}^N H^2(0, \, l_j)) \times V \times H^1(0, \, 1)^{\mathcal{V}_c} : \\ &\sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = -\,(\alpha_1^{(v)} w(v) + \alpha_2^{(v)} z_v(1)) \, \forall v \in \mathcal{V}_c\,; \\ &\sum_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = 0 \, \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c\,; \\ &\frac{\partial u_{j_v}}{\partial n_{j_v}}(v) = 0 \, \forall v \in \mathcal{N}\,; \, z_v(0) = w(v) \, \forall v \in \mathcal{V}_c\}, \end{aligned}$$

where V_c is the number of nodes of $\mathcal{V}_c$.

Now introduce the Hilbert space

$$H := V \times L^2(\mathcal{R}) \times L^2(0,\,1)^{V_c},$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ w \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \right\rangle = \sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial u_j}{\partial x} \frac{\partial \tilde{u}_j}{\partial x} + w_j \tilde{w}_j \right) dx + \sum_{v \in \mathcal{V}_c} \int_0^1 z_v(\rho) \tilde{z}_v(\rho) d\rho.$$

Lemma 1.2.1. *$D(\mathcal{A})$ is dense in H.*

Proof: Let $(f,\,g,\,h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A})$, namely

$$0 = \left\langle \begin{pmatrix} u \\ w \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle = \sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial u_j}{\partial x} \frac{\partial f_j}{\partial x} + w_j g_j \right) dx + \sum_{v \in \mathcal{V}_c} \int_0^1 z_v(\rho) h_v(\rho) d\rho,$$

for all $(u,\,w,\,z)^\top \in D(\mathcal{A})$.

We first take $u = 0$ and $w = 0$ and $z \in \mathcal{D}(0,\,1)^{V_c}$. As $(0,\,0,\,z) \in D(\mathcal{A})$, we get

$$\sum_{v \in \mathcal{V}_c} \int_0^1 z_v(\rho) h_v(\rho) d\rho = 0.$$

Since $\mathcal{D}(0,\,1)$ is dense in $L^2(0,\,1)$, we deduce that $h = 0$.

In the same manner since $\prod_{j=1}^{N} \mathcal{D}(0,\,l_j)$ is dense in $\prod_{j=1}^{N} L^2(0,\,l_j)$, by taking $u = 0$, $z = 0$ and $w \in \prod_{j=1}^{N} \mathcal{D}(0,\,l_j)$ we see that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial u_j}{\partial x} \frac{\partial f_j}{\partial x} dx,\ \forall (u,\,w,\,z) \in D(\mathcal{A}).$$

By restricting ourselves to $w = 0$ and $z = 0$, we obtain

$$\sum_{j=1}^{N} \int_0^{l_j} \frac{\partial u_j}{\partial x} \frac{\partial f_j}{\partial x} dx = 0,\ \forall (u,\,0,\,0) \in D(\mathcal{A}).$$

But we easily check that $(u,\,0,\,0) \in D(\mathcal{A})$ if and only if $u \in D(A)$. Since it is well known that $D(A)$ is dense in V (equipped with the inner product $<.,.>_V$), we conclude that $f = 0$. ■

Let us now suppose that

$$\alpha_2^{(v)} \leq \alpha_1^{(v)},\ \forall v \in \mathcal{V}_c. \tag{1.4}$$

Under this condition, we will show that the operator $\mathcal{A}$ generates a C_0-semi-group in H.

For that purpose, we choose positive real numbers ξ^v such that

$$\tau_v \alpha_2^{(v)} \leq \xi^{(v)} \leq \tau_v(2\alpha_1^{(v)} - \alpha_2^{(v)}),\ \forall v \in \mathcal{V}_c. \tag{1.5}$$

These constants exist due to the condition (1.4).

We now introduce the following inner product on H

$$\left\langle \begin{pmatrix} u \\ w \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \right\rangle_H = \sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial u_j}{\partial x}\frac{\partial \tilde{u}_j}{\partial x} + w_j \tilde{w}_j \right) dx + \sum_{v \in \mathcal{V}_c} \xi^{(v)} \left(\int_0^1 z_v(\rho)\tilde{z}_v(\rho) d\rho \right).$$

This inner product is clearly equivalent to the usual inner product of H.

Theorem 1.2.2. *For an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (1.3). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), H).$$

Proof: By Lumer-Phillips' theorem, it suffices to show that $\mathcal{A}$ is dissipative and maximal monotone.

We first prove that $\mathcal{A}$ is dissipative. Take $U = (u, w, z)^\top \in D(\mathcal{A})$. Then

$$\begin{aligned} (\mathcal{A}U, U) &= \left\langle \begin{pmatrix} w \\ \Delta u \\ -(\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho})_v \end{pmatrix}, \begin{pmatrix} u \\ w \\ z \end{pmatrix} \right\rangle_H \\ &= \sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial w_j}{\partial x}\frac{\partial u_j}{\partial x} + \frac{\partial^2 u_j}{\partial x^2} w_j \right) dx + \sum_{v \in \mathcal{V}_c} \xi^{(v)} \left(\int_0^1 -\frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho}(\rho) z_v(\rho) d\rho \right). \end{aligned}$$

By integrating by parts, we obtain

$$\begin{aligned} (\mathcal{A}U, U) =& \sum_{j=1}^{N} \int_0^{l_j} (-w_j \frac{\partial^2 u_j}{\partial x^2} \\ &+ \frac{\partial^2 u_j}{\partial x^2} w_j) dx + \sum_{j=1}^{N} [w_j \frac{\partial u_j}{\partial x}]_0^{l_j} - \sum_{v \in \mathcal{V}_c} \frac{\xi^{(v)}}{\tau_v} \left(\int_0^1 \frac{\partial z_v}{\partial \rho}(\rho) z_v(\rho) d\rho \right) \\ =& \sum_{j=1}^{N} [w_j \frac{\partial u_j}{\partial x}]_0^{l_j} - \sum_{v \in \mathcal{V}_c} \frac{\xi^{(v)}}{\tau_v} \left(\int_0^1 \frac{\partial z_v}{\partial \rho}(\rho) z_v(\rho) d\rho \right). \end{aligned}$$

Again an integration by parts leads to

$$\int_0^1 \frac{\partial z_v}{\partial \rho}(\rho) z_v(\rho) d\rho = \frac{1}{2}(z_v^2(1) - z_v^2(0)).$$

Moreover by the boundary/transmission conditions satisfied by $(u, w, z)^\top \in D(\mathcal{A})$, we have

$$\begin{aligned}
\sum_{j=1}^{N}[w_j \frac{\partial u_j}{\partial x}]_0^{l_j} &= \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v} w_j(v)\frac{\partial u_j}{\partial n_j}(v) \\
&= \sum_{v\in\mathcal{V}_c}\sum_{j\in\mathcal{E}_v} w_j(v)\tfrac{\partial u_j}{\partial n_j}(v) + \sum_{v\in\mathcal{D}} w_{j_v}(v)\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v) + \sum_{v\in\mathcal{N}} w_{j_v}(v)\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v) \\
&\quad + \sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}\sum_{j\in\mathcal{E}_v} w_j(v)\tfrac{\partial u_j}{\partial n_j}(v) \\
&= \sum_{v\in\mathcal{V}_c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))w_j(v) + \sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v))w_j(v) \\
&= \sum_{v\in\mathcal{V}_c} -(\alpha_1^{(v)}w(v) + \alpha_2^{(v)}z_v(1))z_v(0) \\
&= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}z_v(0)^2 + \alpha_2^{(v)}z_v(1)z_v(0)).
\end{aligned}$$

These properties yield

$$\begin{aligned}
(\mathcal{A}U, U) &= -\sum_{v\in\mathcal{V}_c}\left(\alpha_1^{(v)}z_v(0)^2 + \alpha_2^{(v)}z_v(1)z_v(0)\right) - \sum_{v\in\mathcal{V}_c}\tfrac{\xi^{(v)}}{2\tau_v}(z_v^2(1) - z_v^2(0)) \\
&= -\sum_{v\in\mathcal{V}_c}\left((\alpha_1^{(v)} - \tfrac{\xi^{(v)}}{2\tau_v})z_v(0)^2 + \tfrac{\xi^{(v)}}{2\tau_v}z_v^2(1) + \alpha_2^{(v)}z_v(1)z_v(0)\right).
\end{aligned}$$

By Cauchy-Schwarz's inequality we have

$$-\alpha_2^{(v)}z_v(1)z_v(0) \le \frac{\alpha_2^{(v)}}{2}z_v^2(1) + \frac{\alpha_2^{(v)}}{2}z_v^2(0)$$

and therefore

$$(\mathcal{A}U, U) \le -\sum_{v\in\mathcal{V}_c}\left((\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2})z_v(0)^2 + (\frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2})z_v^2(1)\right)$$

with $\alpha_1^{(v)} - \frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2} \ge 0$ and $\frac{\xi^{(v)}}{2\tau_v} - \frac{\alpha_2^{(v)}}{2} \ge 0$ because $\alpha_1^{(v)}$ and $\alpha_2^{(v)}$ satisfy condition (1.5). This shows that $(\mathcal{A}U, U) \le 0$ and then the dissipativeness of $\mathcal{A}$.

Let us now prove that $\mathcal{A}$ is maximal monotone, i.e. that $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$.

Let $(f, g, h)^\top \in H$. We look for $U = (u, w, z)^\top \in D(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\begin{pmatrix} u \\ w \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \tag{1.6}$$

or equivalently

$$\begin{cases} \lambda u_j - w_j = f_j & \forall j \in \{1, ..., N\}, \\ \lambda w_j - \frac{\partial^2 u_j}{\partial x^2} = g_j & \forall j \in \{1, ..., N\}, \\ \lambda z_v + \frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho} = h_v & \forall v \in \mathcal{V}_c. \end{cases} \tag{1.7}$$

Suppose that we have found u with the appropriate regularity. Then for all $j \in \{1, ..., N\}$, we have

$$w_j := \lambda u_j - f_j \in H^1(0, l_j) \tag{1.8}$$

with $w_{j_v}(v) = \lambda u_{j_v}(v) - f_{j_v}(v) = 0$ for $v \in \mathcal{D}$.

We can then determine z since $w(v) = z_v(0)$. Indeed, for $v \in \mathcal{V}_c$, z_v satisfies the differential equation

$$\lambda z_v + \frac{1}{\tau_v}\frac{\partial z_v}{\partial \rho} = h_v$$

and the boundary condition

$$z_v(0) = w(v) = \lambda u(v) - f(v).$$

Therefore z_v is explicitly given by

$$z_v(\rho) = \lambda u(v)e^{-\lambda\tau_v\rho} - f(v)e^{-\lambda\tau_v\rho} + \tau_v e^{-\lambda\tau_v\rho}\int_0^\rho e^{\lambda\tau_v\sigma}h_v(\sigma)d\sigma.$$

This means that once u is found with the appropriate properties, we can find z and w. Note that in particular we have

$$\begin{aligned} z_v(1) &= \lambda u(v)e^{-\lambda\tau_v} - f(v)e^{-\lambda\tau_v} + \tau_v e^{-\lambda\tau_v}\int_0^1 e^{\lambda\tau_v\sigma}h_v(\sigma)d\sigma \\ &= \lambda u(v)e^{-\lambda\tau_v} + z_v^0(v) \end{aligned}$$

where $z_v^0(v) = -f(v)e^{-\lambda\tau_v} + \tau_v e^{-\lambda\tau_v}\int_0^1 e^{\lambda\tau_v\sigma}h_v(\sigma)d\sigma$ is a fixed real number depending only on f and h.

It remains to find u. By (1.7) and (1.8), u_j must satisfy

$$\lambda^2 u_j - \frac{\partial^2 u_j}{\partial x^2} = g_j + \lambda f_j.$$

Multiplying this identity by a test function ϕ_j, integrating in space and using integration by parts, we obtain

$$\begin{aligned} \sum_{j=1}^N \int_0^{l_j}(\lambda^2 u_j - \frac{\partial^2 u_j}{\partial x^2})\phi_j dx =& \sum_{j=1}^N \int_0^{l_j}(\lambda^2 u_j\phi_j + \frac{\partial u_j}{\partial x}\frac{\partial \phi_j}{\partial x})dx - \sum_{j=1}^N [\frac{\partial u_j}{\partial x}\phi_j]_0^{l_j} \\ =& \sum_{j=1}^N \int_0^{l_j}(\lambda^2 u_j\phi_j + \frac{\partial u_j}{\partial x}\frac{\partial \phi_j}{\partial x})dx - \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v)\phi_j(v). \end{aligned}$$

But using the fact that $(u,\, w,\, z)^\top$ must belong to $D(\mathcal{A})$, we have

$$
\begin{aligned}
\sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v) &= \sum_{v\in\mathcal{V}_c}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)+\sum_{v\in\mathcal{D}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\phi_{j_v}(v)\\
&\quad+\sum_{v\in\mathcal{N}}\tfrac{\partial u_{j_v}}{\partial n_{j_v}}(v)\phi_{j_v}(v)+\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)\\
&= \sum_{v\in\mathcal{V}_c}\Big(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\Big)\phi(v)\\
&= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}w_j(v)+\alpha_2^{(v)}z_v(1))\phi(v).
\end{aligned}
$$

Using the above expression for $z_v(1)$ we arrive at the problem

$$
\begin{aligned}
\sum_{j=1}^{N}\int_0^{l_j}(\lambda^2u_j\phi_j+\frac{\partial u_j}{\partial x}\frac{\partial\phi_j}{\partial x})dx &+ \sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)\phi(v)\\
&= \sum_{j=1}^{N}\int_0^{l_j}(g_j+\lambda f_j)\phi_j dx \\
&\quad+\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}f(v)-\alpha_2^{(v)}z_v^0(v))\phi(v), \forall\phi\in V.
\end{aligned}
\tag{1.9}
$$

This problem has a unique solution $u\in V$ by Lax-Milgram's lemma, because the left-hand side of (1.9) is coercive on V. If we consider $\phi\in\prod_{j=1}^{N}\mathcal{D}(0,\, l_j)\subset V$, then u satisfies

$$
\lambda^2u_j-\frac{\partial^2u_j}{\partial x^2}=g_j+\lambda f_j \text{ in } \mathcal{D}'(0,\, l_j)\quad\forall j=1,\cdots,N.
$$

This directly implies that $u\in\prod_{j=1}^{N}H^2(0,\, l_j)$ and then $u\in V\cap\prod_{j=1}^{N}H^2(0,\, l_j)$. Coming back to (1.9) and by integrating by parts, we find

$$
\begin{aligned}
&\sum_{v\in\mathcal{V}_c}\left(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)+(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)+(\alpha_2^{(v)}z_v^0(v)-\alpha_1^{(v)}f(v))\right)\phi(v)\\
&\qquad=-\sum_{v\in\mathcal{N}}\tfrac{\partial u_j}{\partial n_j}(v)\phi_j(v)-\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}\Big(\sum_{j\in\mathcal{E}_v}\tfrac{\partial u_j}{\partial n_j}(v)\Big)\phi_j(v), \forall\phi\in V.
\end{aligned}
$$

Consequently, by taking particular test functions ϕ, we obtain

$$
\begin{aligned}
\frac{\partial u_{j_v}}{\partial x}(v) &= 0 \quad \forall v\in\mathcal{N},\\
\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v) &= 0 \quad \forall v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v) &= -(\alpha_1^{(v)}+\alpha_2^{(v)}e^{-\lambda\tau_v})\lambda u(v)-(\alpha_2^{(v)}z_v^0(v)-\alpha_1^{(v)}f(v))\\
&= -\alpha_2^{(v)}z_v(1)-\alpha_1^{(v)}(\lambda u(v)+f(v))\\
&= -(\alpha_2^{(v)}z_v(1)+\alpha_1^{(v)}w(v)) \quad \forall v\in\mathcal{V}_c.
\end{aligned}
$$

In summary we have found $(u,\, w,\, z)^\top \in D(\mathcal{A})$ satisfying (1.6). ∎

1.3 The energy

We now restrict the hypothesis (1.4) to obtain the decay of the energy. Namely we suppose that

$$\alpha_2^{(v)} < \alpha_1^{(v)},\ \forall v\in\mathcal{V}_c. \tag{1.10}$$

Let us choose the following energy (which corresponds to the inner product on H)

$$E(t) := \frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}\left(\left(\frac{\partial u_j}{\partial t}\right)^2+\left(\frac{\partial u_j}{\partial x}\right)^2\right)dx+\sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{2}\left(\int_0^1(\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho))^2 d\rho\right) \tag{1.11}$$

where $\xi^{(v)}$ is a positive constant satisfying (that exists due to (1.10))

$$\tau_v\alpha_2^{(v)} < \xi^{(v)} < \tau_v(2\alpha_1^{(v)}-\alpha_2^{(v)}),\ \forall v\in\mathcal{V}_c. \tag{1.12}$$

1.3.1 Decay of the energy

Proposition 1.3.1. *For all regular solution of problem (1.1), the energy is non increasing and there exists two positive constants C_1 and C_2 depending only on $\xi^{(v)}$, $\alpha_1^{(v)}$, $\alpha_2^{(v)}$ and τ_v such that*

$$
\begin{aligned}
&-C_2\sum_{v\in\mathcal{V}_c}\left(\left(\frac{\partial u}{\partial t}(v,\, t)\right)^2+\left(\frac{\partial u}{\partial t}(v,\, t-\tau_v)\right)^2\right)\le E'(t)\\
&\le -C_1\sum_{v\in\mathcal{V}_c}\left(\left(\frac{\partial u}{\partial t}(v,\, t)\right)^2+\left(\frac{\partial u}{\partial t}(v,\, t-\tau_v)\right)^2\right).
\end{aligned} \tag{1.13}
$$

Proof: Deriving (1.11) and integrating by parts in space, we obtain

$$
\begin{aligned}
E'(t) &= \sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial u_j}{\partial t}\frac{\partial^2 u_j}{\partial t^2}+\frac{\partial u_j}{\partial x}\frac{\partial^2 u_j}{\partial x\partial t})dx\\
&\quad+\sum_{v\in\mathcal{V}_c}\xi^{(v)}(\int_0^1\frac{\partial u}{\partial t}(v,\, t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\, t-\tau_v\rho)d\rho)
\end{aligned}
$$

and then

$$\begin{aligned}
E'(t) &= \sum_{j=1}^{N}\int_0^{l_j}(\frac{\partial^2 u_j}{\partial t^2}-\frac{\partial u_j^2}{\partial x^2})\frac{\partial u_j}{\partial t}dx+\sum_{j=1}^{N}[\frac{\partial u_j}{\partial t}\frac{\partial u_j}{\partial x}]_0^{l_j} \\
&\quad +\sum_{v\in\mathcal{V}_c}\xi^{(v)}(\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho) \\
&= \sum_{v\in\mathcal{V}}\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v)\frac{\partial u_j}{\partial t}(v)+\sum_{v\in\mathcal{V}_c}\xi^{(v)}\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho \\
&= \sum_{v\in\mathcal{V}_c}(\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v))\frac{\partial u}{\partial t}(v)+\sum_{v\in\mathcal{D}}\frac{\partial u_{j_v}}{\partial n_{j_v}}(v)\frac{\partial u_{j_v}}{\partial t}(v)+\sum_{v\in\mathcal{N}}\frac{\partial u_{j_v}}{\partial n_{j_v}}(v)\frac{\partial u_{j_v}}{\partial t}(v) \\
&\quad +\sum_{v\in\mathcal{V}_{int}\setminus\mathcal{V}_{int}^c}(\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v))\frac{\partial u}{\partial t}(v) \\
&\quad +\sum_{v\in\mathcal{V}_c}\xi^{(v)}\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho \\
&= -\sum_{v\in\mathcal{V}_c}(\alpha_1^{(v)}(\frac{\partial u}{\partial t}(v,\,t))^2+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,t-\tau_v)\frac{\partial u}{\partial t}(v,\,t)) \\
&\quad +\sum_{v\in\mathcal{V}_c}\xi^{(v)}\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho.
\end{aligned}$$

Now for all $v\in\mathcal{V}_c$, recalling that $z_v(\rho,t)=\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)$, we see that

$$\begin{aligned}
\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho &= \int_0^1 z_v(\rho,t)\frac{\partial z_v}{\partial t}(\rho,t)d\rho \\
&= -\frac{1}{\tau_v}\int_0^1 z_v(\rho,t)\frac{\partial z_v}{\partial \rho}(\rho,t)d\rho.
\end{aligned}$$

By an integration by parts in ρ, we obtain

$$\int_0^1\frac{\partial u}{\partial t}(v,\,t-\tau_v\rho)\frac{\partial^2 u}{\partial t^2}(v,\,t-\tau_v\rho)d\rho=-\frac{1}{2\tau_v}\left((\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2-(\frac{\partial u}{\partial t}(v,\,t))^2\right).$$

Therefore, we have

$$\begin{aligned}
E'(t) &= -\sum_{v\in\mathcal{V}_c}[\alpha_1^{(v)}(\frac{\partial u}{\partial t}(v,\,t))^2+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,t-\tau_v)\frac{\partial u}{\partial t}(v,\,t) \\
&\quad +\frac{\xi^{(v)}}{2\tau_v}((\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2-(\frac{\partial u}{\partial t}(v,\,t))^2)] \\
&= -\sum_{v\in\mathcal{V}_c}[(\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v})(\frac{\partial u}{\partial t}(v,\,t))^2+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,t-\tau_v)\frac{\partial u}{\partial t}(v,\,t) \\
&\quad +\frac{\xi^{(v)}}{2\tau_v}(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2].
\end{aligned}$$

Cauchy-Schwarz's inequality yields

$$E'(t) \leq -\sum_{v\in\mathcal{V}_c}\left((\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)$$
$$E'(t) \geq -\sum_{v\in\mathcal{V}_c}\left((\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}+\frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\xi^{(v)}}{2\tau_v}+\frac{\alpha_2^{(v)}}{2})(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right).$$

The first estimate leads to

$$E'(t) \leq -C_1\sum_{v\in\mathcal{V}_c}\left((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)$$

with

$$C_1=\min\left\{(\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2}),\,(\frac{\xi^{(v)}}{2\tau_v}-\frac{\alpha_2^{(v)}}{2})\;:\;v\in\mathcal{V}_c\right\}$$

which is positive according to the assumption (1.12). The second one yields

$$E'(t) \geq -C_2\sum_{v\in\mathcal{V}_c}\left((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)$$

with

$$C_2=\max\left\{(\alpha_1^{(v)}-\frac{\xi^{(v)}}{2\tau_v}+\frac{\alpha_2^{(v)}}{2}),\,(\frac{\xi^{(v)}}{2\tau_v}+\frac{\alpha_2^{(v)}}{2})\;:\;v\in\mathcal{V}_c\right\}$$

which is also positive due to (1.12). ■

We have just shown that under the assumption (1.10), the energy decays. But we would like to obtain strong stability of the system, in other words, the decay to 0 of the energy. This is the goal of the remainder of this section. But before going on, let us make the next remark that will be useful later on.

Remark 1.3.2. Integrating the expression (1.13) between 0 and T, we obtain

$$\sum_{v\in\mathcal{V}_c}\int_0^T\left((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)dt \lesssim E(0)-E(T)\lesssim E(0))$$

and therefore

$$\int_0^T\left((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)dt \lesssim E(0), \forall v\in\mathcal{V}_c.$$

This estimate implies that $\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\,.)+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,.-\tau_v)$ belongs to $L^2(0,\,T)$ for all $v\in\mathcal{V}_c$, with the estimate

$$\left\|\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\,.)+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,.-\tau_v)\right\|^2_{L^2(0,\,T)} = \int_0^T(\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\,t)+\alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2dt$$
$$\leq 2\max_{v\in\mathcal{V}_c}\{(\alpha_1^{(v)})^2\}\int_0^T(\frac{\partial u}{\partial t}(v,\,t)^2+\frac{\partial u}{\partial t}(v,\,t-\tau_v)^2)dt \lesssim E(0)<+\infty.$$

1.3.2 Problem without damping

In the sequel we need to consider the problem without damping

$$
\left\{
\begin{array}{ll}
\frac{\partial^2\phi_j}{\partial t^2}(x,\, t) - \frac{\partial^2\phi_j}{\partial x^2}(x,\, t) = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., N\} \\
\phi_j(v,\, t) = \phi_l(v,\, t) = \phi(v,t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial\phi_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int}, t > 0, \\
\phi_{j_v}(v,t) = 0 & \forall v \in \mathcal{D}, t > 0, \\
\frac{\partial\phi_{j_v}}{\partial n_{j_v}}(v,t) = 0 & \forall v \in \mathcal{N} \cup \mathcal{V}_{ext}^c, t > 0, \\
\phi(t=0) = u^{(0)},\, \frac{\partial\phi}{\partial t}(t=0) = u^{(1)}. &
\end{array}
\right.
\tag{1.14}
$$

It is well known that this problem is well posed in the natural energy space (see for instance [3]).

Lemma 1.3.3. *Suppose that $(u^{(0)},\, u^{(1)}) \in V \times \prod\limits_{j=1}^{N} L^2(0,\, l_j)$. Then problem (1.14) admits a unique solution*

$$
\phi \in C(0,\, T;\, V) \cap C^1(0,\, T;\, \prod_{j=1}^{N} L^2(0,\, l_j)).
$$

This problem is obviously conservative, its energy is constant.

1.3.3 Decay of the energy to 0

We look at the spectral problem associated with problem (1.14), in other words

$$
\left\{
\begin{array}{ll}
-\lambda^2\phi_j(x) - \frac{\partial^2\phi_j}{\partial x^2}(x) = 0 & 0 < x < l_j,\, \forall j \in \{1, ..., N\}, \\
\varphi_j(v) = \varphi_l(v) = \phi(v) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, \\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial\phi_j}{\partial n_j}(v) = 0 & \forall v \in \mathcal{V}_{int}, \\
\phi_{j_v}(v) = 0 & \forall v \in \mathcal{D}, \\
\frac{\partial\phi_{j_v}}{\partial n_{j_v}}(v) = 0 & \forall v \in \mathcal{N} \cup \mathcal{V}_{ext}^c.
\end{array}
\right.
$$

This system corresponds to an eigenvalue problem of the positive selfadjoint operator A defined above. Let us then denote by $\{\lambda_k^2\}_{k\geq 1}$ the set of eigenvalues counted without their multiplicities, i.e. $\lambda_k \neq \lambda_l$, $\forall k \neq l$, where without any restriction, we may suppose that $\lambda_k > 0$. For all $k \in \mathbb{N}^*$, let l_k be the multiplicity of the eigenvalue λ_k^2 (remark that $l_k \leq 2N$, $\forall k \in \mathbb{N}^*$) and let $\{\varphi_{k,i}\}_{1\leq i\leq l_k}$ be the orthonormal eigenvectors associated with the eigenvalue λ_k^2.

Definition 1.3.4. *For $k \geq 1$ and $v \in \mathcal{V}_c$, we denote by $\mathcal{M}_v(\lambda_k^2)$ the following matrix of size l_k*

$$\mathcal{M}_v(\lambda_k^2) := \begin{pmatrix} \varphi_{k,1}^2(v) & \varphi_{k,1}(v)\varphi_{k,2}(v) & \cdots & \varphi_{k,1}(v)\varphi_{k,l_k}(v) \\ \varphi_{k,1}(v)\varphi_{k,2}(v) & \varphi_{k,2}^2(v) & \cdots & \varphi_{k,2}(v)\varphi_{k,l_k}(v) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{k,1}(v)\varphi_{k,l_k}(v) & \varphi_{k,2}(v)\varphi_{k,l_k}(v) & \cdots & \varphi_{k,l_k}^2(v) \end{pmatrix}.$$

Moreover, let $\mathcal{M}(\lambda_k^2)$ be the matrix of size l_k

$$\mathcal{M}(\lambda_k^2) := \sum_{v \in \mathcal{V}_c} \mathcal{M}_v(\lambda_k^2).$$

Now we recall that the following generalized gap condition holds, namely from Proposition 6.2 of [38], we have

$$\exists \gamma > 0,\ \forall k \geq 1,\ \lambda_{k+N+1} - \lambda_k \geq (N+1)\gamma. \tag{1.15}$$

From this property we will deduce an inequality of Ingham's type. Namely fix a positive real number $\gamma' \leq \gamma$ and denote by $A_k, k = 1, \cdots, N+1$ the set of natural numbers m satisfying (see for instance [13])

$$\begin{cases} \lambda_m - \lambda_{m-1} \geq \gamma' \\ \lambda_n - \lambda_{n-1} < \gamma' & \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{m+k} - \lambda_{m+k-1} \geq \gamma'. \end{cases}$$

Then one easily checks that the sets A_{k+j}, $j = 0, \cdots, k-1$, $k = 1, \cdots, N+1$, form a partition of $\mathbb{N}^*$.

Now for $m \in A_k$, we recall that the finite differences $e_{m+j}(t)$, $j = 0, \cdots, k-1$, corresponding to the exponential functions $e^{i\lambda_{m+j}t}, j = 0, \cdots, k-1$ are given by

$$e_{m+j}(t) = \sum_{p=m}^{m+j} \prod_{q=m,\, q \neq p}^{m+j} (\lambda_p - \lambda_q)^{-1} e^{i\lambda_p t}.$$

Write for shortness, $e_{-n}(t)$ the same finite differences functions corresponding to $-\lambda_n$.

Now we are ready to recall the next inequality of Ingham's type, see for instance Theorem 1.5 of [13] :

Theorem 1.3.5. *If the sequence $(\lambda_n)_{n \geq 1}$ satisfies (1.15), then for all sequence $(a_n)_{n \in \mathbb{Z}^*}$ (where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$), the function*

$$f(t) = \sum_{n \in \mathbb{Z}^*} a_n e_n(t),$$

satisfies the estimates

$$\int_0^T |f(t)|^2 \sim \sum_{n \in \mathbb{Z}^*} |a_n|^2, \tag{1.16}$$

for $T > 2\pi/\gamma$.

Going back to the original functions $e^{i\lambda_n t}$, the above equivalence (1.16) means that, for $T > 2\pi/\gamma$, the function (from now on $\lambda_{-n} = -\lambda_n$)

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_n t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 \sim \sum_{k=1}^{N+1} \sum_{|n|\in A_k} \|B_n^{-1} C_n\|_2^2, \tag{1.17}$$

where $\|\cdot\|_2$ means the Euclidean norm of the vector, for $n \in A_k$, the vector C_n is given by

$$C_n = (\alpha_n, \cdots, \alpha_{n+k-1})^\top,$$

and the $k \times k$ matrix B_n allows to pass from the coefficients a_n to α_n, namely

$$C_n = B_n \cdot (a_n, \cdots, a_{n+k-1})^\top,$$

and is given by $B_n = (B_{n,ij})_{1\le i,j\le k}$ with

$$B_{n,ij} = \begin{cases} \displaystyle\prod_{q=n,\, q\neq n+i-1}^{n+j-1} (\lambda_{n+i-1} - \lambda_q)^{-1} & \text{if } i \le j,\ (i,j) \neq (1,1), \\ 1 & \text{if } i = j = 1, \\ 0 & \text{if } i > j. \end{cases}$$

We proceed similarly for $n \le -1$, but the indices being decreasing from n to $n - k + 1$.

Remark 1.3.6. Notice that if the standard gap condition

$$\exists \gamma > 0,\ \forall k \ge 1,\ \lambda_{k+1} - \lambda_k \ge \gamma \tag{1.18}$$

holds, then $A_1 = \mathbb{Z}^*$ and $B_1 = 1$ and in that case the next equivalence holds (see [57]) :

$$\int_0^T |f(t)|^2 \sim \sum_{n\in\mathbb{Z}^*} |\alpha_n|^2.$$

We are now ready to give a necessary and sufficient condition that guarantees the decay to 0 of the energy.

Proposition 1.3.7. *For all initial data in H, we have*

$$\lim_{t\to\infty} E(t) = 0 \tag{1.19}$$

if and only if the operator A satisfies

$$\lambda_{min}(\mathcal{M}(\lambda_k^2)) > 0, \forall k \in \mathbb{N}^*, \tag{1.20}$$

where $\lambda_{min}(\mathcal{M})$ denotes the smallest eigenvalue of the matrix $\mathcal{M}$.

Proof: $\boxed{\Leftarrow}$ Let us show that (1.20) implies (1.19) : Let $S(t)$ be the semi-group of contractions generated by the operator $\mathcal{A}$.

It suffices to show that

$$\lim_{t\to\infty} S(t)\begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v\cdot))_v \end{pmatrix} = 0,\ \forall \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v\cdot))_v \end{pmatrix} \in D(\mathcal{A}).$$

Let us fix $U_0 = \begin{pmatrix} u^{(0)} \\ u^{(1)} \\ (f^0(-\tau_v\cdot))_v \end{pmatrix} \in D(\mathcal{A})$. Since $D(\mathcal{A})$ is compactly embedded into H, the set

$$orb(U_0) = \bigcup_{t\geq 0} S(t)U_0$$

is precompact in H. Indeed, for any sequence $(t_n)_n$, since $U_0 \in D(\mathcal{A})$, one has $S(t_n)U_0 \in D(\mathcal{A})$ and

$$\begin{aligned} \|S(t_n)U_0\|_{D(\mathcal{A})} &= \|S(t_n)U_0\|_H + \|\mathcal{A}S(t_n)U_0\|_H = \|S(t_n)U_0\|_H + \|S(t_n)\mathcal{A}U_0\| \\ &\leq \|U_0\|_H + \|\mathcal{A}U_0\|_H = cst. \end{aligned}$$

Therefore the sequence $S(t_n)U_0$ is bounded in D($\mathcal{A}$) and by the compact embedding of $D(\mathcal{A})$ into H, there exists a subsequence, still denote by $S(t_n)U_0$ which converges in H. In this case, the ω-limit of U_0 defined by

$$\omega(U_0) = \{U \in H\ :\ \exists(t_n),\, t_n \to \infty,\, S(t_n)U_0 \to U,\, t \to \infty\}$$

is non empty.

On the other hand, if $\Phi \in \omega(U_0)$, then

$$S(t)\Phi \in \omega(U_0).$$

Note further that one readily checks that $S(t)\Phi$ is of the form

$$S(t)\Phi = \begin{pmatrix} \phi(.,t) \\ \frac{\partial\phi}{\partial t}(.,t) \\ \psi \end{pmatrix},$$

for some $\phi \in C([0,\infty); V) \cap C^1([0,\infty); L^2(\mathcal{R}))$ and $\psi \in C([0,\infty); L^2(0,1)^{V_c})$.

We can now apply LaSalle's invariance principle [32] with the relatively compact set $\bigcup_{t\geq 0} S(t)U_0$ and the Lyapounov functional $\phi = \|.\|_H$. Since $\begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}$ and $S(t)\begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} = \begin{pmatrix} \phi(.,t) \\ \frac{\partial\phi}{\partial t}(.,t) \\ \psi \end{pmatrix}$ belong to $\omega(U_0)$, we find that

$$\left\|\begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}\right\|_H = \left\|\begin{pmatrix} \phi(.,t) \\ \frac{\partial\phi}{\partial t}(.,t) \\ \psi \end{pmatrix}\right\|_H = L \quad \forall t \geq 0.$$

Therefore ϕ satisfies problem (1.2) with initial conditions $\phi(\cdot, 0) = \phi^{(0)}$ and $\frac{\partial \phi}{\partial t}(\cdot, 0) = \phi^{(1)}$. Moreover by (1.13) we have

$$
\begin{aligned}
0 = L - L &= \left\| \begin{pmatrix} \phi(., t) \\ \frac{\partial \phi}{\partial t}(., t) \\ \psi \end{pmatrix} \right\|_H - \left\| \begin{pmatrix} \phi^{(0)} \\ \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \right\|_H \\
&\leq -C_1 \sum_{v \in \mathcal{V}_c} \int_0^t (\psi_v(0, s)^2 + \psi_v(1, s)^2) ds \leq 0.
\end{aligned}
$$

In other words, it holds

$$
\sum_{v \in \mathcal{V}_c} \int_0^t (\psi_v(0, s)^2 + \psi_v(1, s)^2) ds = 0
$$

which implies that

$$
\psi_v(0, t) = \psi_v(1, t) = 0 \quad \forall t \geq 0, \forall v \in \mathcal{V}_c. \tag{1.21}
$$

In particular, this implies that ϕ is solution of problem (1.14) with initial data $\phi(\cdot, 0) = \phi^{(0)}$ and $\frac{\partial \phi}{\partial t}(\cdot, 0) = \phi^{(1)}$ because $0 = \psi_v(0, t) = \frac{\partial \phi}{\partial t}(v, t)$ and $0 = \psi_v(1, t) = \frac{\partial \phi}{\partial t}(v, t - \tau_v)$, which means that in (1.2) the damping terms disappear.

Let us now write

$$
\begin{aligned}
\phi^{(0)} &= \sum_{k \geq 1} \sum_{i=1}^{l_k} a_{k, i} \varphi_{k, i}, \\
\phi^{(1)} &= \sum_{k \geq 1} \sum_{i=1}^{l_k} b_{k, i} \varphi_{k, i},
\end{aligned}
$$

where $(\lambda_{k, i} a_{k, i})_{i, k}$, $(b_{k, i})_{i, k} \in l^2(\mathbb{N}^*)$. Then ϕ is given by

$$
\phi(\cdot, t) = \sum_{k \geq 1} \sum_{i=1}^{l_k} \left(a_{k, i} \cos(\lambda_k t) + \frac{b_{k, i}}{\lambda_k} \sin(\lambda_k t) \right) \varphi_{k, i}.
$$

Consequently by (1.21) for $v \in \mathcal{V}_c$ and $j \in \mathcal{E}_v$

$$
0 = \frac{\partial \phi_j}{\partial t}(v, t) = \sum_{k \geq 1} \sum_{i=1}^{l_k} \left(-a_{k, i} \lambda_k \sin(\lambda_k t) + b_{k, i} \cos(\lambda_k t) \right) \varphi_{k, i}(v).
$$

By grouping the terms corresponding to the same eigenvalue, we get

$$
\begin{aligned}
0 = \frac{\partial \phi_j}{\partial t}(v, t) &= \sum_{k \geq 1} \left(\sum_{i=1}^{l_k} - a_{k, i} \varphi_{k, i}(v) \right) \lambda_k \sin(\lambda_k t) + \sum_{k \geq 1} \left(\sum_{i=1}^{l_k} b_{k, i} \varphi_{k, i}(v) \right) \cos(\lambda_k t) \\
&= \sum_{n \in \mathbb{Z}^*} \alpha_n(v) e^{i \lambda_n t},
\end{aligned}
$$

where

$$\alpha_k(v) = \frac{1}{2}\left(\left(\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}(v)\right) + i\left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}(v)\right)\lambda_k\right), \forall k \geq 1,$$
$$\alpha_{-k}(v) = \frac{1}{2}\left(\left(\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}(v)\right) - i\left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}(v)\right)\lambda_k\right), \forall k \geq 1.$$

Integrating this identity between 0 and $T > 0$ sufficiently large and using Ingham's inequality (1.17), we obtain (with the notation introduced above)

$$0 = \int_0^T \left(\frac{\partial \phi_j}{\partial t}(v, t)\right)^2 dt \gtrsim \sum_{k=1}^{N+1} \sum_{|n|\in A_k} \|B_n^{-1}C_n(v)\|_2^2,$$

where $C_n(v)$ is defined as C_n using $\alpha_n(v)$ instead of α_n. Summing on $v \in \mathcal{V}_c$:

$$0 \gtrsim \sum_{v\in\mathcal{V}_c} \sum_{k=1}^{N+1} \sum_{|n|\in A_k} \|B_n^{-1}C_n(v)\|_2^2 \geq 0.$$

This implies that for all $k = 1, \cdots, N+1$ and all $|n| \in A_k$, we have

$$\sum_{v\in\mathcal{V}_c} \|B_n^{-1}C_n(v)\|_2^2 = 0.$$

Since B_n^{-1} is invertible, there exits $\gamma_n > 0$ such that

$$\|B_n^{-1}C_n(v)\|_2 \geq \gamma_n \|C_n(v)\|_2.$$

From this estimate we deduce that for all $k = 1, \cdots, N+1$ and all $|n| \in A_k$, we have

$$\sum_{v\in\mathcal{V}_c} \|C_n(v)\|_2^2 = 0.$$

Since $\lambda_k \neq 0$, we necessarily have

$$\forall k \geq 1, \sum_{v\in\mathcal{V}_c} \left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}(v)\right)^2 = 0 \text{ and } \sum_{v\in\mathcal{V}_c} \left(\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}(v)\right)^2 = 0.$$

For a fixed $k \geq 1$, if we set $b = \begin{pmatrix} b_{k,1} \\ \vdots \\ b_{k,l_k} \end{pmatrix}$, then

$$\sum_{v\in\mathcal{V}_c} \left(\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}(v)\right)^2 = {}^t b \mathcal{M}(\lambda_k^2) b.$$

As a consequence if $\lambda_{min}(\mathcal{M}(\lambda_k^2)) > 0$, we obtain that $b_{k,1} = ... = b_{k,l_k} = 0$. In the same manner we have $a_{k,1} = ... = a_{k,l_k} = 0$.

We have proved that $\phi^{(0)} = 0 = \phi^{(1)}$.

Moreover $\phi^{(2)} = 0$ because ψ_v satisfies the transport equation

$$\begin{cases} \frac{\partial \psi_v}{\partial t} = -\frac{1}{\tau_v}\frac{\partial \psi_v}{\partial \rho}, \\ \psi_v(0,\, t) = \psi_v(1,\, t) = 0, \\ \psi_v(\rho,\, 0) = \phi^{(2)}, \end{cases}$$

for all $v \in \mathcal{V}_c$.

We have shown that for all $\left(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}\right)^\top \in \omega(U_0)$, we have $\phi^{(0)} = 0 = \phi^{(1)} = \phi^{(2)}$. Consequently $\lim_{t\to\infty} S(t)U_0 = 0$ and then $\lim_{t\to\infty} E(t) = 0$.

$\boxed{\Rightarrow}$ Let us show that (1.19) implies (1.20). For that purpose we use a contradiction argument. Suppose that there exists $k > 0$ such that $\lambda_{min}(\mathcal{M}(\lambda_k^2)) = 0$. This means that there exists $a = (a_{k,1}, \cdots, a_{k,l_k})^\top \neq 0$, such that

$$\sum_{v\in\mathcal{V}_c}\left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}(v)\right)^2 = 0.$$

Let us set

$$u(.,\, t) = \left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}\right)\cos(\lambda_k t).$$

Then u is solution of (1.1) and satisfies

$$\begin{aligned} E(t) &= \tfrac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}\left((\frac{\partial u_j}{\partial t})^2 + (\frac{\partial u_j}{\partial x})^2\right)dx + \sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{2}\left(\int_0^1 (\frac{\partial u_j}{\partial t}(v,\, t - \tau_v\rho))^2 d\rho\right) \\ &= \tfrac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}\left((\frac{\partial u_j}{\partial t})^2 + (\frac{\partial u_j}{\partial x})^2\right)dx \\ &= E(u(0)) \end{aligned}$$

because

$$\forall v \in \mathcal{V}_c,\, \forall t,\, \frac{\partial u_j}{\partial t}(v,\, t) = \left(\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}(v)\right)(-\lambda_k)\sin(\lambda_k t) = 0.$$

This means that we have obtained a solution of problem (1.1) with a constant energy, which contradicts (1.19). ∎

Remark 1.3.8. 1. Notice that the condition (1.20) is independent of the choice of the orthonormal basis of eigenvectors associated with the eigenvalue λ_k^2. Indeed if $\{\tilde{\varphi}_{k,i}\}_{i=1}^{l_k}$ is another orthonormal basis of eigenvectors associated with the eigenvalue λ_k^2, then there exists an orthogonal matrix $O \in \mathbb{R}^{l_k \times l_k}$ such that

$$\begin{pmatrix} \tilde{\varphi}_{k,1} \\ \vdots \\ \tilde{\varphi}_{k,l_k} \end{pmatrix} = O \begin{pmatrix} \varphi_{k,1} \\ \vdots \\ \varphi_{k,l_k} \end{pmatrix}.$$

Consequently the matrix $\tilde{\mathcal{M}}(\lambda_k^2)$ built as $\mathcal{M}(\lambda_k^2)$ by using $\{\tilde{\varphi}_{k,i}\}_{i=1}^{l_k}$ instead of $\{\varphi_{k,i}\}_{i=1}^{l_k}$ is given by

$$\tilde{\mathcal{M}}(\lambda_k^2) = O\mathcal{M}(\lambda_k^2)O^\top,$$

and therefore $\lambda_{min}(\tilde{\mathcal{M}}(\lambda_k^2)) = \lambda_{min}(\mathcal{M}(\lambda_k^2))$.
2. If $l_k = 1$, then the condition (1.20) is reduced to

$$\forall k \in \mathbb{N}^*, \sum_{v\in\mathcal{V}_c} |\varphi_k(v)|^2 > 0$$

because $\mathcal{M}(\lambda_k^2)$ is the matrix of size 1 equals to $\sum_{v\in\mathcal{V}_c} |\varphi_k(v)|^2$.

1.3.4 Counterexample to the stability of the system

In this section (and in the remainder of the chapter) we have made the hypothesis (1.10). As in [110], we may expect non-stability results if this condition fails.

In [110], the authors consider the wave equation on a string of length π and used a boundary control. They show that if (1.10) does not hold then non-stabilities appear. Since their problem enters in our framework, this is a first counterexample. As a second counterexample, we consider the wave equation on a string of length π but with an interior control. Namely we consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 & 0 < x < \pi,\, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-,t) - \frac{\partial u}{\partial x}(\xi_+,t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi,t) + \alpha_2 \frac{\partial u}{\partial t}(\xi, t-\tau)) & t > 0, \\ u(\xi_-,t) = u(\xi_+,t) & t > 0, \\ u(0,t) = 0 & t > 0, \\ \frac{\partial u}{\partial x}(\pi,t) = 0 & t > 0, \\ u(t=0) = u^{(0)},\ \frac{\partial u}{\partial t}(t=0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi, t-\tau) = f^0(t-\tau) & 0 < t < \tau. \end{cases} \tag{1.22}$$

Lemma 1.3.9. *A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue associated with system (1.22) if and only if λ satisfies*

$$(\alpha_1 + \alpha_2 e^{-\lambda\tau})\cosh(\lambda(\xi - \pi))\sinh(\lambda\xi) + \cosh(\lambda\pi) = 0. \tag{1.23}$$

Proof: Setting $u(t,.) = e^{\lambda t}\varphi$, we see that u is solution of (1.22) if and only if φ satisfies

$$\begin{cases} \lambda^2\varphi(x) - \frac{\partial^2\varphi}{\partial x^2}(x) = 0 & 0 < x < \pi, \\ \frac{\partial\varphi}{\partial x}(\xi_-) - \frac{\partial\varphi}{\partial x}(\xi_+) = -(\alpha_1 + \alpha_2 e^{-\lambda\tau})\lambda\varphi(\xi), \\ \varphi(\xi_-) = \varphi(\xi_+), \\ \varphi(0) = 0, \\ \frac{\partial\varphi}{\partial x}(\pi) = 0. \end{cases}$$

We then obtain

$$\varphi(x) = \begin{cases} A\sinh(\lambda x) & \text{in } (0,\,\xi) \\ A_1\cosh(\lambda(x-\pi)) & \text{in } (\xi,\,\pi) \end{cases}$$

where A, A_1 are real constants. The continuity $\varphi(\xi_-) = \varphi(\xi_+)$ and $\frac{\partial\varphi}{\partial x}(\xi_-) - \frac{\partial\varphi}{\partial x}(\xi_+) = -(\alpha_1 + \alpha_2 e^{-\lambda\tau})\lambda\varphi(\xi)$ imply that

$$\begin{pmatrix} \sinh(\lambda\xi) & -\cosh(\lambda(\xi-\pi)) \\ \cosh(\lambda\xi) + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh(\lambda\xi) & -\sinh(\lambda(\xi-\pi)) \end{pmatrix} \begin{pmatrix} A \\ A_1 \end{pmatrix} = 0.$$

Therefore a non trivial solution exists if and only if

$$\det\begin{pmatrix} \sinh(\lambda\xi) & -\cosh(\lambda(\xi-\pi)) \\ \cosh(\lambda\xi) + (\alpha_1 + \alpha_2 e^{-\lambda\tau})\sinh(\lambda\xi) & -\sinh(\lambda(\xi-\pi)) \end{pmatrix} = 0,$$

and simple calculations lead to the characteristic equation (1.23). ∎

The characteristic equation (1.23) is equivalent to

$$\begin{aligned} \Delta(\lambda) :=& (\alpha_1 + \alpha_2 e^{-\lambda\tau})e^{2\lambda(\xi-\pi)} - (\alpha_1 + \alpha_2 e^{-\lambda\tau} - 2)e^{-2\lambda\pi} \\ &- (\alpha_1 + \alpha_2 e^{-\lambda\tau})e^{-2\lambda\xi} + \alpha_2 e^{-\lambda\tau} + 2 + \alpha_1 = 0. \end{aligned}$$

Take an interior control ξ and a delay τ such that

$$\frac{\xi}{\pi} = \frac{2m+1}{2n+1}, \quad \tau = \frac{2(2k+1)}{2n+1}\pi$$

where $n,\, m,\, k \in \mathbb{Z}$. Now we look for λ in the form

$$\lambda = \eta + i\,\frac{2n+1}{2},\ \eta \in \mathbb{R}.$$

For such a λ, we have

$$\begin{aligned} \Delta(\lambda) &= (\alpha_1 - \alpha_2 e^{-\eta\tau})e^{2\eta(\xi-\pi)} + (\alpha_1 - \alpha_2 e^{-\eta\tau} - 2)e^{-2\eta\pi} \\ &\quad + (\alpha_1 - \alpha_2 e^{-\eta\tau})e^{-2\eta\xi} - \alpha_2 e^{-\eta\tau} + 2 + \alpha_1 \\ &=: \Delta_0(\eta). \end{aligned}$$

If we suppose that

$$\alpha_2 \geq \alpha_1 \geq 0.$$

Then

$$\Delta_0(0) = (\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_2 - 2) + (\alpha_1 - \alpha_2) - \alpha_2 + 2 + \alpha_1 = 4(\alpha_1 - \alpha_2) \leq 0$$

and

$$\lim_{\eta \longrightarrow +\infty} \Delta_0(\eta) = \alpha_1 + 2 > 0.$$

By the mean value theorem, there exists $\eta \geq 0$ such that $\Delta_0(\eta) = 0$. Therefore $\lambda = \eta + \frac{2n+1}{2} i$, $n \in \mathbb{Z}$, is an eigenvalue of system (1.22) with $Re(\lambda) = \eta \geq 0$. The system is then unstable for the countable set of delays τ and of control points ξ in the above form.

1.4 A regularity result and an a priori estimate

We consider the following problem with non-homogeneous transmission conditions

$$\begin{cases} \frac{\partial^2 w_j}{\partial t^2}(x,\, t) - \frac{\partial^2 w_j}{\partial x^2}(x,\, t) = 0 & 0 < x_j < l_j,\, t > 0,\, \forall j \in \{1, ..., N\} \\ w_j(v,\, t) = w_l(v,\, t) = w(v, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial w_j}{\partial n_j}(v,\, t) = k_v(t) & \forall v \in \mathcal{V}_c, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial w_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}_{int}^c, t > 0, \\ w_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial w_{j_v}}{\partial n_{j_v}}(v, t) = 0 & \forall v \in \mathcal{N}, t > 0, \\ w(t=0) = 0,\, \frac{\partial w}{\partial t}(t=0) = 0 & \end{cases} \tag{1.24}$$

This system modelizes the vibrations of a network of strings with local forces at the nodes of $\mathcal{V}_c$. The next proposition gives existence and regularity results for the solution of problem (1.24).

Proposition 1.4.1. *Let $T > 0$ be fixed. Suppose that $k_v \in L^2(0, T)$ for all $v \in \mathcal{V}_c$. Then the problem (1.24) admits a unique solution $w \in \prod_{j=1}^{N} H^1((0,\, l_j) \times (0,\, T))$. Moreover $w(v,\, .) \in H^1(0, T)$ for all $v \in \mathcal{V}_c$ and*

$$\sum_{v \in \mathcal{V}_c} \|w(v,\, .)\|_{H^1(0,T)} \lesssim \sum_{v \in \mathcal{V}_c} \|k_v\|_{L^2(0,T)} \quad \forall v \in \mathcal{V}_c. \tag{1.25}$$

The proof of this proposition is relatively technical and requires some preliminary results.

We first consider the following problem

$$\begin{cases} w_{tt}(x,\, t) - w_{xx}(x,\, t) = 0 & \text{in } (0,\, 1) \times (0,\, T), \\ w(1,\, t) = 0 & \text{on } (0,\, T), \\ w_x(0,\, t) = k(t) & \text{on } (0,\, T), \\ w(x,\, 0) = 0,\;\; w_t(x,\, 0) = 0 & \text{on } (0,\, 1). \end{cases} \tag{1.26}$$

Lemma 1.4.2. *Assume that $k \in L^2(0, T)$. Then problem (1.26) has a unique solution $w \in H^1((0, 1) \times (0, T))$ which satisfies*

$$\|w\|_{H^1((0,1)\times(0,T))} \lesssim \|k\|_{L^2(0,T)}.$$

Moreover $w(0, .) \in H^1(0, T)$ and satisfies

$$\|w(0, .)\|_{H^1(0,T)} \lesssim \|k\|_{L^2(0,T)}.$$

Proof: We extend k by 0 on $\mathbb{R} \setminus [0, T]$ because (1.26) is reversible in time.

Let $\hat{w}(x, \lambda)$ where $\lambda = \gamma + i\eta$, $\gamma > 0$, $\eta \in \mathbb{R}$, be the Laplace transform of w with respect to t. Then $\hat{w}$ satisfies

$$\begin{cases} \lambda^2 \hat{w}(x, \lambda) - \frac{\partial^2 \hat{w}}{\partial x^2}(x, \lambda) = 0 \quad \text{in } (0, 1), \\ \hat{w}(1, \lambda) = 0, \\ \frac{\partial \hat{w}}{\partial x}(0, \lambda) = \hat{k}(\lambda), \end{cases}$$

where $\Re\lambda > 0$. Consequently $\hat{w}(x, \lambda) = a\cosh(\lambda(x-1)) + b\sinh(\lambda(x-1))$, with two complex numbers a and b.
Since $\hat{w}(1, \lambda) = a = 0$ and $\frac{\partial \hat{w}}{\partial x}(0, \lambda) = b\lambda\cosh(\lambda) = \hat{k}(\lambda)$, we deduce that

$$\hat{w}(x, \lambda) = \frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)}\hat{k}(\lambda).$$

because $\cosh(\lambda) \neq 0$.

• Existence of w : $w(x, t) = \mathcal{L}^{-1}(\frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)}) \star k$ where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

• Uniqueness of w : if w_1, w_2 are two solutions of (1.26), then $w = w_1 - w_2$ satisfies the wave equation in $(0, 1)$ with homogeneous boundary and initial conditions. Therefore $w_1 - w_2 = 0$, which proves the uniqueness of the solution of (1.26).

• Regularity : Let $\gamma > 0$ be fixed and set $C_\gamma := \{\lambda \in \mathbb{C}\,;\, \Re\lambda = \gamma\}$. Define $H(x, \lambda) = \frac{\sinh(\lambda(x-1))}{\cosh(\lambda)}$, for $\lambda \in C_\gamma$. We clearly have

$$\|H\|_{L^\infty((0,1)\times C_\gamma)} \leq \coth(\gamma).$$

Therefore

$$\|\hat{w}(x, \lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} = \left\|\frac{H(x, \lambda)}{\lambda}\hat{k}(\lambda)\right\|_{L^2((0,1)\times\mathbb{R}_\eta)} \leq \frac{\coth(\gamma)}{\gamma}\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}, \tag{1.27}$$

which implies that $w \in L^2((0, 1) \times (0, T))$ with

$$\|w\|_{L^2((0,1)\times(0,T))} \lesssim \|k\|_{L^2(0,T)}. \tag{1.28}$$

Indeed, we have

$$\begin{aligned} \hat{w}(x, \lambda) &= \int_0^\infty e^{-(\gamma+i\eta)t} w(x, t)dt = \int_{-\infty}^{+\infty} e^{-\gamma t} e^{-i\eta t} w(x, t)dt \\ &= \mathcal{F}(e^{-\gamma\cdot} w)(\eta) = \mathcal{F}(w_1)(\eta) \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform and where we have set $w_1 = e^{-\gamma \cdot} w$. Therefore

$$\begin{array}{rcll} \|w\|_{L^2((0,1)\times(0,T))} & \sim & \|w_1\|_{L^2((0,1)\times(0,T))} & \text{as } e^{-\gamma T} \leq e^{-\gamma t} \leq 1 \text{ on } (0,T) \\ & \leq & \|w_1\|_{L^2((0,1)\times\mathbb{R})} & \\ & \sim & \|\mathcal{F}(w_1)(\eta)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula} \\ & \sim & \|\hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \end{array}$$

while

$$\begin{array}{rcl} \|k\|_{L^2(0,T)} & = & \|k\|_{L^2(\mathbb{R})} \sim \|ke^{-\gamma \cdot}\|_{L^2(\mathbb{R})} \\ & \sim & \|\mathcal{F}(e^{-\gamma \cdot} k)(\eta)\|_{L^2(\mathbb{R}_\eta)} \\ & = & \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}. \end{array}$$

These two equivalences and the estimate (1.27) lead to (1.28).

Since

$$\|\lambda \hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} \leq \coth(\gamma) \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

we deduce that $w \in H^1(0,T; L^2(0,1))$ and

$$\|w\|_{H^1(0,T;L^2(0,1))} \lesssim \|k\|_{L^2(0,T)}.$$

Indeed

$$\begin{array}{rcll} \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & \leq & \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times\mathbb{R})} & \\ & \sim & \left\|\mathcal{F}(\frac{\partial w_1}{\partial t})(\eta)\right\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula} \\ & \sim & \|(i\eta)\mathcal{F}(w_1)(\eta)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \\ & \sim & \|(i\eta)\hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} & \end{array}$$

and

$$\begin{array}{rcl} \|\lambda \hat{w}(x,\lambda)\|^2_{L^2((0,1)\times\mathbb{R}_\eta)} & = & \int_0^1 \int_{\mathbb{R}_\eta} |\lambda|^2 \hat{w}(x,\lambda)^2 d\lambda dx \\ & = & \int_0^1 \int_{\mathbb{R}_\eta} (\gamma^2+\eta^2)\hat{w}(x,\lambda)^2 d\lambda dx \\ & \geq & \|(i\eta)\hat{w}(x,\lambda)\|^2_{L^2((0,1)\times\mathbb{R}_\eta)}. \end{array}$$

Therefore

$$\begin{array}{rcl} \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & \lesssim & \|\lambda\hat{w}(x,\lambda)\|_{L^2((0,1)\times\mathbb{R}_\eta)} \\ & \lesssim & \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)} \lesssim \|k\|_{L^2(0,T)}. \end{array}$$

We finally conclude that

$$\begin{array}{rcl} \left\|\frac{\partial w}{\partial t}\right\|_{L^2((0,1)\times(0,T))} & = & \left\|e^{\gamma \cdot}\frac{\partial w_1}{\partial t} + \gamma w\right\|_{L^2((0,1)\times(0,T))} \\ & \lesssim & \left\|\frac{\partial w_1}{\partial t}\right\|_{L^2((0,1)\times(0,T))} + \|w\|_{L^2((0,1)\times(0,T))} \\ & \lesssim & \|k\|_{L^2(0,T)}. \end{array}$$

In a similar manner we have

$$\|\lambda\hat{w}(0,\lambda)\|_{L^2(\mathbb{R}_\eta)} \leq \coth(\gamma) \left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

which implies that $w(0,\,.) \in H^1(0,\,T)$ and satisfies

$$\|w(0,\,.)\|_{H^1(0,\,T)} \lesssim \|k\|_{L^2(0,\,T)}\,.$$

Finally $\frac{\partial \hat{w}}{\partial x}(x,\,\lambda) = \frac{\cosh(\lambda(x-1))}{\cosh(\lambda)}\hat{k}(\lambda)$. But the standard estimate $|\cosh z| \leq \cosh(Re\, z),\ \forall z \in \mathbb{C}$ implies that

$$\left|\frac{\cosh(\lambda(x-1))}{\cosh(\lambda)}\right| \leq \frac{\cosh(\gamma(1-x))}{\sinh(\gamma)} \leq \frac{\cosh(\gamma)}{\sinh(\gamma)} = \coth\gamma\,;$$

therefore

$$\left\|\frac{\partial \hat{w}(x,\,\lambda)}{\partial x}\right\|_{L^2((0,\,1)\times\mathbb{R}_\eta)} \leq \coth(\gamma)\left\|\hat{k}(\lambda)\right\|_{L^2(\mathbb{R}_\eta)}$$

which leads to $\frac{\partial w}{\partial x} \in L^2((0,\,1)\times(0,\,T))$ with

$$\left\|\frac{\partial w}{\partial x}\right\|_{L^2((0,\,1)\times(0,\,T))} \lesssim \|k\|_{L^2(0,\,T)}\,.$$

Indeed

$$\begin{array}{lll} \left\|\frac{\partial w}{\partial x}\right\|_{L^2((0,\,1)\times(0,\,T))} & \sim \left\|\frac{\partial w_1}{\partial x}\right\|_{L^2((0,\,1)\times(0,\,T))} & \leq \left\|\frac{\partial w_1}{\partial x}\right\|_{L^2((0,\,1)\times\mathbb{R})} \\ & \sim \left\|\mathcal{F}(\frac{\partial w_1}{\partial x})(\eta)\right\|_{L^2((0,\,1)\times\mathbb{R}_\eta)} & \text{by Plancherel's formula} \\ & \sim \left\|\frac{\partial \hat{w}}{\partial x}\right\|_{L^2((0,\,1)\times\mathbb{R}_\eta)}\,. & \end{array}$$

It remains to check the initial conditions. We remark that

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{\sinh(\lambda(x-1))}{\lambda\cosh(\lambda)}\right) =& \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2n-1}\sin\left(\frac{(2n-1)\pi(x-1)}{2}\right)\sin\left(\frac{(2n-1)\pi t}{2}\right) \\ =& F(x,\,t).\end{aligned}$$

Note further that $F \in L^\infty([0,\,T]\,;\,L^2(0,\,1))$. Therefore

$$w(x,\,t) = \int_0^t F(x,\,t-s)k(s)ds.$$

Consequently we directly see that

$$w(x,\,0) = 0$$

(the trace having a meaning because $w \in H^1((0,\,1)\times(0,\,T))$). Moreover

$$\frac{\partial w}{\partial t}(x,\,t) = F(x,\,0)k(t) + \int_0^t \frac{\partial F}{\partial t}(x,\,t-s)k(s)ds = \int_0^t \frac{\partial F}{\partial t}(x,\,t-s)k(s)ds.$$

Consequently

$$\begin{array}{lll} \left\|\frac{\partial w}{\partial t}(.,\,t)\right\|_{\tilde{V}'} & = & \left\|\int_0^t \frac{\partial F}{\partial t}(x,\,t-s)k(s)ds\right\|_{\tilde{V}'} \\ & \leq & \int_0^t \left\|\frac{\partial F}{\partial t}(x,\,t-s)\right\|_{\tilde{V}'}|k(s)|\,ds, \end{array}$$

where $\tilde{V}'$ is the dual space of $\tilde{V}$ with $\tilde{V} = \{u \in H^1(0, 1);\, u(1) = 0\}$. But we may write

$$\begin{aligned}
\frac{\partial F}{\partial t}(x,\, t) &= \frac{4}{\pi}\frac{\pi}{2}\sum_{n=1}^{+\infty}(-1)^{n+1}\sin\left(\frac{(2n-1)\pi(x-1)}{2}\right)\cos\left(\frac{(2n-1)\pi t}{2}\right)\\
&= \sum_{n=1}^{+\infty} a_n(t)\sin\left(\frac{(2n-1)\pi(x-1)}{2}\right),
\end{aligned}$$

where $a_n(t) = 2(-1)^{n+1}\cos(\frac{(2n-1)\pi t}{2})$ and $\{\sin(\frac{(2n-1)\pi(x-1)}{2})\}_n$ is an orthogonal basis of $L^2(0, 1)$. This implies that

$$\left\|\frac{\partial F}{\partial t}(x,\, t)\right\|_{\tilde{V}'} \sim \sum_{n=1}^{+\infty} a_n^2(t) n^{-2},$$

with $a_n^2(t) \leq 4$. Therefore

$$\left\|\frac{\partial F}{\partial t}(x,\, t)\right\|_{\tilde{V}'} \lesssim 1$$

and consequently

$$\begin{aligned}
\|\tfrac{\partial w}{\partial t}(\cdot,\, t)\|_{\tilde{V}'} &\lesssim \textstyle\int_0^t |k(s)|\, ds\\
&\lesssim (\textstyle\int_0^t ds)^{\frac{1}{2}}(\int_0^t |k(s)|^2\, ds)^{\frac{1}{2}}\\
&\lesssim \sqrt{t}\, \|k\|_{L^2(0,\,1)}\, .
\end{aligned}$$

This shows that

$$\frac{\partial w}{\partial t} \in C([0,\, T]\,;\, \tilde{V}')$$

and moreover

$$\frac{\partial w}{\partial t}(x,\, 0) = 0.$$

This ends the proof of the Lemma. ∎

We now make a local construction. Namely for a fixed $v \in \mathcal{V}_c$, we consider $\tilde{w}^{(v)}$ solution of

$$\left\{\begin{array}{ll}
\frac{\partial^2 \tilde{w}_j^{(v)}}{\partial t^2}(x,\, t) - \frac{\partial^2 \tilde{w}_j^{(v)}}{\partial x^2}(x,\, t) = 0 & 0 < x < l_v,\, t > 0,\, \forall j \in \mathcal{E}_v,\\
\tilde{w}_j^{(v)}(v,\, t) = \tilde{w}_l^{(v)}(v,\, t) & \forall j,\, l \in \mathcal{E}_v,\, t > 0,\\
\sum_{j \in \mathcal{E}_v} \frac{\partial \tilde{w}_j^{(v)}}{\partial n_j}(v,\, t) = k_v(t) & t > 0,\\
\tilde{w}_j^{(v)}(l_v, t) = 0 & \forall j \in \mathcal{E}_v,\, t > 0,\\
\tilde{w}^{(v)}(t=0) = 0,\, \frac{\partial \tilde{w}^{(v)}}{\partial t}(t=0) = 0, &
\end{array}\right. \tag{1.29}$$

where $l_v = \min_{j \in \mathcal{E}_v} l_j$ (without loss of generality we may identify v with the extremity 0 for all edges of $\mathcal{E}_v$).

The unique solution of this system is simply $\tilde{w}_j^{(v)}(x,t) = w(x/l_v, t/l_v)$, where w is solution of problem (1.26) with $k(t) = -\frac{l_v}{E_v}k_v(l_v t)$, when E_v is the cardinal of $\mathcal{E}_v$. By Lemma 1.4.2, we directly obtain the

Lemma 1.4.3. *The system (1.29) admits a unique solution $\tilde{w}_j^{(v)} \in H^1((0,\, l_v) \times (0,\, T))$, $j \in \mathcal{E}_v$ such that*

$$\left\|\tilde{w}_j^{(v)}\right\|_{H^1((0,\, l_v)\times(0,\, T))} \lesssim \|k_v\|_{L^2(0,T)} .$$

Moreover $\tilde{w}_j^{(v)}(v,\, .) \in H^1(0,\, T)$ with the estimate

$$\left\|\tilde{w}_j^{(v)}(v,\, .)\right\|_{H^1(0,T)} \lesssim \|k_v\|_{L^2(0,T)} .$$

Let us now set (assuming for the moment that w exists)

$$\omega(x,\, t) := w(x,t) - \sum_{v\in\mathcal{V}_c} \eta^{(v)}(x)\tilde{w}^{(v)}(x,\, t)$$

where $\tilde{w}^{(v)}$ is solution of problem (1.29) and $\eta^{(v)}$ is a cut-off function such that supp $\eta^{(v)} = \mathcal{E}_v$, $\eta^{(v)}$ is equal to 1 in a neighborhood of v and is 0 outside a larger neighborhood of v. Then we easily see that ω is solution of the following system

$$\left\{\begin{array}{ll}
\frac{\partial^2 \omega_j}{\partial t^2}(x,\, t) - \frac{\partial^2 \omega_j}{\partial x^2}(x,\, t) = h_j(x,\, t) & 0 < x_j < l_j,\, t > 0,\, \forall j \in \{1, ..., N\}, \\
\omega_j(v,\, t) = \omega_l(v,\, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int},\, t > 0, \\
\sum_{j\in\mathcal{E}_v} \frac{\partial \omega_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_c \cup \mathcal{V}_{int}^c,\, t > 0, \\
\omega_{j_v}(v,\, t) = 0 & \forall v \in \mathcal{D},\, t > 0, \\
\frac{\partial \omega_{j_v}}{\partial n_{j_v}}(v,\, t) = 0 & \forall v \in \mathcal{N},\, t > 0, \\
\omega(t = 0) = 0,\ \frac{\partial \omega}{\partial t}(t = 0) = 0, &
\end{array}\right. \tag{1.30}$$

where h_j is given by

$$h_j(x,\, t) := \sum_{v\in\mathcal{V}_c\,:\, j\in\mathcal{E}_v} \left(\frac{\partial^2 \eta_j^{(v)}}{\partial x^2}(x)\tilde{w}_j{}^{(v)}(x,\, t) + 2\frac{\partial \eta_j^{(v)}}{\partial x}(x)\frac{\partial \tilde{w}_j^{(v)}}{\partial x}(x,\, t) \right).$$

We have then transformed the system with nonhomogeneous transmission conditions to a system with nonhomogeneous right-hand sides. This system can be written in the form

$$\frac{\partial^2 \omega}{\partial t^2} + A\omega = h,$$

where the operator A was defined before. Since A is a positive selfadjoint operator on $L^2(\mathcal{R})$, we directly obtain the

Lemma 1.4.4. *The solution ω of (1.30) has the regularity*

$$\omega \in C(0,\, T;\, V) \cap C^1(0,\, T;\, L^2(\mathcal{R}))$$

and is given by the so-called constant variation formula

$$\omega(\cdot,\, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k} \frac{1}{\lambda_k} \left(\int_0^t \sin((t-s)\lambda_k)(h(\cdot,\, s),\, \varphi_{k,i})_{L^2(\mathcal{R})} ds \right) \varphi_{k,i}.$$

Lemmas 1.4.3 and 1.4.4 guarantee the existence of a unique solution $w \in \prod_{j=1}^{N} H^1((0,\, l_j) \times (0,\, T))$ of problem (1.24) given by

$$w(x,\, t) = \omega(x,\, t) + \sum_{v\in\mathcal{V}_c} \eta^{(v)}(x)\tilde{w}^{(v)}(x,\, t),$$

and satisfying

$$\sum_{j=1}^{N} \|w_j\|_{H^1((0,l_j)\times(0,T))} \lesssim \sum_{v\in\mathcal{V}_c} \|k_v\|_{L^2(0,T)}. \tag{1.31}$$

It remains to show the regularity at the nodes of $\mathcal{V}_c$. Fix one vertex $v \in \mathcal{V}_c$ and a cut-off function $\chi^{(v)}$ such that

$$\chi_j^{(v)} \equiv 1 \text{ on } [0, l_v/3], \quad \chi_j^{(v)} \equiv 0 \text{ on } [2l_v/3, l_j], \forall j \in \mathcal{E}_v,$$

where we have identified v to 0. Let us now set

$$W = \chi^{(v)} w.$$

This function W is solution of the following wave equation :

$$\left\{ \begin{array}{ll} \frac{\partial^2 W_j}{\partial t^2}(x,\, t) - \frac{\partial^2 W_j}{\partial x^2}(x,\, t) = \tilde{h}_j(x, t) & 0 < x < l_v,\, t > 0, \forall j \in \mathcal{E}_v, \\ W_j(0,\, t) = W_l(0,\, t) & \forall j,\, l \in \mathcal{E}_v,\, t > 0, \\ \sum_{j\in\mathcal{E}_v} \frac{\partial W_j}{\partial x}(0,\, t) = k_v(t) & t > 0, \\ W_j(l_v,\, t) = 0 & \forall j \in \mathcal{E}_v,\, t > 0, \\ W(t=0) = 0,\, \frac{\partial W}{\partial t}(t=0) = 0, & \end{array} \right. \tag{1.32}$$

where $\tilde{h}_j$ is given by

$$\tilde{h}_j(x,\, t) := -\sum_{v\in\mathcal{V}_c : j\in\mathcal{E}_v} \left(\frac{\partial^2 \chi_j^{(v)}}{\partial x^2}(x) w_j(x,\, t) + 2\frac{\partial \chi_j^{(v)}}{\partial x}(x) \frac{\partial w_j}{\partial x}(x,\, t) \right).$$

According to the estimate (1.31) satisfied by w, we have

$$\|\tilde{h}_j\|_{L^2((0,l_v)\times(0,T))} \lesssim \sum_{v'\in\mathcal{V}_c} \|k_{v'}\|_{L^2(0,T)}.$$

Recalling that E_v is the cardinal of $\mathcal{E}_v$, we then may write

$$\mathcal{E}_v = \{j_i\}_{i=1,\cdots,E_v}.$$

Introduce

$$\begin{aligned} V_1 &= \sum_{i=1}^{E_v} W_{j_i}, \\ V_i &= W_{j_i} - W_{j_1}, \forall i = 2, \cdots, E_v. \end{aligned}$$

We remark that $V_i, i = 2, \cdots, E_v$ is solution of the wave equation with Dirichlet boundary condition at 0 and l_v, while V_1 is solution of the wave equation with Dirichlet boundary condition at l_v and Neumann boundary condition at 0, namely

$$\left\{\begin{array}{ll} \frac{\partial^2 V_1}{\partial t^2}(x,\,t) - \frac{\partial^2 V_1}{\partial x^2}(x,\,t) = g(x,t) & 0 < x < l_v,\, t > 0, \\ \frac{\partial V_1}{\partial x}(0,\,t) = k_v(t) & t > 0, \\ V_1(l_v,\,t) = 0 & t > 0, \\ V_1(t=0) = 0,\ \frac{\partial V_1}{\partial t}(t=0) = 0, & \end{array}\right. \tag{1.33}$$

where $g = \sum_{j\in\mathcal{E}_v} \tilde{h}_j$ and then satisfies

$$\|g\|_{L^2((0,l_v)\times(0,T))} \lesssim \sum_{v'\in\mathcal{V}_c} \|k_{v'}\|_{L^2(0,T)}. \tag{1.34}$$

But Lemma 1.4.5 below shows that

$$\|V_1(0,\cdot)\|_{H^1(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}.$$

Since

$$\begin{aligned} W_{j_1} &= \frac{1}{E_v}(V_1 - \sum_{i=2}^{E_v} V_i), \\ W_{j_i} &= V_i + W_{j_1}, \forall i = 2, \cdots, E_v, \end{aligned}$$

we conclude that

$$\|W_{j_i}(0,\cdot)\|_{H^1(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}, \forall i = 1, \cdots, E_v.$$

Using (1.34), we obtain the estimate (1.25) from Proposition 1.4.1.

Lemma 1.4.5. *Let $V_1 \in H^1((0,l_v)\times(0,T))$ be a solution of (1.33) with $g \in L^2((0,l_v)\times(0,T))$ and $k_v \in L^2(0,T)$. Then*

$$\|\frac{\partial V_1}{\partial t}(0,\cdot)\|_{L^2(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}.$$

Proof: Let us denote by V the solution of

$$\left\{\begin{array}{ll} \frac{\partial^2 V}{\partial t^2}(x,\,t) - \frac{\partial^2 V}{\partial x^2}(x,\,t) = \tilde{g}(x,t) & 0 < x < l_v,\, t > 0, \\ \frac{\partial V}{\partial x}(0,\,t) = \tilde{k}_v(t) & t > 0, \\ V(l_v,t) = 0 & t > 0, \\ V(\cdot,t=0) = 0,\ \frac{\partial V}{\partial t}(\cdot,t=0) = 0, & \end{array}\right. \tag{1.35}$$

where $\tilde{g}$ (resp. $\tilde{k}_v$) means the extension of g (resp. k_v) by zero outside $(0,T)$. Since $V = V_1$ on $(0,l_v)\times(0,T)$, it suffices to show that

$$\|\frac{\partial V}{\partial t}(0,\cdot)\|_{L^2(0,T)} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)}. \tag{1.36}$$

For that purpose we use a multiplier technique. Namely we multiply the wave equation satisfied by V by $(l_v - x)(2T - t)V_x(x,t)$ (here and below for shortness we write $V_x = \frac{\partial V}{\partial x}$) and integrate the result on $(0, l_v) \times (0, 2T)$. This yields

$$\int_0^{l_v} \int_0^{2T} \tilde{g}(x,t)(l_v - x)(2T - t)V_x(x,t)\, dxdt = I_1 - I_2, \tag{1.37}$$

where

$$\begin{aligned} I_1 &= \int_0^{l_v} \int_0^{2T} V_{tt}(l_v - x)(2T - t)V_x\, dxdt, \\ I_2 &= \int_0^{l_v} \int_0^{2T} V_{xx}(l_v - x)(2T - t)V_x\, dxdt. \end{aligned}$$

For the first term I_1 an integration by parts in time yields (no boundary terms occur because $V_t(t=0) = 0$)

$$I_1 = \int_0^{l_v} \int_0^{2T} V_t(l_v - x)V_x(x,t)\, dxdt - I_3, \tag{1.38}$$

where

$$I_3 = \int_0^{l_v} \int_0^{2T} V_t(l_v - x)(2T - t)V_{xt}(x,t)\, dxdt.$$

For I_3, an integration by parts in space and Leibniz's rule lead to

$$\begin{aligned} I_3 &= -\int_0^{l_v} \int_0^{2T} \frac{\partial}{\partial x}(V_t(l_v - x))(2T - t)V_t\, dxdt \\ &\quad + \left[\int_0^{2T} (l_v - x))(2T - t)|V_t|^2\, dt\right]_0^{l_v} \\ &= \int_0^{l_v} \int_0^{2T} |V_t|^2(2T - t)\, dxdt - I_3 \\ &\quad - \int_0^{2T} (2T - t)l_v|V_t(0,t)|^2\, dt. \end{aligned}$$

This shows that

$$I_3 = \frac{1}{2}\int_0^{l_v} \int_0^{2T} |V_t|^2(2T - t)\, dxdt - \frac{l_v}{2}\int_0^{2T} (2T - t)|V_t(0,t)|^2\, dt.$$

Inserting this expression in (1.38) we find that

$$\begin{aligned} I_1 &= \int_0^{l_v} \int_0^{2T} V_t(l_v - x)V_x(x,t)\, dxdt \\ &\quad -\frac{1}{2}\int_0^{l_v} \int_0^{2T} |V_t|^2(2T - t)\, dxdt \\ &\quad +\frac{l_v}{2}\int_0^{2T} (2T - t)|V_t(0,t)|^2\, dt. \end{aligned} \tag{1.39}$$

Similarly for the second term I_2 an integration by parts in space yields

$$\begin{aligned}
I_2 &= -\int_0^{l_v}\int_0^{2T} V_x(2T-t)\frac{\partial}{\partial x}((l_v-x)V_x)\,dxdt \\
&\quad + \left[\int_0^{2T}(2T-t)(l_v-x)|V_x|^2\,dt\right]_0^{l_v} \\
&= -I_2 + \int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt \\
&\quad -l_v\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt.
\end{aligned}$$

This means that

$$\begin{aligned}
I_2 &= \frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt \\
&\quad -\frac{l_v}{2}\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt.
\end{aligned} \tag{1.40}$$

Inserting (1.39) and (1.40) into the identity (1.37), we have obtained that

$$\begin{aligned}
\int_0^{l_v}\int_0^{2T}\tilde{g}(x,t)(l_v-x)(2T-t)V_x(x,t)\,dxdt &= \int_0^{l_v}\int_0^{2T}V_t(l_v-x)V_x(x,t)\,dxdt \\
&\quad -\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_t|^2(2T-t)\,dxdt \\
&\quad +\frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt. \\
&\quad -\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt \\
&\quad +\frac{l_v}{2}\int_0^{2T}(2T-t)|V_x(0,t)|^2\,dt.
\end{aligned}$$

Reminding the Neumann boundary condition satisfied by V we get

$$\begin{aligned}
\frac{l_v}{2}\int_0^{2T}(2T-t)|V_t(0,t)|^2\,dt &= \int_0^{l_v}\int_0^{2T}\tilde{g}(x,t)(l_v-x)(2T-t)V_x(x,t)\,dxdt \\
&\quad -\int_0^{l_v}\int_0^{2T}V_t(l_v-x)V_x(x,t)\,dxdt \\
&\quad +\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_t|^2(2T-t)\,dxdt \\
&\quad +\frac{1}{2}\int_0^{l_v}\int_0^{2T}|V_x|^2(2T-t)\,dxdt \\
&\quad -\frac{l_v}{2}\int_0^{T}(2T-t)|k_v(t)|^2\,dt.
\end{aligned}$$

By Cauchy-Schwarz's inequality we obtain finally

$$\int_0^{2T} (2T-t)|V_t(0,t)|^2\,dt \lesssim \|g\|^2_{L^2((0,l_v)\times(0,T))} + \|V\|^2_{H^1((0,l_v)\times(0,2T))}.$$

This leads to the conclusion due to the estimate (1.34) and the estimate (consequence of our previous considerations, see (1.31))

$$\|V\|_{H^1((0,l_v)\times(0,2T))} \lesssim \|g\|_{L^2((0,l_v)\times(0,T))} + \|k_v\|_{L^2(0,T)},$$

and since $2T-t \geq T$ on $(0,T)$. ■

Now we prove an a priori estimate which uses the trace regularity result of Proposition 1.4.1 and that will be useful to prove our stability results for problem (1.1).

Let $u \in C(0,T;V) \cap C^1(0,T;L^2(\mathcal{R}))$ be the solution of (1.1). Then it can be splitted up in the form

$$u = \phi + \psi$$

where ϕ is solution of problem without damping (1.14), and ψ satisfies

$$\left\{\begin{array}{ll}
\frac{\partial^2\psi_j}{\partial t^2}(x,t) - \frac{\partial^2\psi_j}{\partial x^2}(x,t) = 0 & 0<x<l_j,\ \ t>0, \forall j \in \{1,...,N\},\\
\psi_j(v,t) = \psi_l(v,t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int},\, t>0,\\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial\psi_j}{\partial n_j}(v,t) = -(\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,t) + \alpha_2^{(v)}\frac{\partial u}{\partial t}(v,t-\tau_v)) & \forall v \in \mathcal{V}_c,\, t>0,\\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial\psi_j}{\partial n_j}(v,t) = 0 & \forall v \in \mathcal{V}_{int}\backslash\mathcal{V}_{int}^c,\, t>0,\\
\psi_{j_v}(v,t) = 0 & \forall v \in \mathcal{D},\, t>0,\\
\frac{\partial\psi_{j_v}}{\partial n_{j_v}}(v,t) = 0 & \forall v \in \mathcal{N},\, t>0,\\
\psi(t=0) = 0,\ \frac{\partial\psi}{\partial t}(t=0) = 0. &
\end{array}\right. \tag{1.41}$$

In other words ψ is solution of problem (1.24) with

$$k_v(t) = -\left(\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,t) + \alpha_2^{(v)}\frac{\partial u}{\partial t}(v,t-\tau_v)\right). \tag{1.42}$$

For all $v \in \mathcal{V}_c$, we may write

$$\left\|\frac{\partial\phi}{\partial t}(v,.)\right\|_{L^2(0,T)} \leq \left\|\frac{\partial u}{\partial t}(v,.)\right\|_{L^2(0,T)} + \left\|\frac{\partial\psi}{\partial t}(v,.)\right\|_{L^2(0,T)}.$$

Now by Remark 1.3.2, $k_v = -(\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\cdot) + \alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\cdot-\tau_v)) \in L^2(0,T)$ for all $v \in \mathcal{V}_c$. Then we can apply Proposition 1.4.1 to ψ and obtain

$$\begin{aligned}
\sum_{v\in\mathcal{V}_c} \|\tfrac{\partial\psi}{\partial t}(v,.)\|_{L^2(0,T)} &\lesssim \sum_{v\in\mathcal{V}_c} \left\|\alpha_1^{(v)}\tfrac{\partial u}{\partial t}(v,.) + \alpha_2^{(v)}\tfrac{\partial u}{\partial t}(v,.-\tau_v)\right\|_{L^2(0,T)}\\
&\lesssim \sum_{v\in\mathcal{V}_c} (\|\tfrac{\partial u}{\partial t}(v,.)\|_{L^2(0,T)} + \|\tfrac{\partial u}{\partial t}(v,.-\tau_v)\|_{L^2(0,T)}).
\end{aligned}$$

The two above estimates yield

$$\begin{aligned}\sum_{v\in\mathcal{V}_c}\|\tfrac{\partial\phi}{\partial t}(v,\,.)\|_{L^2(0,T)} &\le \sum_{v\in\mathcal{V}_c}\|\tfrac{\partial u}{\partial t}(v,\,.)\|_{L^2(0,T)}+\sum_{v\in\mathcal{V}_c}\|\tfrac{\partial\psi}{\partial t}(v,\,.)\|_{L^2(0,T)}\\ &\lesssim \sum_{v\in\mathcal{V}_c}(\|\tfrac{\partial u}{\partial t}(v,\,.)\|_{L^2(0,T)}+\|\tfrac{\partial u}{\partial t}(v,\,.-\tau_v)\|_{L^2(0,T)}).\end{aligned}$$

This directly leads to the next a priori bound :

Lemma 1.4.6. *Suppose that $(u^{(0)},\,u^{(1)},\,(f^0(-\tau_v\cdot))_{v\in\mathcal{V}_c})\in H$. Then the solutions u of (1.1) with initial data $(u^{(0)},\,u^{(1)},\,(f^0(-\tau_v\cdot))_{v\in\mathcal{V}_c})$ and ϕ of (1.14) with initial data $(u^{(0)},\,u^{(1)})$ (which belongs to $V\times L^2(\mathcal{R})$) satisfy*

$$\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial\phi}{\partial t}(v,\,t)\right)^2dt\lesssim\sum_{v\in\mathcal{V}_c}\int_0^T\left((\frac{\partial u}{\partial t}(v,\,t))^2+(\frac{\partial u}{\partial t}(v,\,t-\tau_v))^2\right)dt.$$

1.5 The exponential stability

Our approach is based (as for the polynomial stability) on a trace regularity result (Proposition 1.4.1 and Lemma 1.4.6) and on an observability inequality of problem without damping (1.14).

1.5.1 An observability inequality

Proposition 1.5.1. *Let $(\varphi_{k,i})_{1\le i\le l_k,k\ge 1}$ be the orthonormal basis of the operator A. Let ϕ be the solution of (1.14) with $(u^{(0)},\,u^{(1)})\in V\times L^2(\mathcal{R})$. Then there exists a time $T>0$ and a constant $C>0$ (depending on T) such that*

$$\|u^{(0)}\|_V^2+\|u^{(1)}\|_{L^2(\mathcal{R})}^2\le C\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial\phi}{\partial t}(v,\,t)\right)^2dt \tag{1.43}$$

if and only if

$$\exists\alpha>0,\,\forall k=1,\cdots,N+1,\,\forall n\in A_k,\,\lambda_{min}(\tilde{\mathcal{M}}^n)\ge\alpha, \tag{1.44}$$

where the matrix $\tilde{\mathcal{M}}^n$ is defined by

$$\tilde{\mathcal{M}}^n=\sum_{v\in\mathcal{V}_c}\Phi_n(v)^\top B_n^{-\top}B_n^{-1}\Phi_n(v),$$

the matrix $\Phi_n(v)$ of size $k\times L_n$, where $L_n=\sum_{i=1}^k l_{n+i-1}$, is given as follows : for all $i=1,\cdots,k$, we set

$$(\Phi_n(v))_{ij}=\begin{cases}\varphi_{n+i-1,j-L_{n,i-1}}(v) & \text{if } L_{n,i-1}<j\le L_{n,i},\\ 0 & \text{else,}\end{cases}$$

where $L_{n,0}=0$ and for $i\ge 1$, $L_{n,i}=\sum_{i'=1}^{i}l_{n+i'-1}$.

Proof: We first show that (1.44)$\Rightarrow$(1.43). Writting $u^{(0)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}$ and $u^{(1)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}$ where $(\lambda_k a_{k,i})_{i,k}$, $(b_{k,i})_{i,k} \in l^2(\mathbb{N}^*)$, then the solution ϕ of problem (1.14) is given by

$$\phi(\cdot, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k}\left(a_{k,i}\cos(\lambda_k t) + \frac{b_{k,i}}{\lambda_k}\sin(\lambda_k t)\right)\varphi_{k,i}.$$

Consequently for any $v \in \mathcal{V}_c$, we get

$$\frac{\partial \phi}{\partial t}(v, t) = \sum_{k\geq 1}\sum_{i=1}^{l_k}(-a_{k,i}\lambda_k\sin(\lambda_k t) + b_{k,i}\cos(\lambda_k t))\varphi_{k,i}(v).$$

Putting together the terms corresponding to the same eigenvalue, we obtain

$$\frac{\partial \phi}{\partial t}(v, t) = \sum_{k\geq 1}\left(\sum_{i=1}^{l_k} - a_{k,i}\varphi_{k,i}(v)\right)\lambda_k\sin(\lambda_k t) + \sum_{k\geq 1}\left(\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}(v)\right)\cos(\lambda_k t).$$

Using the notation introduced in Proposition 1.3.7, this is equivalent to

$$\frac{\partial \phi}{\partial t}(v, t) = \sum_{n\in\mathbb{Z}^*}\alpha_n(v)e^{i\lambda_n t}.$$

Integrating the square of this identity between 0 and $T > 0$ and using Ingham's inequality (1.17) for T large enough, and summing on $v \in \mathcal{V}_c$, we get

$$\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial \phi}{\partial t}(v, t)\right)^2 dt \gtrsim \sum_{k=1}^{N+1}\sum_{|n|\in A_k}\sum_{v\in\mathcal{V}_c}\|B_n^{-1}C_n(v)\|_2^2.$$

But for all $n \in A_k$, setting

$$\begin{aligned}
\tilde{A}_n &= \big(\lambda_n a_{n,1}, \cdots, \lambda_n a_{n,l_n}, \lambda_{n+1}a_{n+1,1}, \cdots, \lambda_{n+1}a_{n+1,l_{n+1}}, \cdots,\\
&\qquad \lambda_{n+k-1}a_{n+k-1,1}, \cdots, \lambda_{n+k-1}a_{n+k-1,l_{n+k-1}}\big)^\top,\\
\tilde{B}_n &= \big(b_{n,1}, \cdots, b_{n,l_n}, b_{n+1,1}, \cdots, b_{n+1,l_{n+1}}, \cdots, b_{n+k-1,1}, \cdots, b_{n+k-1,l_{n+k-1}}\big)^\top,
\end{aligned}$$

we readily check that

$$\sum_{v\in\mathcal{V}_c}\|B_n^{-1}C_n(v)\|_2^2 = \frac{1}{4}(\tilde{B}_n^\top\tilde{\mathcal{M}}^n\tilde{B}_n + \tilde{A}_n^\top\tilde{\mathcal{M}}^n\tilde{A}_n).$$

Hence the assumption (1.44) yields (because $\tilde{\mathcal{M}}^n$ is a symmetric matrix)

$$\sum_{v\in\mathcal{V}_c}\|B_n^{-1}C_n(v)\|_2^2 \gtrsim \|\tilde{B}_n\|_2^2 + \|\tilde{A}_n\|_2^2.$$

Therefore under our hypothesis, we have

$$\begin{aligned}\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial\phi}{\partial t}(v,t)\right)^2 dt &\gtrsim \sum_{k\geq 1}\sum_{i=1}^{l_k}(a_{k,i}^2\lambda_k^2+b_{k,i}^2)\\ &\gtrsim \left\|u^{(0)}\right\|_V^2+\left\|u^{(1)}\right\|_{L^2(\mathcal{R})}^2\end{aligned}$$

because $(\varphi_{k,i})_{k,i}$ is an orthonormal basis associated with the operator A.

It remains to show that (1.43)$\Rightarrow$(1.44).

Let $k = 1, \cdots, N+1$ and $n \in A_k$ be fixed. Take $u^{(0)} = \sum_{m=n}^{n+k-1}\sum_{i=1}^{l_m} a_{m,i}\varphi_{m,i}$ and $u^{(1)} = \sum_{m=n}^{n+k-1}\sum_{i=1}^{l_m} b_{m,i}\varphi_{m,i}$. Then the solution ϕ of problem (1.14) is given by

$$\phi(\cdot,t) = \sum_{m=n}^{n+k-1}\sum_{i=1}^{l_m}\left(a_{m,i}\cos(\lambda_m t)+\frac{b_{m,i}}{\lambda_m}\sin(\lambda_m t)\right)\varphi_{m,i}.$$

Then for $v \in \mathcal{V}_c$

$$\frac{\partial\phi}{\partial t}(v,t) = \sum_{m=n}^{n+k-1}\sum_{i=1}^{l_m}(-a_{m,i}\lambda_m\sin(\lambda_m t)+b_{m,i}\cos(\lambda_m t))\varphi_{m,i}(v).$$

Applying again Ingham's inequality, we get for T large enough

$$\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial\phi}{\partial t}(v,t)\right)^2 dt \sim \tilde{B}_n^\top\tilde{\mathcal{M}}^n\tilde{B}_n+\tilde{A}_n^\top\tilde{\mathcal{M}}^n\tilde{A}_n.$$

By (1.43), we obtain

$$\tilde{B}_n^\top\tilde{\mathcal{M}}^n\tilde{B}_n+\tilde{A}_n^\top\tilde{\mathcal{M}}^n\tilde{A}_n \geq C\sum_{m=n}^{n+k-1}\sum_{i=1}^{l_m}(a_{m,i}^2\lambda_n^2+b_{m,i}^2),$$

for some $C > 0$. Hence we conclude that

$$\lambda_{min}(\tilde{\mathcal{M}}^n) \geq C.$$

This ends the proof. ■

Remark 1.5.2. If the standard gap condition (1.18) holds, then the condition (1.44) reduces to

$$\exists\alpha > 0,\ \forall k \geq 1,\ \lambda_{min}(\mathcal{M}(\lambda_k^2)) \geq \alpha. \tag{1.45}$$

In particular if $l_k = 1$, then the condition (1.45) becomes

$$\forall k \geq 1,\ \sum_{v\in\mathcal{V}_c}|\varphi_k(v)|^2 \geq \alpha.$$

1.5.2 The stability result

We are now ready to give a sufficient condition that guarantees the exponential stability of (1.1).

Theorem 1.5.3. *The system (1.1) is exponentially stable in the energy space if (1.43) holds.*

Proof: Let u be a solution of (1.1) with initial data $(u^{(0)}, u^{(1)}, (f^0(-\tau_v\cdot))_{v\in\mathcal{V}_c}) \in D(\mathcal{A})$.

Integrating the inequality (1.13) of Proposition 1.3.1 between 0 and T where T is sufficiently large, we obtain

$$\begin{aligned}
E(0)-E(T) &\gtrsim \sum_{v\in\mathcal{V}_c}\int_0^T [(\frac{\partial u}{\partial t}(v,t))^2 + (\frac{\partial u}{\partial t}(v,t-\tau_v))^2]dt \\
&\gtrsim \frac{1}{2}\sum_{v\in\mathcal{V}_c}\int_0^T [(\frac{\partial u}{\partial t}(v,t))^2 + (\frac{\partial u}{\partial t}(v,t-\tau_v))^2]dt + \frac{1}{2}\sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial u}{\partial t}(v,t-\tau_v))^2dt \\
&\gtrsim \sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial \phi}{\partial t}(v,t))^2dt + \sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial u}{\partial t}(v,t-\tau_v))^2dt \text{ by Lemma 1.4.6} \\
&\gtrsim \|u^{(0)}\|_V^2 + \|u^{(1)}\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial u}{\partial t}(v,t-\tau_v))^2dt \\
&\qquad \text{by assumption} \\
&\gtrsim \|u^{(0)}\|_V^2 + \|u^{(1)}\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c}\tau_v\int_0^1 (\frac{\partial u}{\partial t}(v,-\tau_v\rho))^2d\rho.
\end{aligned}$$

Indeed, for $T > \tau_v$, by changes of variables, we have

$$\begin{aligned}
\int_0^T (\frac{\partial u}{\partial t}(v,t-\tau_v))^2dt &= \int_{-\tau_v}^{T-\tau_v} (\frac{\partial u}{\partial t}(v,t))^2dt \\
&\geq \int_{-\tau_v}^{0} (\frac{\partial u}{\partial t}(v,t))^2dt = \tau_v\int_0^1 (\frac{\partial u}{\partial t}(v,-\tau_v\rho))^2d\rho.
\end{aligned} \tag{1.46}$$

The previous inequality directly implies that

$$E(0)-E(T) \gtrsim \frac{1}{2}\left(\|u^{(0)}\|_V^2 + \|u^{(1)}\|_{L^2(\mathcal{R})}^2 + \sum_{v\in\mathcal{V}_c}\frac{\xi^{(v)}}{2}\int_0^1 (\frac{\partial u}{\partial t}(v,-\tau_v\rho))^2d\rho\right).$$

This means that for T large enough

$$E(0)-E(T) \geq CE(0)$$

for some $C > 0$. Since our system is invariant by translation and the energy is decreasing, it is well known that the above estimate implies the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$E(t) \leq C_1E(0)e^{-C_2t}, \forall t \geq 0. \tag{1.47}$$

Hence the energy decays exponentially. By density of $D(\mathcal{A})$ into H, we deduce that (1.47) holds for any initial data in H. ■

1.6 The polynomial stability

1.6.1 An observability estimate

Proposition 1.6.1. *Let $(\varphi_{k,i})_{1\leq i\leq l_k, k\geq 1}$ be an orthonormal basis of eigenvectors of the operator A. Let $m \in \mathbb{N}^*$. Let ϕ be the solution of (1.14) with $(u^{(0)}, u^{(1)}) \in V \times L^2(\mathcal{R})$. Then there exists a time $T > 0$ and a constant $C > 0$ such that*

$$\sum_{v\in\mathcal{V}_c} \int_0^T \left(\frac{\partial\phi}{\partial t}(v, t)\right)^2 dt \geq C\left(\sum_{k\geq 1}\frac{1}{k^{2m}}\sum_{i=1}^{l_k}(a_{k,i}^2\lambda_k^2 + b_{k,i}^2)\right) \tag{1.48}$$

where $u^{(0)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} a_{k,i}\varphi_{k,i}$ *and* $u^{(1)} = \sum_{k\geq 1}\sum_{i=1}^{l_k} b_{k,i}\varphi_{k,i}$ *if and only if*

$$\exists m \in \mathbb{N}^*,\ \exists\alpha > 0,\ \forall k = 1, \cdots, N+1,\ \forall n \in A_k,\ \lambda_{min}(\tilde{\mathcal{M}}^n) \geq \frac{\alpha}{n^{2m}}. \tag{1.49}$$

Proof: The proof is similar to the one of Proposition 1.5.1 and is therefore omitted. ∎

Remark 1.6.2. As before, if the standard gap condition (1.18) holds, then the condition (1.49) reduces to

$$\exists m \in \mathbb{N}^*,\ \exists\alpha > 0,\ \forall k \geq 1,\ \lambda_{min}(\mathcal{M}(\lambda_k^2)) \geq \frac{\alpha}{k^{2m}}, \tag{1.50}$$

and if $l_k = 1$, then condition (1.50) is nothing else than

$$\exists\alpha > 0,\ \forall k \geq 1,\ \sum_{v\in\mathcal{V}_c} |\varphi_k(v)|^2 \geq \frac{\alpha}{k^{2m}}.$$

Let $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$. By the so-called Weyl's formula (see for instance [20, 82]), we have

$$\lambda_k \sim \frac{k\pi}{L}$$

where $L = \sum_{j=1}^N l_j$. This implies that

$$\begin{aligned}\sum_{k\geq 1}\tfrac{1}{k^{2m}}\sum_{i=1}^{l_k}(a_{k,i}^2\lambda_k^2 + b_{k,i}^2) &\sim \sum_{k\geq 1}\sum_{i=1}^{l_k}(a_{k,i}^2\lambda_k^{2(1-m)} + b_{k,i}^2\lambda_k^{-2m})\\ &\sim \left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})}\end{aligned}$$

because, for $u = \sum_{k\geq 1}\sum_{i=1}^{l_k} u_{k,i}\varphi_{k,i}$, we have $\|u\|^2_{D(A^s)} \sim \sum_{k\geq 1}\sum_{i=1}^{l_k}\lambda_k^{4s}u_{k,i}^2$ for all $s \in \mathbb{R}$. Therefore the observability estimate (1.48) is equivalent to

$$\sum_{v\in\mathcal{V}_c}\int_0^T\left(\frac{\partial\phi}{\partial t}(v, t)\right)^2 dt \gtrsim \left\|u^{(0)}\right\|^2_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|^2_{D(A^{-\frac{m}{2}})}.$$

Now using Lemma 1.4.6, we obtain

$$\|u^{(0)}\|^2_{D(A^{\frac{1-m}{2}})} + \|u^{(1)}\|^2_{D(A^{-\frac{m}{2}})} \lesssim \sum_{v\in\mathcal{V}_c}\int_0^T \left((\frac{\partial u}{\partial t}(v,\, t))^2 + (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2 \right) dt.$$

On the other hand integrating the inequality (1.13) of Proposition 1.3.1 between 0 and T where T is sufficiently large, we have

$$\begin{aligned} E(0) - E(T) &\gtrsim \sum_{v\in\mathcal{V}_c}\int_0^T ((\frac{\partial u}{\partial t}(v,\, t))^2 + (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2)dt \\ &\gtrsim \frac{1}{2}\sum_{v\in\mathcal{V}_c}\int_0^T ((\frac{\partial u}{\partial t}(v,\, t))^2 + (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2)dt \\ &\quad +\frac{1}{2}\sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2 dt. \end{aligned}$$

Therefore

$$\begin{aligned} E(0) - E(T) &\gtrsim \|u^{(0)}\|^2_{D(A^{\frac{1-m}{2}})} + \|u^{(1)}\|^2_{D(A^{-\frac{m}{2}})} + \sum_{v\in\mathcal{V}_c}\int_0^T (\frac{\partial u}{\partial t}(v,\, t-\tau_v))^2 dt \\ &\gtrsim \|(u^{(0)},\, u^{(1)})\|^2_{X_{-m}} + \sum_{v\in\mathcal{V}_c}\tau_v\int_0^1 (\frac{\partial u}{\partial t}(v,\, -\tau_v\rho))^2 d\rho, \end{aligned}$$

where $X_{-m} = D(A^{\frac{1-m}{2}}) \times D(A^{-\frac{m}{2}})$ and due to (1.46). This finally shows that the observability estimate (1.48) implies that

$$E(T) \le E(0) - K_1 \left(\|(u^{(0)},\, u^{(1)})\|^2_{X_{-m}} + \sum_{v\in\mathcal{V}_c}\int_0^1 (\frac{\partial u}{\partial t}(v,\, -\tau_v\rho))^2 d\rho \right), \tag{1.51}$$

for some $K_1 > 0$.

Now we recall the following interpolation result.

Lemma 1.6.3. *For $(u^{(0)},\, u^{(1)}) \in D(A) \times V$, we have*

$$\begin{aligned} \|u^{(0)}\|^{m+1}_{D(A^{\frac{1}{2}})} &\lesssim \|u^{(0)}\|^m_{D(A)} \|u^{(0)}\|_{D(A^{\frac{1-m}{2}})}, \\ \|u^{(1)}\|^{m+1}_{D(A^0)} &\lesssim \|u^{(1)}\|^m_{D(A^{\frac{1}{2}})} \|u^{(1)}\|_{D(A^{\frac{-m}{2}})}. \end{aligned}$$

Proof: Direct consequence of the equivalence $\|u\|^2_{D(A^s)} \sim \sum_{k\ge1}\sum_{i=1}^{l_k} |u_{k,i}|^2 \lambda_k^{4s}$ for all $s \in \mathbb{R}$, when $u = \sum_{k\ge1}\sum_{i=1}^{l_k} u_{k,i}\varphi_{k,i}$ and of Hölder's inequality. ∎

Corollary 1.6.4. *For all $u \in X$, it holds*

$$\|u\|^{m+1}_V \lesssim \|u\|^m_X \|u\|_{D(A^{\frac{1-m}{2}})}. \tag{1.52}$$

Proof: We fix a sequence of cut-off functions $\eta_v, v \in \mathcal{V}$ satisfying

$$\sum_{v\in\mathcal{V}} \eta_v = 1 \text{ on } \mathcal{R}, \quad \eta_v \equiv 1 \text{ near } v,$$

and such that the support of η_v is included into $\mathcal{S}_v$, where $\mathcal{S}_v$ is the star-shaped network made of the edges $e_j, j \in \mathcal{E}_v$.

Denote by A_v the Laplace operator defined on $\mathcal{S}_v$ with Dirichlet boundary condition on all nodes of $\mathcal{S}_v$ except at v, where we impose Neumann or transmission conditions. Let us show that if (1.52) holds on $\mathcal{S}_v$ for $\eta_v u$, then it holds on $\mathcal{R}$. Indeed a convex inequality yields

$$\|u\|_V^{m+1} \lesssim \sum_{v\in\mathcal{V}} \|\eta_v u\|_V^{m+1}.$$

Now using (1.52) on $\mathcal{S}_v$ for $\eta_v u$, we get

$$\|u\|_V^{m+1} \lesssim \sum_{v\in\mathcal{V}} \|\eta_v u\|_{X_v}^m \|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})}, \tag{1.53}$$

the norm X_v being defined as the norm X but on $\mathcal{S}_v$. By Leibniz's rule, we directly have

$$\|\eta_v u\|_{X_v} \lesssim \|u\|_X.$$

Therefore it remains to estimate $\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})}$. For that purpose, we use a duality argument, namely

$$\begin{aligned}
\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} &= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{S}_v} \eta_v u w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \\
&= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{S}_v} u \eta_v w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \\
&= \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\int_{\mathcal{R}} u \eta_v w}{\|w\|_{D(A_v^{\frac{m-1}{2}})}},
\end{aligned}$$

by extending $\eta_v w$ by zero outside $\mathcal{S}_v$. Using the duality in $\mathcal{R}$, we obtain

$$\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} \leq \|u\|_{D(A^{\frac{1-m}{2}})} \sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\|\eta_v w\|_{D(A^{\frac{m-1}{2}})}}{\|w\|_{D(A_v^{\frac{m-1}{2}})}}.$$

But again Leibniz's rule yields

$$\sup_{w\in D(A_v^{\frac{m-1}{2}})} \frac{\|\eta_v w\|_{D(A^{\frac{m-1}{2}})}}{\|w\|_{D(A_v^{\frac{m-1}{2}})}} \lesssim 1,$$

which shows that

$$\|\eta_v u\|_{D(A_v^{\frac{1-m}{2}})} \lesssim \|u\|_{D(A^{\frac{1-m}{2}})}.$$

This estimate in (1.53) shows that (1.52) holds.

We are reduced to show that (1.52) holds on $\mathcal{S}_v$ for $\eta_v u$. For that purpose, without loss of generality we may assume that v is identified with 0 for all edges of $\mathcal{E}_v$ and that supp $\eta_v \subset \cup_{j\in\mathcal{E}_v}[0, 2l_j/3]$. For shortness write $U = \eta_v u$. Now for any $j \in \mathcal{E}_v$, we introduced the following extension of U_j :

$$E_j U_j(x) = \begin{cases} U_j(x) & \text{if } x \in (0, l_j), \\ \sum_{i=0}^{n-1} \nu_i U_j(-2^i x) & \text{if } x \in (-2^{-(n-1)} l_j, 0), \end{cases}$$

where U is extended by zero outside its support and the real numbers ν_i are the unique solution of the system

$$\begin{cases} \sum_{i=0}^{n-1} \nu_i = 1 \\ -\sum_{i=0}^{n-1} 2^i \nu_i = 1 \\ \sum_{i=0}^{n-1} 2^{2i} \nu_i = 1 \\ \sum_{i=0}^{n-1} 2^{-2ki} \nu_i = 1, \forall k = 1, \cdots, n-3, \end{cases}$$

and finally $n = \frac{m+5}{2}$ if m is odd and $n = \frac{m+4}{2}$ if m is even.

With the help of these extension operators, we obtain an extension of $U \in X_v$ to a function EU, which belongs to $D(\tilde{A}_v)$ (due to the three first properties of the ν_i), where $\tilde{A}_v$ is the positive Laplace operator on the star shaped network $\tilde{\mathcal{S}}_v = \cup_{j\in\mathcal{E}_v}(0, l_j) \bigcup \cup_{j\in\mathcal{E}_v}(-2^{-(n-1)} l_j, 0)$, with interior vertex v and Dirichlet boundary conditions at all other vertices. Applying Lemma 1.6.3 to EU on the network $\tilde{\mathcal{S}}_v$, we may write

$$\|EU\|^{m+1}_{D(\tilde{A}_v^{\frac{1}{2}})} \lesssim \|EU\|^m_{D(\tilde{A}_v)} \|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})}.$$

Now using the fact that the norm in $D(\tilde{A}_v^{\frac{1}{2}})$ is equivalent to the H^1 semi-norm and since EU is equal to U on $\mathcal{S}_v$, we obtain

$$\|U\|^{m+1}_{D(A_v^{\frac{1}{2}})} \lesssim \|EU\|^m_{D(\tilde{A}_v)} \|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})}. \tag{1.54}$$

This means that it remains to estimate the right-hand side of this estimate. First by construction, we easily check that

$$\|EU\|_{D(\tilde{A}_v)} \lesssim \|U\|_{X_v}. \tag{1.55}$$

As before to estimate the second factor, we use a duality argument. First we remark that for $w \in D(\tilde{A}_v^{\frac{m-1}{2}})$, we have

$$\int_{\tilde{\mathcal{S}}_v} EUw = \sum_{j\in\mathcal{E}_v} \int_0^{l_j} U_j(x) w_j(x)\, dx + \sum_{j\in\mathcal{E}_v} \int_{-2^{-(n-1)}l_j}^0 (Eu)_{-j}(x) w_{-j}(x)\, dx,$$

where for shortness we write w_{-j} the restriction of w to the edge $(-2^{-(n-1)}l_j, 0)$. By changes of variables, we obtain

$$\int_{\tilde{S}_v} Euw = \sum_{j\in\mathcal{E}_v} \int_0^{l_j} U_j(x)(Fw)_j(x)\, dx,$$

where

$$(Fw)_j(x) = w_j(x) + \chi_j(x)\sum_{i=0}^{n-1} \nu_i 2^{-i} U_j(-2^{-i}x), \forall x \in (0, l_j),$$

the cut-off function χ_j being fixed such that $\chi_j \equiv 1$ on $[0, 2l_j/3]$ and $\chi_j \equiv 0$ on $[5l_j/6, l_j]$ (reminding that $U_j(x) \equiv 0$ for $x > 2l_j/3$). Now we notice that the conditions on ν_i guarantees that Fw belongs to $D(A_v^{\frac{m-1}{2}})$ and by Leibniz's rule we have

$$\|Fw\|_{D(A_v^{\frac{m-1}{2}})} \lesssim \|w\|_{D(\tilde{A}_v^{\frac{m-1}{2}})}.$$

By duality we conclude that

$$\|EU\|_{D(\tilde{A}_v^{\frac{1-m}{2}})} \lesssim \|U\|_{D(A_v^{\frac{1-m}{2}})}.$$

This estimate and (1.55) in (1.54) show the requested estimate (1.52) on $\mathcal{S}_v$ for $\eta_v u$. ∎

Now let $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$ be fixed (and different from 0). By a convexity inequality and since $V \times L^2(\mathcal{R}) = D(A^{\frac{1}{2}}) \times D(A^0)$, we have

$$\left\|(u^{(0)}, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{m+1} \lesssim \left\|u^{(0)}\right\|_{D(A^{\frac{1}{2}})}^{m+1} + \left\|u^{(1)}\right\|_{D(A^0)}^{m+1}.$$

Using Lemma 1.6.3 and Corollary 1.6.4, we get

$$\begin{aligned}
\left\|(u^{(0)}, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{m+1} &\lesssim \left\|u^{(0)}\right\|_X^m \left\|u^{(0)}\right\|_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|_{D(A^{\frac{1}{2}})}^m \left\|u^{(1)}\right\|_{D(A^{\frac{-m}{2}})}\\
&\lesssim \left(\left\|u^{(0)}\right\|_X^m + \left\|u^{(1)}\right\|_{D(A^{\frac{1}{2}})}^m\right)\left(\left\|u^{(0)}\right\|_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|_{D(A^{\frac{-m}{2}})}\right)\\
&\lesssim \left(\left\|u^{(0)}\right\|_X + \left\|u^{(1)}\right\|_{D(A^{\frac{1}{2}})}\right)^m\left(\left\|u^{(0)}\right\|_{D(A^{\frac{1-m}{2}})} + \left\|u^{(1)}\right\|_{D(A^{\frac{-m}{2}})}\right)\\
&\lesssim \left\|(u^{(0)}, u^{(1)})\right\|_{X\times D(A^{\frac{1}{2}})}^m \left\|(u^{(0)}, u^{(1)})\right\|_{D(A^{\frac{1-m}{2}})\times D(A^{-\frac{m}{2}})}\\
&\lesssim \left\|(u^{(0)}, u^{(1)})\right\|_{X\times D(A^{\frac{1}{2}})}^m \left\|(u^{(0)}, u^{(1)})\right\|_{X_{-m}}.
\end{aligned}$$

This inequality is equivalent to

$$\left\|(u^{(0)}, u^{(1)})\right\|_{X_{-m}}^2 \gtrsim \frac{\left\|(u^{(0)}, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\left\|(u^{(0)}, u^{(1)})\right\|_{X\times D(A^{\frac{1}{2}})}^{2m}}.$$

Using the trivial inequality

$$\begin{aligned}
\left\|(u^{(0)}, u^{(1)})\right\|_{X\times D(A^{\frac{1}{2}})} &\leq \left\|(u^{(0)}, u^{(1)}, (f^0(-\tau_v.)))\right\|_{X\times D(A^{\frac{1}{2}})\times H^1(0,1)^{V_c}}\\
&\lesssim \left\|(u^{(0)}, u^{(1)}, (f^0(-\tau_v.)))\right\|_{D(\mathcal{A})},
\end{aligned}$$

we finally obtain

$$\left\|(u^{(0)},\, u^{(1)})\right\|_{X_{-m}}^2 \gtrsim \frac{\left\|(u^{(0)},\, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\|(u^{(0)},\, u^{(1)},\, (f^0(-\tau_v\cdot)))\|_{D(\mathcal{A})}^{2m}}. \tag{1.56}$$

1.6.2 Polynomial decay of the energy

The proof of our stability result requires the next technical Lemma proved in Lemma 5.2 of [10].

Lemma 1.6.5. *Let $(\varepsilon_k)_k$ be a sequence of positive real numbers satisfying*

$$\varepsilon_{k+1} \le \varepsilon_k - C\varepsilon_{k+1}^{2+\alpha},\ \forall k \ge 0, \tag{1.57}$$

where $C > 0$ and $\alpha > -1$. Then there exists a positive constant M (depending on α and C) such that

$$\varepsilon_k \le \frac{M}{(1+k)^{\frac{1}{1+\alpha}}},\ \forall k \ge 0.$$

Theorem 1.6.6. *Let u be a solution of (1.1) with $(u^{(0)},\, u^{(1)},\, (f^0(-\tau_v\cdot))) \in D(\mathcal{A})$. If (1.48) holds, then the energy decays polynomially, i.e.*

$$E(t) \le \frac{C}{(1+t)^{\frac{1}{m}}} \left\|(u^{(0)},\, u^{(1)},\, (f^0(-\tau_v\cdot)))\right\|_{D(\mathcal{A})}^2,\ \forall t \ge 0, \tag{1.58}$$

for some $C > 0$.

Proof: Introduce the modified energy

$$\tilde{E}(t) = \frac{1}{2}\|U(t)\|_{D(\mathcal{A})}^2 = \frac{1}{2}(\|U(t)\|_H^2 + \|\mathcal{A}U(t)\|_H^2).$$

As in Proposition 1.3.1, this energy is decaying.

Combining the estimates (1.51) and (1.56), we obtain

$$E(T) \le E(0) - K_2 \left(\frac{\left\|(u^{(0)},\, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\tilde{E}(0)^m} + \sum_{v\in\mathcal{V}_c} \int_0^1 (\frac{\partial u}{\partial t}(v,\, -\tau_v\rho))^2 d\rho \right),$$

for some $K_2 > 0$, or equivalently

$$E(T) \le E(0) - K_2 \left(\frac{\left\|(u^{(0)},\, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^{2m+2}}{\tilde{E}(0)^m} + \left\|(f_v^0(-\tau_v\cdot))\right\|_{L^2(0,\,1)^{\mathcal{V}_c}}^2 \right).$$

Using the trivial estimate

$$\begin{aligned}\left\|(f_v^0(-\tau_v\cdot))\right\|_{L^2(0,\,1)^{\mathcal{V}_c}}^{2m+2} &\le \left\|(f_v^0(-\tau_v\cdot))\right\|_{L^2(0,\,1)^{\mathcal{V}_c}}^2 \left\|(f_v^0(-\tau_v\cdot))\right\|_{H^1(0,\,1)^{\mathcal{V}_c}}^{2m} \\ &\lesssim \left\|(f_v^0(-\tau_v\cdot))\right\|_{L^2(0,\,1)^{\mathcal{V}_c}}^2 \tilde{E}(0)^m,\end{aligned}$$

the above inequality becomes

$$
\begin{aligned}
E(T) &\leq E(0) - K_2 \frac{\|(u^{(0)}, u^{(1)})\|^{2m+2}_{V\times L^2(\mathcal{R})} + \|(f_v^0(-\tau_v \cdot))\|^{2m+2}_{L^2(0,1)^{V_c}}}{\tilde{E}(0)^m} \\
&\leq E(0) - K_3 \frac{E(0)^{m+1}}{\tilde{E}(0)^m},
\end{aligned}
$$

with $K_3 > 0$. Since the energy of our system is decaying, we obtain

$$
E(T) \leq E(0) - K_3 \frac{E(T)^{m+1}}{\tilde{E}(0)^m}. \tag{1.59}
$$

We now follow the method used in [10]. The estimate (1.59) being valid on the intervals $[kT, (k+1)T]$, for any $k \geq 0$, we have

$$
E((k+1)T) \leq E(kT) - K_3 \frac{E((k+1)T)^{m+1}}{\tilde{E}(kT)^m}. \tag{1.60}
$$

We set

$$
\varepsilon_k = \frac{E(kT)}{\tilde{E}(0)}.
$$

By dividing (1.60) by $\tilde{E}(0)$, we obtain

$$
\varepsilon_{k+1} \leq \varepsilon_k - K_3 \varepsilon_{k+1}^{m+1}, \tag{1.61}
$$

because $\tilde{E}(kT) \leq \tilde{E}(0)$. By Lemma 1.6.5 with $\alpha = m - 1 > -1$ (since $m > 0$), there exists a constant $M > 0$ such that

$$
\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{m}}}, \ \forall k \geq 0,
$$

or equivalently

$$
E(kT) \leq \frac{M}{(1+k)^{\frac{1}{m}}} \tilde{E}(0).
$$

This estimate and again the decay of the energy lead to the estimate (1.58). ■

1.7 Examples

Our aim is to give some concrete examples that illustrate our stability results.

1.7.1 One string with an interior damping

We consider a homogeneous string of length π with an interior damping at ξ. Two types of boundary conditions will be considered.

Mixed boundary conditions

We study the following problem (see Fig. 1.1)

$$\left\{\begin{array}{ll}\frac{\partial^2 u}{\partial t^2}(x,\, t) - \frac{\partial^2 u}{\partial x^2}(x,\, t) = 0 & 0 < x < \pi,\, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-,\, t) - \frac{\partial u}{\partial x}(\xi_+,\, t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi,\, t) + \alpha_2 \frac{\partial u}{\partial t}(\xi,\, t-\tau)) & t > 0, \\ u(0,\, t) = 0,\, \frac{\partial u}{\partial x}(\pi,\, t) = 0 & t > 0, \\ u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi,\, t-\tau) = f^0(t-\tau) & 0 < t < \tau. \end{array}\right. \tag{1.62}$$

Here contrary to subsection 1.3.4, we suppose that $\alpha_2 < \alpha_1$.

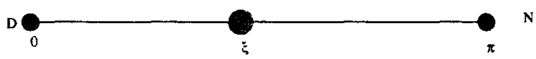

FIG. 1.1 –

It is well known that the eigenvectors associated with problem (1.62) without damping are $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin((k + \frac{1}{2})x)$ of eigenvalue $(k + \frac{1}{2})^2$, $k \geq 0$ of multiplicity 1. We then have to look at

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = \frac{2}{\pi} \sin^2\left((k + \frac{1}{2})\xi\right).$$

For the exponential decay we will use the next result proved in Lemma 2.9 of [97].

Lemma 1.7.1. *s is a rational number with an irreducible fraction*

$$s = \frac{p}{q}, \quad \textit{where } p \textit{ is odd} \tag{1.63}$$

if and only if there exists $\alpha > 0$ such that

$$\left|\sin\left((\frac{\pi}{2} + k\pi)s\right)\right| > \alpha, \forall k \in \mathbb{N}. \tag{1.64}$$

Applying Proposition 1.3.7 and Theorem 1.5.3 (and the above Lemma), we obtain

Theorem 1.7.2. *1) The energy of system (1.62) tends to 0 for all initial data in H if and only if*

$$\frac{\xi}{\pi} \neq \frac{2p}{2q+1}, \forall p,\, q \in \mathbb{N}.$$

2) The system (1.62) is exponentially stable in the energy space if $\frac{\xi}{\pi}$ is a rational number with an irreducible fraction

$$\frac{\xi}{\pi} = \frac{p}{q}, \quad \textit{where } p \textit{ is odd.}$$

If we consider the system (1.62) without delay (i.e. $\alpha_2 = 0$), we find the results of [6] obtained by a similar method.

Dirichlet boundary conditions

We here consider the problem (see Fig. 1.2)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,\,t) - \frac{\partial^2 u}{\partial x^2}(x,\,t) = 0 & 0 < x < \pi,\, t > 0, \\ \frac{\partial u}{\partial x}(\xi_-,\,t) - \frac{\partial u}{\partial x}(\xi_+,\,t) = -(\alpha_1 \frac{\partial u}{\partial t}(\xi,\,t) + \alpha_2 \frac{\partial u}{\partial t}(\xi,\,t-\tau)) & t > 0, \\ u(0,\,t) = 0,\, u(\pi,\,t) = 0 & t > 0, \\ u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(\xi,\,t-\tau) = f^0(t-\tau) & 0 < t < \tau. \end{cases} \tag{1.65}$$

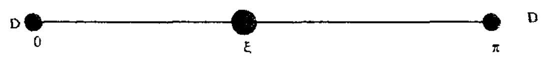

FIG. 1.2 –

The eigenvectors of problem without damping associated with problem (1.65) are $\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$ of eigenvalue k^2, $k \geq 1$ of multiplicity 1. We then have to consider

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = \frac{2}{\pi}\sin^2(k\xi).$$

Denote by $\mathcal{S}$ the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0,\, a_1, ...,\, a_n, ...]$ is the expansion of ρ as a continued fraction, then the sequence (a_n) is bounded. It is well known that $\mathcal{S}$ is uncountable and that its Lebesgue measure is zero. Roughly speaking, the set $\mathcal{S}$ contains all irrational numbers which are badly approximated by rational numbers. In particular, by the Euler-Lagrange theorem, $\mathcal{S}$ contains all irrational quadratic numbers (i.e. the roots of a second order equation with rational coefficients). By a classical result, we have the

Lemma 1.7.3. *If $s \in \mathcal{S}$, then there exists a positive constant C such that*

$$|\sin(k\pi s)| \geq \frac{C}{k},\ \forall k \geq 1.$$

By Proposition 1.3.7 and Theorem 1.6.6, we then obtain

Theorem 1.7.4. *1) The energy of system (1.65) decays to 0 for all initial data in H if and only if*

$$\frac{\xi}{\pi} \notin \mathbb{Q}.$$

2) If $\frac{\xi}{\pi} \in \mathcal{S}$, then for all initial data in $D(\mathcal{A})$, the energy of system (1.65) decays polynomially like $1/t$.

Without delay, we find the results of [11].

Remark 1.7.5. These two examples show that the boundary conditions influence the stability of the system because we do not have the same hypotheses for the decay to 0 of the energy ; moreover for mixed boundary conditions, we may have an exponential stability, while for Dirichlet boundary conditions, we cannot have an exponential stability but have a polynomial stability.

1.7.2 A star shaped network

Dirichlet boundary conditions at all extremities

We take Dirichlet boundary conditions at all extremities and put a damping at the interior node (see Fig. 1.3).

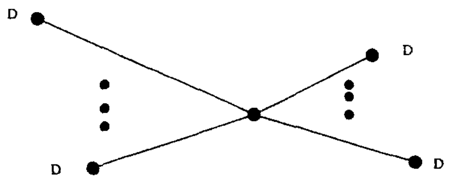

FIG. 1.3 –

In appropriated coordinates, any eigenvector of the problem without damping is of the form

$$\varphi_j(x) = a_j \sin(\lambda x), \quad 1 \leq j \leq N,$$

for some constants a_j. The transmission conditions at the interior node then lead to

$$a_1 \sin(\lambda l_1) = ... = a_N \sin(\lambda l_N), \tag{1.66}$$

$$\sum_{j=1}^{N} a_j \cos(\lambda l_j) = 0. \tag{1.67}$$

We now suppose that

$$\forall i,\, j \in \{1, ..., N\},\, i \neq j,\, \frac{l_i}{l_j} \notin \mathbb{Q}. \tag{1.68}$$

In that case we cannot have $\sin(\lambda l_i) = \sin(\lambda l_j) = 0$ for $i \neq j$. Indeed if $\lambda l_i = p\pi$, $p \in \mathbb{Z}$ and $\lambda l_j = q\pi$, $q \in \mathbb{Z}$, then $\frac{l_i}{l_j} = \frac{p}{q} \in \mathbb{Q}$, which contradicts our assumption.

Therefore, there exists $j \in \{1, ..., N\}$, say $j = 1$ such that $\sin(\lambda l_j) \neq 0$. But then $a_1 \neq 0$. Indeed if $a_1 = 0$, then

$$a_2 \sin(\lambda l_2) = ... = a_N \sin(\lambda l_N) = 0.$$

As we cannot have $\sin(\lambda l_i) = \sin(\lambda l_j) = 0$ for $i \neq j$, all a_j are equal to zero except one, say a_N for example and then $\sin(\lambda l_N) = 0$. By the last transmission condition (1.67) we would have $a_N \cos(\lambda l_N) = 0$ and then $a_N = 0$, which is impossible.

Under the assumption (1.68), we have

$$\sin(\lambda l_j) \neq 0, \forall j = 1, \cdots, N,$$

and the transmission condition (1.67) yields the characteristic equation

$$\sum_{j=1}^{N} \cot(\lambda l_j) = 0. \tag{1.69}$$

From this equation, we deduce that all eigenvalues are simple. Hence

$$\sum_{v \in \mathcal{V}_c} |\varphi_k(v)|^2 = a_1^2 \sin^2(\lambda l_1) > 0,$$

when φ_k is the eigenvector associated with λ and by Proposition 1.3.7 we directly conclude the

Proposition 1.7.6. *If (1.68) holds, then the energy of our system tends to 0 for all initial data in H.*

Without delay we find the result of Proposition 2.1 of [7].

After calculations and normalization, we find that

$$a_1^2 \sin^2(\lambda l_1) = \frac{2}{\sum_{j=1}^{N} \frac{l_j}{\sin^2(\lambda l_j)}}.$$

In its full generality, it is difficult to find the behavior of $a_1^2 \sin^2(\lambda l_1)$ from the characteristic equation (1.69). We therefore restrict ourselves to some particular cases.

We first suppose that the lengths of the edges are equal to 1. Then easy calculations allow to show that the set of eigenvalues of the operator A is made of two sequences : First $\lambda_k^2 = (\frac{\pi}{2} + k\pi)^2, k \in \mathbb{N}$ is an eigenvalue of multiplicity 1 with associated eigenvector

$$(\varphi_k)_j(x) = \sqrt{\frac{1}{2}} \sin\left((\frac{\pi}{2} + k\pi)x\right), \forall j \in \{1, ..., N\}.$$

Secondly, $\lambda_k^2 = k^2\pi^2, k \in \mathbb{N}^*$ is of multiplicity $N-1$ with orthonormal eigenvectors given by

$$\varphi_{k,1} = \sin(k\pi x) \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \varphi_{k,2} = \frac{2}{\sqrt{3}} \sin(k\pi x) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \cdots,$$

$$
\varphi_{k,i} = \frac{\sqrt{2i}}{\sqrt{1+i}} \sin(k\pi x) \begin{pmatrix} \frac{1}{i} \\ \frac{1}{i} \\ \vdots \\ \frac{1}{i} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \varphi_{k,N-1} = \frac{\sqrt{2(N-1)}}{\sqrt{N}} \sin(k\pi x) \begin{pmatrix} \frac{1}{N-1} \\ \frac{1}{N-1} \\ \vdots \\ \frac{1}{N-1} \\ -1 \end{pmatrix}.
$$

where for shortness we write $\varphi_{k,1}$ as a vector with N components, the j^{th} component corresponding to the restriction of $\varphi_{k,1}$ to the edge j.

If the feedback law is applied only at the interior node, as for the eigenvalue $\lambda_k^2 = k^2\pi^2$, $\mathcal{M}_{v_{int}}(k^2\pi^2) = 0$ (because $\sin(k\pi) = 0$), the eigenvalues of $\mathcal{M}_{v_{int}}(k^2\pi^2)$ are 0. Therefore Proposition 1.3.7 yields

Proposition 1.7.7. *If the lengths of the star shaped network are one and the feedback law is applied at the interior node, the energy does not decay to 0.*

We then need to add some interior controls to stabilize the system. On each edge e_j except one (for instance the first one), we put a control at ξ_j (see Fig. 1.4).

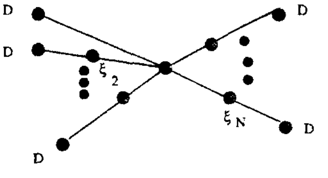

FIG. 1.4 –

Then for the eigenvalue $\lambda_k^2 = k^2\pi^2$, we readily check that

$$
\mathcal{M}(k^2\pi^2) = \mathcal{M}_{v_{int}}(k^2\pi^2) + \sum_{j=2}^{N} \mathcal{M}_{\xi_j}(k^2\pi^2)
$$

satifies

$$
\eta^\top \mathcal{M}(k^2\pi^2)\eta = \sum_{j=2}^{N} \sin^2(k\pi\xi_j)(\chi_j^\top \eta)^2,
$$

where the vectors $\chi_j = ((\chi_j)_i)_{i=2}^N, j = 2, \ldots, N$ are non zero vectors, which are independent of k and of $\xi_j, j = 2, \ldots, N$ and satisfy

$$
(\chi_j)_i = 0, \forall i < j.
$$

Consequently if for all $j \in \{2, ..., N\}$, $\xi_j \in \mathcal{S}$, then $\sin^2(k\pi\xi_j) \gtrsim \frac{1}{k^2}$ and therefore

$$\eta^\top \mathcal{M}(k^2\pi^2)\eta \gtrsim \frac{1}{k^2}\sum_{j=2}^{N}(\chi_j^\top \eta)^2.$$

The above properties of χ_j imply that $(\sum_{j=2}^{N}(\chi_j^\top \eta)^2)^{1/2}$ is a norm on $\mathbb{R}^{N-1}$ and therefore

$$\eta^\top \mathcal{M}(k^2\pi^2)\eta \gtrsim \frac{1}{k^2}\|\eta\|_2^2,$$

or equivalently $\lambda_{min}(\mathcal{M}(k^2\pi^2)) \gtrsim \frac{1}{k^2}$.

On the other hand for $\lambda_k = \frac{\pi}{2} + k\pi$, we see that

$$\sum_{v\in\mathcal{V}_c} |\varphi_k(v)|^2 = \frac{1}{2}\sin^2\left(\frac{\pi}{2} + k\pi\right) + \frac{1}{2}\sum_{j=2}^{N}\sin^2\left((\frac{\pi}{2} + k\pi)\xi_j\right) \geq \frac{1}{2}.$$

We then conclude the

Proposition 1.7.8. *If the lengths of the star shaped network are one and the feedback law is applied at the interior node, and at $N-1$ interior points ξ_j and if*

$$\xi_j \in \mathcal{S},\ \forall j \in \{2, ..., N\},$$

then for any $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy of the system decays like $\frac{1}{t}$.

Now we assume that $N = 2n$ is even and that

$$l_i = l, \forall i = 1, \cdots, n,\ l_i = l', \forall i = n+1, \cdots, 2n \text{ and } \frac{l}{l'} \notin \mathbb{Q}. \tag{1.70}$$

Under this assumption, (1.69) is equivalent to

$$\cot(\lambda l) + \cot(\lambda l') = 0,$$

and then to

$$\sin(\lambda(l + l')) = 0.$$

This means that a first set of eigenvalues is given by

$$\lambda_k = \frac{k\pi}{l + l'}, k \in \mathbb{N}^*.$$

A second set is made of the roots of $\sin(\lambda l) = 0$. Since $\frac{l}{l'} \notin \mathbb{Q}$, we deduce that $a_i = 0$, for all $i = n+1, \cdots, 2n$. Consequently for all $k \in \mathbb{N}^*$, $\frac{k^2\pi^2}{l^2}$ is of multiplicity $n-1$, its associated orthonormal eigenvectors being given by :

$$\begin{aligned}
\varphi_{k,1}(x) &= \frac{1}{\sqrt{l}}\sin\left(\frac{k\pi x}{l}\right)(1,-1,0,\cdots,0,0,\cdots,0)^\top, \\
\varphi_{k,2}(x) &= \frac{2}{\sqrt{3l}}\sin\left(\frac{k\pi x}{l}\right)\left(\frac{1}{2},\frac{1}{2},-1,0,\cdots,0,0,\cdots,0\right)^\top, \\
&\vdots \\
\varphi_{k,n-1}(x) &= \frac{\sqrt{2(n-1)}}{\sqrt{nl}}\sin\left(\frac{k\pi x}{l}\right)\left(\frac{1}{n-1},\frac{1}{n-1},\cdots,\frac{1}{n-1},-1,0,\cdots,0\right)^\top.
\end{aligned}$$

By symmetry a third set of eigenvalues is made of the numbers $\frac{k^2\pi^2}{(l')^2}$ of multiplicity $n-1$ for all $k \in \mathbb{N}^*$.

Note that for this example, the standard gap condition holds.

Since the eigenvectors associated with the eigenvalues of the second and third sets are zero at the interior node, if we impose a damping only at this interior node, the energy will not tend to zero. Therefore some interior control points have to be added. More precisely we impose a feedback law at some points ξ_i of the edge e_i, for $i = 2, \cdots, n$ and for $i = n+2, \cdots, 2n$. By direct calculations, the matrix

$$\mathcal{M}\left(\frac{k^2\pi^2}{l^2}\right) = \mathcal{M}_{v_{int}}\left(\frac{k^2\pi^2}{l^2}\right) + \sum_{j=2}^{n}\mathcal{M}_{\xi_j}\left(\frac{k^2\pi^2}{l^2}\right) + \sum_{j=n+2}^{2n}\mathcal{M}_{\xi_j}\left(\frac{k^2\pi^2}{l^2}\right)$$

satisfies

$$\eta^\top\mathcal{M}\left(\frac{k^2\pi^2}{l^2}\right)\eta = l^{-1}\sum_{j=2}^{n}\sin^2\left(\frac{k\pi\xi_j}{l}\right)(\chi_j^\top\eta)^2,$$

where the vectors $\chi_j = ((\chi_j)_i)_{i=2}^n, j = 2, \ldots, n$ satisfy the same property than before. Hence if for all $j \in \{2, ..., n\}$, $\frac{\xi_j}{l} \in \mathcal{S}$, then $\sin^2(\frac{k\pi\xi_j}{l}) \gtrsim \frac{1}{k^2}$ and therefore $\lambda_{min}(\mathcal{M}(\frac{k^2\pi^2}{l^2})) \gtrsim \frac{1}{k^2}$.

By symmetry if for all $j \in \{n+2, ..., 2n\}$, $\frac{\xi_j}{l'} \in \mathcal{S}$, then the matrix $\mathcal{M}(\frac{k^2\pi^2}{(l')^2})$ satisfies $\lambda_{min}(\mathcal{M}(\frac{k^2\pi^2}{(l')^2})) \gtrsim \frac{1}{k^2}$.

Finally for the eigenvalue $\lambda_k^2 = \frac{k^2\pi^2}{(l+l')^2}, k \in \mathbb{N}^*$, since we have $\sin(\lambda_k l') = (-1)^{k+1}\sin(\lambda_k l)$, we deduce that

$$a_1^2\sin^2(\lambda_k l_1) = \frac{2\sin^2(\lambda_k l)}{n(l+l')} = \frac{2}{n(l+l')}\sin^2\left(\frac{k\pi l}{l+l'}\right).$$

By Lemma 1.7.3 if $\frac{l}{l+l'} \in \mathcal{S}$, then its associated matrix $\mathcal{M}(\frac{k^2\pi^2}{(l+l')^2})$ satisfies $\lambda_{min}(\mathcal{M}(\frac{k^2\pi^2}{(l+l')^2})) \gtrsim \frac{1}{k^2}$.

Theorem 1.6.6 then leads to the

Theorem 1.7.9. *Consider a star shaped network with Dirichlet boundary condition satisfying (1.70) and feedback laws at the interior node as well as at the point ξ_i of the edge e_i,* *for* $i = 2, \cdots, n$ *and for* $i = n+2, \cdots, 2n$. *If* $\frac{\xi_i}{l} \in \mathcal{S}$, *for all* $i = 2, \cdots, n$, $\frac{\xi_i}{l'} \in \mathcal{S}$, *for all* $i = n+2, \cdots, 2n$ *and* $\frac{l}{l+l'} \in \mathcal{S}$, *then for all* $(u^{(0)}, u^{(1)}, (f^0(-\tau_v \cdot))) \in D(\mathcal{A})$, *the energy of our system decays polynomially like* $\frac{1}{t}$.

Note that the two previous examples do not enter into the setting of [7].

Mixed boundary conditions

We suppose that the star shaped network is made of 4 edges, with e_1 and e_4 of same length l, and e_2 and e_3 of same length l'. At the extremities of e_1 and e_2, we impose Dirichlet condition and at the extremities of e_3 and e_4, Neumann condition. We control at the interior node. In this case the eigenvalues of the problem without damping are $\frac{k^2\pi^2}{4(l+l')^2}$ of multiplicity 1.

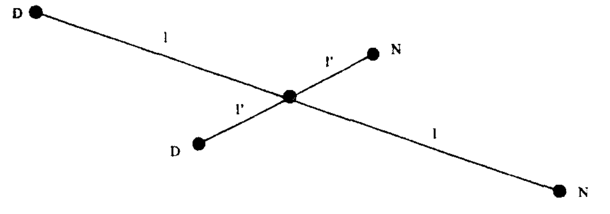

FIG. 1.5 –

Similar considerations as above allow to show the following results :

Proposition 1.7.10. *1) If* $\frac{l}{l'} \notin \mathbb{Q}$, *then the energy of our system tends to 0.*
2) Suppose that $\frac{l}{l'} \notin \mathbb{Q}$ *and that* $\frac{l}{l+l'} \in \mathcal{S}$ *and* $\frac{l'}{l+l'} \in \mathcal{S}$, *then for all* $(u^{(0)}, u^{(1)}, (f^0(-\tau_v \cdot))) \in D(\mathcal{A})$, *the energy decays as* $\frac{1}{\sqrt{t}}$.

Remark 1.7.11. For instance if $l = \sqrt{2}$ and $l' = 1-\sqrt{2}$, then $\frac{l}{l'} \notin \mathbb{Q}$, $\frac{l}{l+l'} \in \mathcal{S}$ and $\frac{l'}{l+l'} \in \mathcal{S}$.

1.7.3 More complex networks

In this subsection, we assume that $\mathcal{R}$ is a network whose edges are all of same length, i.e. $l_j = 1$, for all $j \in \{1, ..., N\}$. In that case, the spectrum of the Laplace operator is explicitly know via the algebraic properties of the network [20, 81]. More precisely, introduce the adjacency matrix C of the vertices of the network

$$C = (c_{s,t})_{s,t \in \mathcal{V} \setminus \mathcal{D}},$$

where

$$\forall s,\, t \in \mathcal{V} \setminus \mathcal{D},\ c_{s,t} = \begin{cases} \frac{card(\mathcal{E}_s \cap \mathcal{E}_t)}{\sqrt{card(\mathcal{E}_s)}\sqrt{card(\mathcal{E}_t)}} & \text{if } \mathcal{E}_s \cap \mathcal{E}_t \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We may now recall the following result proved in [81] :

Theorem 1.7.12. *Under the above assumption, we have*

$$Sp(A) = S_1 \cup S_2, \text{ where}$$

$S_1 = \{k^2\pi^2$ *with multiplicity* $N - \#(\mathcal{V} \setminus \mathcal{D})$, $k \in \mathbb{N}^*\}$ *of associated eigenvector* $\varphi_{k,j}(x) = c_j \sin(k\pi x)$, *for some constants* c_j, *and*
$S_2 = \{\lambda^2 : \cos\lambda \in Sp(\mathcal{C}) \cap] -1, 1[\}$. *Moreover* φ *is a eigenvector of* A *associated with* λ^2 *if and only if* $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ *is an eigenvector of the matrix* $\mathcal{C}$ *of eigenvalue* $\cos\lambda$.

Note that this Theorem implies that the standard gap condition holds for networks with edges which are of length one (or more generally rational numbers).

Before going on, let us make a more precise relation between the orthonormal eigenvectors of an eigenvalue λ^2 from S_2 and the eigenvectors of the matrix $\mathcal{C}$ of eigenvalue $\cos\lambda$:

Lemma 1.7.13. *Let* $\lambda^2 \in S_2$. *There exists a positive definite symmetric matrix* $\mathcal{E}(\lambda) \in \mathbb{R}^{m\times m}$, *where* m *is the cardinal of* $\mathcal{V}\setminus\mathcal{D}$, *such that for all eigenvectors* φ, φ' *associated with* λ^2, *we have*

$$(\varphi, \varphi')_{L^2(\mathcal{R})} = (\varphi'(v))^{\top}_{v\in\mathcal{V}\setminus\mathcal{D}} \mathcal{E}(\lambda)(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}. \tag{1.71}$$

Moreover this matrix $\mathcal{E}(\lambda)$ *is uniformly positive definite and uniformly bounded, in the sense that there exists a positive constant* C *(independent of* λ*) such that*

$$\lambda_{min}(\mathcal{E}(\lambda)) \geq C, \quad \|\mathcal{E}(\lambda)\|_2 \leq C^{-1}.$$

Proof: For all $j = 1, \cdots, N$ we may write

$$\varphi_j(x) = a_j \sin(\lambda x) + b_j \sin(\lambda(1-x)), \forall x \in (0,1), \tag{1.72}$$

where

$$a_j = \frac{\varphi(v)}{\sin\lambda}, b_j = \frac{\varphi(v')}{\sin\lambda}, \tag{1.73}$$

when v (resp. v') is the vertex corresponding to the extremity 1 (resp. 0) of the edge e_j. We use the same relations for φ' using a'_j and b'_j.

Now by direct calculations, we see that

$$(\varphi, \varphi')_{L^2(\mathcal{R})} = \frac{1}{2}\sum_{j=1}^{N} (a_j \quad b_j)(\mathcal{B}(\lambda) + \mathcal{B}_r(\lambda)) \begin{pmatrix} a'_j \\ b'_j \end{pmatrix},$$

where the 2×2 matrices $\mathcal{B}(\lambda)$ and $\mathcal{B}_r(\lambda)$ are given by

$$\mathcal{B}(\lambda) = \begin{pmatrix} 1 & -\cos\lambda \\ -\cos\lambda & 1 \end{pmatrix}, \quad \mathcal{B}_r(\lambda) = \frac{1}{2\lambda}\begin{pmatrix} \sin(2\lambda) & 2\sin\lambda \\ 2\sin\lambda & \sin(2\lambda) \end{pmatrix}.$$

This proves the identity (1.71) due to the relation (1.73).

Let us now remark that $\mathcal{B}(\lambda)$ depends only on $\cos\lambda$ and then on $\cos\lambda_0$ with $\lambda_0 \in (0, \pi)$, when $\lambda = \lambda_0 + 2k\pi$ or $\lambda = -\lambda_0 + 2(k+1)\pi$, for some $k \in \mathbb{N}$. The eigenvalues of $\mathcal{B}(\lambda)$ being $1 \pm \cos\lambda$, this matrix is uniformly positive definite, i.e. there exists a positive constant C (independent of λ) such that

$$(a \quad b)\mathcal{B}(\lambda) \begin{pmatrix} a \\ b \end{pmatrix} \geq C(a^2 + b^2), \forall a, b \in \mathbb{R}.$$

We further remark that the matrix $\mathcal{B}_r(\lambda)$ is a remainder since

$$\|\mathcal{B}_r(\lambda)\|_2 \lesssim \frac{1}{\lambda}.$$

Therefore we introduce the matrix $\mathcal{F}(\lambda_0) \in \mathbb{R}^{m\times m}$ as follows :

$$(\xi_v)^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{F}(\lambda_0)(\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} = \frac{1}{2}\sum_{j=1}^{N}(a_j \quad b_j)\mathcal{B}(\lambda) \begin{pmatrix} a'_j \\ b'_j \end{pmatrix},$$
$$\forall(\xi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}, (\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} \in \mathbb{R}^m,$$

with the relation

$$a_j = \frac{\xi_v}{\sin\lambda}, b_j = \frac{\xi_{v'}}{\sin\lambda}. \tag{1.74}$$

From the uniform positiveness of $\mathcal{B}(\lambda)$ and the above relations between a_j, b_j and ξ_v, we directly deduce that $\mathcal{F}(\lambda_0)$ is uniformly positive definite.

The previous considerations clearly show that

$$\mathcal{E}(\lambda) = \mathcal{F}(\lambda_0) + \mathcal{F}_r(\lambda),$$

with

$$\|\mathcal{F}_r(\lambda)\|_2 \lesssim \frac{1}{\lambda}.$$

Therefore there exists $\Lambda > 0$ such that $\mathcal{E}(\lambda)$ is uniformly positive definite, for all $\lambda > \Lambda$.

It remains to consider the case $0 < \lambda \leq \Lambda$. But in that case we see that

$$(\xi_v)^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{E}(\lambda)(\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} = (\varphi, \varphi')_{L^2(\mathcal{R})}, \forall(\xi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}, (\xi'_v)_{v\in\mathcal{V}\setminus\mathcal{D}} \in \mathbb{R}^m,$$

when φ is given by (1.72) when a_j, b_j are defined by (1.74) (and similarly for φ'). As the above right-hand side is an inner product on $L^2(\mathcal{R})$, the left-hand side is also an inner product in $\mathbb{R}^m$. Hence the matrix $\mathcal{E}(\lambda)$ is positive definite. The uniformness follows from the fact that the interval $(0, \Lambda]$ contains a finite number of λ such that $\lambda^2 \in S_2$.

The uniform boundedness of $\mathcal{E}(\lambda)$ is proved in the same manner. ∎

Corollary 1.7.14. *Assume that $\mathcal{V}_c = \mathcal{V} \setminus \mathcal{D}$. Then for any $\lambda^2 \in S_2$, its associated matrix $\mathcal{M}(\lambda^2)$ is uniformly positive definite, i.e.*

$$\lambda_{min}(\mathcal{M}(\lambda^2)) \gtrsim 1.$$

Proof: Assume that λ^2 is of multiplicity l and denote by $\varphi_i, i = 1, \cdots, l$ the associated orthonormal eigenvectors. Now we introduce the vectors

$$(\tilde{\varphi}_i(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \mathcal{E}(\lambda)^{1/2}(\varphi_i(v))_{v\in\mathcal{V}\setminus\mathcal{D}}, \forall i = 1, \cdots, l.$$

According to the relation (1.71) these vectors are orthonormal :

$$(\tilde{\varphi}_i(v))^{\top}_{v\in\mathcal{V}\setminus\mathcal{D}}(\tilde{\varphi}_j(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \delta_{ij}, \forall i, j = 1, \cdots, l.$$

Now we simply remark that

$$(\xi_i)^{\top}\mathcal{M}(\lambda^2)(\xi_i) = \sum_{i,j}\xi_i\xi_j(\varphi_j(v))^{\top}_v(\varphi_i(v))_v,$$

and consequently

$$\begin{aligned}
(\xi_i)^{\top}\mathcal{M}(\lambda^2)(\xi_i) &= \sum_{i,j}\xi_i\xi_j(\tilde{\varphi}_j(v))^{\top}_v\,\mathcal{E}(\lambda)^{-1}(\tilde{\varphi}_i(v))_v \\
&= (\tilde{\varphi}(v))^{\top}_v\mathcal{E}(\lambda)^{-1}(\tilde{\varphi}(v))_v,
\end{aligned}$$

where $(\tilde{\varphi}(v))_v = \sum_i \xi_i(\tilde{\varphi}_i(v))_v$. By the uniform boundedness of $\mathcal{E}(\lambda)$, we deduce that

$$\begin{aligned}
(\xi_i)^{\top}\mathcal{M}(\lambda^2)(\xi_i) &\gtrsim (\tilde{\varphi}(v))^{\top}_v(\tilde{\varphi}(v))_v \\
&\gtrsim \sum_{i,j}\xi_i\xi_j(\tilde{\varphi}_j(v))^{\top}_v(\tilde{\varphi}_i(v))_v = \sum_i \xi_i^2,
\end{aligned}$$

this last identity following from the orthonormality of the vectors $(\tilde{\varphi}_i(v))_v$. ∎

From this Corollary we see that if we control at least at all nodes of $\mathcal{V}\setminus\mathcal{D}$ then the assumption for the exponential stability (and obviously polynomial stability) holds for all eigenvalues of S_2. In that case we only need to manage the eigenvalues of S_1. Note further that if S_1 is not empty, then some additional interior controls are necessary since the eigenvectors associated with such eigenvalues are zero at the nodes.

Now if we want to control on a subset of $\mathcal{V}\setminus\mathcal{D}$ then the assumption for the exponential stability does not necessary hold for the eigenvalues of S_2. Let us then describe how we proceed in that case. For a fixed $\lambda^2 \in S_2$, we denote by $(\varphi_i^{app}(v))_{v\in\mathcal{V}\setminus\mathcal{D}} \in \mathbb{R}^m$, $i = 1, \cdots, l$, the eigenvectors of $\mathcal{C}$ of eigenvalue $\cos\lambda$ such that their corresponding eigenvectors on the network are approximated orthonormal, namely

$$(\varphi_i^{app}(v))^{\top}_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{F}(\lambda_0)(\varphi_j^{app}(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \delta_{ij},$$

This basis can be computed for all λ since it depends only on λ_0 (which form a finite set). Denote by $\mathcal{M}^{app}(\lambda_0^2)$, the matrix build as $\mathcal{M}(\lambda^2)$ by replacing $(\varphi_i(v))_{v\in\mathcal{V}_c}$ by $(\varphi_i^{app}(v))_{v\in\mathcal{V}_c}$, namely

$$\mathcal{M}^{app}(\lambda_0^2) = \begin{pmatrix} (\varphi_1^{app}(v))^{\top}_{v\in\mathcal{V}_c} \\ \vdots \\ (\varphi_l^{app}(v))^{\top}_{v\in\mathcal{V}_c} \end{pmatrix} \cdot \left((\varphi_1^{app}(v))_{v\in\mathcal{V}_c} \quad \cdots \quad (\varphi_l^{app}(v))_{v\in\mathcal{V}_c}\right).$$

Now we can state the

Lemma 1.7.15. *Let $\lambda^2 \in S_2$ be fixed. If $\mathcal{M}^{app}(\lambda_0^2)$ is positive definite and if $\mathcal{C}$ has no eigenvector associated with* $\cos\lambda_0$ *identically equal to zero at the nodes of $\mathcal{V}_c$, then $\mathcal{M}(\lambda^2)$ is uniformly positive definite.*

Proof: As before we show that

$$\mathcal{M}(\lambda^2) = \mathcal{M}^{app}(\lambda_0^2) + \mathcal{M}_r(\lambda^2),$$

where

$$\|\mathcal{M}_r(\lambda^2)\|_2 \lesssim \frac{1}{\lambda}.$$

Consequently for λ large enough, the uniform positive definiteness follows from the positive definiteness of $\mathcal{M}^{app}(\lambda_0^2)$. On the contrary for small λ, it suffices to remark that $\mathcal{M}(\lambda)$ has an eigenvalue equal to zero if and only if $\mathcal{C}$ has an eigenvector associated with $\cos\lambda = \cos\lambda_0$ identically equal to zero at the nodes of $\mathcal{V}_c$. Since we have assumed that such eigenvectors do not exist, $\mathcal{M}(\lambda^2)$ is positive definite and the conclusion follows by finiteness. ∎

Note that our above Lemma has only to be used for multiple eigenvalues. Indeed assume that $\lambda^2 \in S_2$ is simple, then denote by $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ its eigenvector such that

$$(\varphi(v))^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{E}(\lambda)(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = 1.$$

Now consider any computed eigenvector $(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}$ of $\mathcal{C}$ that depends only on $\cos\lambda$ and then on λ_0. This eigenvector then satisfies

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} = \mu(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}},$$

with $\mu \in \mathbb{R}$ such that $\mu^2 = \frac{1}{c(\lambda)}$, when

$$c(\lambda) = (\psi_v)^\top_{v\in\mathcal{V}\setminus\mathcal{D}}\mathcal{E}(\lambda)(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}.$$

Since $c(\lambda)$ remains uniformly bounded from below and from above, it is equivalent to check the uniform definite positiveness of $\mathcal{M}(\lambda^2)$ using $(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}}$ or using $(\psi_v)_{v\in\mathcal{V}\setminus\mathcal{D}}$.

A first example

Consider the tree described by figure 1.6.

By Theorem 1.7.12, the eigenvalues of A are $k^2\pi^2$ of multiplicity $5-4=1$, and λ^2 such that $\cos(\lambda) \in Sp(\mathcal{C})\cap]-1,1[$.

1st case : We easily check that the eigenvector associated with $k^2\pi^2$ is given by

$$\varphi_k(x) = \sqrt{\frac{2}{3}}\sin(k\pi x)\begin{pmatrix} 1 \\ 0 \\ (-1)^k \\ 1 \\ 0 \end{pmatrix}.$$

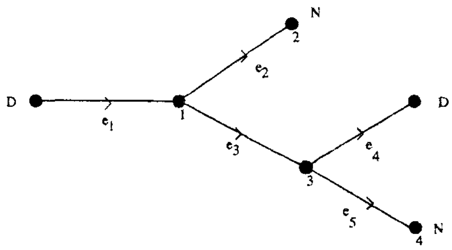

FIG. 1.6 –

Since the eigenvector is zero at all nodes of the network, feedbacks at the nodes are not sufficient to stabilize the system. But if we take a control at ξ_1 on the edge e_1 for instance, we have

$$\varphi_1^2(\xi_1) = \frac{2}{3}\sin^2(k\pi\xi_1) \gtrsim \frac{1}{k^2} \text{ for } \xi_1 \in \mathcal{S}.$$

<u>2d case</u> : The eigenvalues of the matrix

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{3} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

are $-\frac{1}{6}+\frac{1}{6}\sqrt{13}$, $-\frac{1}{6}-\frac{1}{6}\sqrt{13}$, $\frac{1}{6}+\frac{1}{6}\sqrt{13}$, $\frac{1}{6}-\frac{1}{6}\sqrt{13} \in]-1,\ 1[$ (of multiplicity 1) of associated eigenvectors

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} \simeq \begin{pmatrix} -0.75 \\ -1 \\ 0.75 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.3 \\ -1 \\ -1.3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.3 \\ 1 \\ 1.3 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.75 \\ 1 \\ -0.75 \\ 1 \end{pmatrix},$$

respectively. Then we can consider a feedback at any node of $\mathcal{V} \setminus \mathcal{D}$, for instance at the node number 4 (see Fig. 1.7).

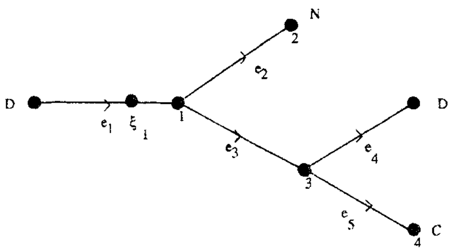

FIG. 1.7 –

We then have the next result.

Proposition 1.7.16. *If we control at $\xi_1 \in \mathcal{S}$ on the edge e_1 and at one of the vertices of $\mathcal{V} \setminus \mathcal{D}$, for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v\cdot))) \in D(\mathcal{A})$, the energy decays like $\frac{1}{t}$.*

A second example

Consider the tree as described in figure 1.8.

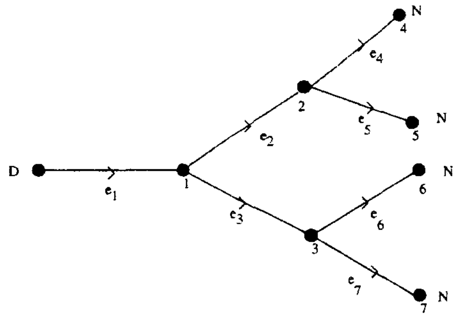

FIG. 1.8 –

By Theorem 1.7.12, the operator A has only the eigenvalues λ^2 such that $\cos\lambda \in Sp(\mathcal{C})\cap]-1, 1[$. Therefore by Corollary 1.7.14, if we control at all nodes except the Dirichlet one, then we obtain an exponential decay. Without delay we find the result of [109].

Note that it suffices to control at the nodes 5, 6 and 7. Indeed the eigenvalues of the matrix

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

are approximatively $0.9, -0.9, 0.8, -0.8, 0, 0, 0 \in [-1, 1]$ of eigenvector

$$(\varphi(v))_{v\in\mathcal{V}\setminus\mathcal{D}} \simeq \begin{pmatrix} 1.15 \\ 1.6 \\ 1.6 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1.15 \\ -1.6 \\ -1.6 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1.4 \\ 1.4 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.4 \\ -1.4 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1.7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -1.7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

respectively. Therefore if we control at the nodes 5, 6 and 7 the assumption for the exponential stability holds for the simple eigenvalues corresponding to 0.9, -0.9, 0.8, -0.8. Now for $\cos\lambda = 0$, i.e. $\lambda = \frac{\pi}{2} + k\pi$, we see that $\mathcal{C}$ has no eigenvectors which are zero at the nodes 5, 6 and 7. Furthermore we easily check that $\mathcal{F}(\frac{\pi^2}{4})$ is the diagonal matrix with entries $(3/2, 3/2, 3/2, 1/2, 1/2, 1/2, 1/2)$ and then

$$\varphi_1^{app} = \sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \varphi_2^{app} = \sqrt{2}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}, \varphi_3^{app} = \sqrt{2}\begin{pmatrix} -2/9 \\ 0 \\ 0 \\ 1/9 \\ 1/9 \\ -7/9 \\ 1 \end{pmatrix}.$$

By direct calculations we obtain

$$\mathcal{M}^{app}(\frac{\pi^2}{4}) = 2\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & \frac{5}{4} & -\frac{13}{18} \\ \frac{1}{9} & -\frac{13}{18} & -\frac{131}{81} \end{pmatrix}.$$

Since this matrix is positive definite, by Lemma 1.7.15 we deduce the uniform positive definiteness of $\mathcal{M}((\frac{\pi}{2} + k\pi)^2)$.

In other words, we have proved the

Proposition 1.7.17. *If the feedback law is at the vertices 5, 6 and 7, then the energy decays exponentially in the energy space.*

Note that if we impose Dirichlet boundary conditions at the nodes 4 and 6, then the system is no more exponentially stable.

A network with a circuit

Similar considerations than before allow to prove the following result :

Proposition 1.7.18. *Consider the network described by figure 1.9. If the feedback law is at $\xi_1 \in \mathcal{S}$ on the edge e_2, at $\xi_2 \in \mathcal{S}$ on the edge e_3, at $\xi_3 \in \mathcal{S}$ on the edge e_4 and at the vertex 6 or 7, then for all $(u^{(0)}, u^{(1)}, (f^0(-\tau_v.))) \in D(\mathcal{A})$, the energy decays like $\frac{1}{t}$.*

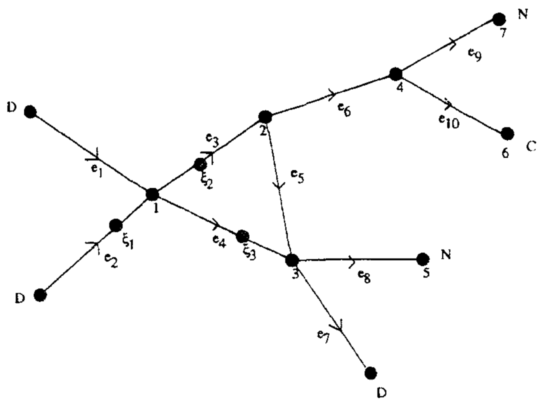

FIG. 1.9 –

Chapitre 2

Weak stabilization of the wave equation on 1-d networks

2.1 Introduction and main results

In this chapter, we consider a planar network of elastic strings that undergoes small perpendicular vibrations. Recently, the control, observation and stabilization problems of these networks have been the object of intensive research (see [38, 65] and the references therein).

Here we are interested in the problem of stabilization of the network by means of a damping term located on one single exterior node. The aim of this chapter is to develop a systematic method to address this issue and to give a general result allowing to transform an observability result for the corresponding conservative system into a stabilization one for the damped one.

Before going on, let us recall some definitions and notation about $1-d$ networks used in the chapter. We refer to [2, 20, 82] for more details.

A $1-d$ network $\mathcal{R}$ is a connected set of $\mathbb{R}^n$, $n \geq 1$, defined by

$$\mathcal{R} = \bigcup_{j=1}^{N} e_j$$

where e_j is a curve that we identify with the interval $(0,\, l_j)$, $l_j > 0$, and such that for $k \neq j$, $\overline{e_j} \cap \overline{e_k}$ is either empty or a common end called a vertex or a node (here $\overline{e_j}$ stands for the closure of e_j).

For a function $u : \mathcal{R} \longrightarrow \mathbb{R}$, we set $u_j = u_{|e_j}$ the restriction of u to the edge e_j.

We denote by $\mathcal{E} = \{e_j\, ;\, 1 \leq j \leq N\}$ the set of edges of $\mathcal{R}$ and by $\mathcal{V}$ the set of vertices of $\mathcal{R}$. For a fixed vertex v, let

$$\mathcal{E}_v \;=\; \{j \in \{1, ..., N\}\, ;\, v \in \overline{e_j}\}$$

be the set of edges having v as vertex. If card $(\mathcal{E}_v) = 1$, v is an exterior node, while if card

$(\mathcal{E}_v) \geq 2$, v is an interior one. We denote by $\mathcal{V}_{ext}$ the set of exterior nodes and by $\mathcal{V}_{int}$ the set of interior ones. For $v \in \mathcal{V}_{ext}$, the single element of $\mathcal{E}_v$ is denoted by j_v.

We now fix a partition of $\mathcal{V}_{ext} : \mathcal{V}_{ext} = \mathcal{D} \cup \{v_1\}$. In this way, we distinguish the conservative exterior nodes, $\mathcal{D}$, in which we impose Dirichlet homogeneous boundary condition, and the one in which the damping term is effective, v_1. To simplify the notation, we will assume that v_1 is located at the end 0 of the edge e_1.

Let $u_j = u_j(t, x) : \mathbb{R} \times [0, l_j] \to \mathbb{R}$ be the function describing the transversal displacement in time t of the string e_j of length l_j. Let us denote by L the sum of the lengths of all edges of the network, the total length of the network.

We assume that the displacements u_j satisfy the following system

$$\left\{ \begin{array}{ll} \frac{\partial^2 u_j}{\partial t^2}(x, t) - \frac{\partial^2 u_j}{\partial x^2}(x, t) = 0 & 0 < x < l_j,\ t > 0,\ \forall j \in \{1, ..., N\}, \\ u_j(v, t) = u_l(v, t) & \forall j,\ l \in \mathcal{E}_v,\ v \in \mathcal{V}_{int}, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v, t) = 0 & \forall v \in \mathcal{V}_{int}, t > 0, \\ u_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial u_1}{\partial x}(0, t) = \frac{\partial u_1}{\partial t}(0, t) & \forall t > 0, \\ u(t=0) = u^{(0)},\ \frac{\partial u}{\partial t}(t=0) = u^{(1)}, & \end{array} \right. \tag{2.1}$$

where $\partial u_j / \partial n_j(v, .)$ stands for the outward normal (space) derivative of u_j at the vertex v. We denote by u the vector $u = (u_j)_{j=1,...,N}$.

The above system has been considered by several authors in some particular situations. We refer, for instance, to [11], [6], [7,8], [9] and [109], where explicit decay rates are obtained for networks with some special structures. We also refer to chapter 1 or [89] where the problem is considered in the presence of delay terms in the feedback law.

The object of this chapter is not to give an additional result in a particular case, but rather to develop a systematic method allowing to address the issue in a general context. We do this transfering known observability results for the corresponding conservative system into stabilization results for the dissipative one. This provides a new proof for the existing results mentioned above and allows getting new ones.

As mentioned above, in this chapter, we consider the case where the dissipation is located on an external node of the network, but the method can be adapted to treat the case where the damping term is located in several nodes, both exterior and interior ones.

In order to study system (2.1) we need a proper functional setting. We define the Hilbert spaces

$$L^2(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R}; u_j \in L^2(0, l_j), \forall j = 1, \cdots, N\},$$

and

$$V := \{\phi \in \prod_{j=1}^{N} H^1(0, l_j)\ :\ \phi_j(v) = \phi_k(v)\, \forall j,\ k \in \mathcal{E}_v, \forall v \in \mathcal{V}_{int}\, ;\ \phi_{j_v}(v) = 0\, \forall v \in \mathcal{D}\},$$

equipped with the natural inner products

$$< \phi, \tilde{\phi} >_{L^2(\mathcal{R})} = \sum_{j=1}^{N} \int_0^{l_j} \phi_j \tilde{\phi}_j dx \text{ and } < \phi, \tilde{\phi} >_V = \sum_{j=1}^{N} \int_0^{l_j} \frac{\partial \phi_j}{\partial x} \frac{\partial \tilde{\phi}_j}{\partial x} dx$$

respectively.

It is well known that system (2.1) may be rewritten as the first order evolution equation

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u^{(0)}, u^{(1)}) = U_0, \end{cases} \tag{2.2}$$

in the Hilbert space

$$H := V \times L^2(\mathcal{R}),$$

equipped with the usual inner product

$$\left\langle \left(\begin{array}{c} u \\ w \end{array} \right), \left(\begin{array}{c} \tilde{u} \\ \tilde{w} \end{array} \right) \right\rangle := \sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial u_j}{\partial x} \frac{\partial \tilde{u}_j}{\partial x} + w_j \tilde{w}_j \right) dx.$$

Here U is the vector $U = (u, \frac{\partial u}{\partial t})^T$, the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \left(\begin{array}{c} u \\ w \end{array} \right) := \left(\begin{array}{c} w \\ \Delta u \end{array} \right),$$

Δ being the Laplace operator : $\Delta u = \left(\frac{\partial^2 u_j}{\partial x^2} \right)_{j=1,\ldots,N}$, and where the domain $D(\mathcal{A})$ of the operator $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{(u, w) \in (V \cap \prod_{j=1}^{N} H^2(0, l_j)) \times V \ : \ \frac{\partial u_1}{\partial x}(0) = w_1(0) \, ; \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = 0, \forall v \in \mathcal{V}_{int}\}.$$

Therefore, we know that for an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (2.2). Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), H).$$

We define the natural energy of u by

$$E_u(t) := \frac{1}{2} \sum_{j=1}^{N} \int_0^{l_j} \left(\left(\frac{\partial u_j}{\partial t} \right)^2 + \left(\frac{\partial u_j}{\partial x} \right)^2 \right) dx. \tag{2.3}$$

This energy satisfies

$$E_u'(t) = - \left(\frac{\partial u_1}{\partial t}(0, t) \right)^2 \leq 0, \tag{2.4}$$

and therefore it is decreasing.

Remark 2.1.1. *Integrating the expression (2.4) between* 0 *and* T, *we obtain*

$$\int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, t)\right)^2 dt = E_u(0) - E_u(T) \leq E_u(0).$$

Consequently, this estimate implies that $\frac{\partial u_1}{\partial t}(0,\,.)$ *belongs to* $L^2(0,\, T)$ *for finite energy solutions. This is a "hidden" regularity property in the sense that it is not a direct consequence of the regularity of finite energy solutions.*

In [6–9, 11, 89] and in chapter 1, the method to obtain the stabilization of (2.1) in particular cases is based on the use of observability estimates for the solutions ϕ of the conservative problem without damping, with Neumann boundary condition at the node $x = 0$:

$$\begin{cases} \frac{\partial^2 \phi_j}{\partial t^2}(x,\, t) - \frac{\partial^2 \phi_j}{\partial x^2}(x,\, t) = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., N\}, \\ \phi_j(v,\, t) = \phi_l(v,\, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum\limits_{j \in \mathcal{E}_v} \frac{\partial \phi_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int}, t > 0, \\ \phi_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \frac{\partial \phi_1}{\partial x}(0,\, t) = 0 & \forall t > 0, \\ \phi(t = 0) = u^{(0)},\, \frac{\partial \phi}{\partial t}(t = 0) = u^{(1)}. & \end{cases} \tag{2.5}$$

It is well known that this problem is well posed in the natural energy space. If we suppose that $(u^{(0)},\, u^{(1)}) \in H$, then problem (2.5) admits a unique solution

$$\phi \in C(0,\, T;\, V) \cap C^1(0,\, T;\, L^2(\mathcal{R})).$$

This system is obviously conservative and its energy is constant : $E_\phi(t) = E_\phi(0)$ for all $t > 0$. Let us then denote by $(\lambda_n^2)_{n \geq 1}$ the sequence of eigenvalues of (2.5) and let $(\varphi_n)_{n \geq 1}$ be the corresponding eigenvectors forming an orthonormal basis of $L^2(\mathcal{R})$. In particular cases, in [6–9, 11, 89], under appropriate conditions, observability inequalities of the form

$$\sum_{n \geq 1} c_n^2 (\lambda_n^2 a_n^2 + b_n^2) \leq C \int_0^T \left|\frac{\partial \phi_1}{\partial t}(0,\, t)\right|^2 dt, \tag{2.6}$$

are proved for system (2.5), where a_n, b_n are the Fourier coefficients of the initial data of ϕ in the basis $(\varphi_n)_n$, and with weights $c_n^2 > 0$ depending on the network. To obtain the stabilization result from the observability inequality (2.6), the authors decompose the solution u as the sum of ϕ, solution of (2.5) with the same initial data than u, and a rest with vanishing initial data. Then they show regularity results of trace type for this rest (see section 1.4). However, in these articles and in chapter 1, the analysis is limited to strong observability inequalities (leading to exponential or polynomial decay results) which hold only for a restricted class of networks. In the present chapter we are able to deal with arbitrary networks in which weaker observability inequalities may hold leading to arbitrarily slow decay rates.

Our analysis is based on the existing observability results for the following system with Dirichlet boundary condition at all exterior nodes

$$\begin{cases} \frac{\partial^2 \psi_j}{\partial t^2}(x, t) - \frac{\partial^2 \psi_j}{\partial x^2}(x, t) = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., N\}, \\ \psi_j(v, t) = \psi_l(v, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial \psi_j}{\partial n_j}(v, t) = 0 & \forall v \in \mathcal{V}_{int}, t > 0, \\ \psi_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \psi_1(0, t) = 0 & t > 0, \\ \psi(t = 0) = \psi^{(0)},\ \frac{\partial \psi}{\partial t}(t = 0) = \psi^{(1)}. & \end{cases} \tag{2.7}$$

The study of the observability of this system is motivated by control problems (see [33–38]).

The difference between systems (2.5) and (2.7) is the boundary condition at the end 0 of e_1 : Dirichlet boundary condition for (2.7) and Neumann boundary condition for (2.5).

If we suppose that $(\psi^{(0)}, \psi^{(1)}) \in \tilde{H} = \tilde{V} \times L^2(\mathcal{R})$ where

$$\tilde{V} = \{\psi \in \prod_{j=1}^{N} H^1(0, l_j) \,:\, \psi_j(v) = \psi_k(v) \forall j,\, k \in \mathcal{E}_v,\, \forall v \in \mathcal{V}_{int};\, \psi_{j_v}(v) = 0,\, \forall v \in \mathcal{D};\, \psi_1(0) = 0\},$$

then problem (2.7) admits a unique solution

$$\psi \in C(0, T; \tilde{V}) \cap C^1(0, T; L^2(\mathcal{R})).$$

This system is obviously also conservative.

In this chapter, we show the link between the existing observability results for (2.7) and the stabilization of (2.1). To be more precise, let us denote by $(\tilde{\lambda}_n^2)_{n\geq 1}$ the sequence of eigenvalues corresponding to problem (2.7) and let $(\varphi_n^D)_{n\geq 1}$ be the corresponding eigenvectors forming an orthonormal basis of $L^2(\mathcal{R})$. Under some conditions on the topology of the network and the lengths of the strings entering in it, Dager and Zuazua [33–38] proved weighted observability inequalities for (2.7) of the following form

$$E_*^D(\psi, 0) := \sum_{n\geq 1} c_n^2(\tilde{\lambda}_n^2 \psi_{0,n}^2 + \psi_{1,n}^2) \leq C \int_0^T \left| \frac{\partial \psi_1}{\partial x}(0, t) \right|^2 dt, \tag{2.8}$$

for a positive constant C, where $\psi_{0,n}$, $\psi_{1,n}$ are the Fourier coefficients of the initial data of ψ in the basis $(\varphi_n^D)_n$ and with positive weights $(c_n^2)_{n\geq 1}$ depending on the properties of the network.

To obtain this weighted observability estimate (2.8), several methods have been developed in those articles. The first one, that uses the D'Alembert representation formula, applies to tree-shaped networks. Its main advantage is that it does not require computing the spectrum of the network. This method has been later adapted in [7] and [9] to analyze the stabilization of stars and trees. A second method uses the Fourier expansion of solutions and the Beurling-Malliavin Theorem (see for instance [37, 38, 89]). This applies for general networks and avoids the difficulties related to applying the D'Alembert formula to more complex networks that may contain circuits.

The key observation of this chapter is that the weighted observability estimates for u solution of (2.1) can be obtained directly from (2.8). This method is of systematic application and avoids remaking all the fine analysis already carried out in [33, 38], that uses, in particular, tools from Number Theory, to analyze the properties of the weights $\{c_n^2\}_{n\geq 1}$ in terms of those of the network under consideration.

Let us now explain how (2.8) can be applied directly in our context. For that we decompose u as the sum of w, a solution of (2.7) with appropriate initial data $(u^{(0)} - u_1^{(0)}(0)\varphi,\, u^{(1)})$ (where φ is a given smooth function such that $\varphi_1(0) = 1$), and a rest. Applying (2.8) to the solution w of (2.7), we obtain the following weighted observability estimate for u solution of (2.1)

$$E_*^D(w,\, 0) + u_1^{(0)}(0)^2 \leq C_T \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, t)\right)^2 dt, \tag{2.9}$$

where $E_*^D(w,\, 0)$ is defined by (2.8), with weights $(c_n^2)_n$ depending on the network. If the weights c_n^2 are non-trivial for all $n \in \mathbb{N}^*$, the energy of the dissipative system tends to 0 as $t \to \infty$. However, in general, the weights tend to 0 as $n \to \infty$, the observed quantity is weaker than the $V \times L^2(\mathcal{R})$-norm and the decay rate is not exponential.

It is important to underline that (2.9) holds under the same assumptions on the network needed for (2.8) to hold for the Dirichlet problem (2.7). Thus, no further analysis is required to establish the existence of the weights $\{c_n^2\}_{n\geq 1}$ in terms of the properties of the network.

To derive decay properties out of (2.9), we view this inequality as a weak observability estimate in which the observed energy $E_-(0)$ is equal, roughly speaking, to $E_*^D(w,\, 0) + u_1^{(0)}(0)^2$. In practice we often take, if necessary, the lower convex envelop of c_n^2 instead of the weights c_n^2 in the definition of E_*^D (see Section 3.1). The observed energy E_- is weaker than the $V \times L^2(\mathcal{R})$-norm of the initial data that would be required to prove exponential decay and, consequently, we obtain weaker decay rates. To obtain explicit decay rates out of this weak observability inequality we use an interpolation inequality which is a variant of the one from Bégout and Soria [19] and which is a generalization of Hölder's inequality. For this to be done we need to assume more regularity of the initial data. To be more precise we shall consider initial data $(u^{(0)},\, u^{(1)}) \in X_s := [D(\mathcal{A}),\, D(\mathcal{A}^0)]_{1-s}$ for $0 < s < 1/2$. In this way we deduce an interpolation inequality of the form

$$1 \leq \Phi_s\left(\frac{E_-(0)}{CE_u(0)}\right) \frac{\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}{C'E_u(0)},$$

where Φ_s is an increasing function which depends on s and on the energy E_- under consideration.

The previous interpolation inequality implies

$$E_-(0) \geq CE_u(0)\Phi_s^{-1}\left(\frac{E_u(0)}{C' \left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}\right).$$

With (2.4) and (2.9), we obtain

$$E_u(0) - E_u(T) \geq CE_u(0)\Phi_s^{-1}\left(\frac{E_u(0)}{C' \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2}\right),$$

which implies, by the semigroup property (see Ammari and Tucsnak [11])

$$\forall t > 0,\ E_u(t) \leq C\Phi_s\left(\frac{1}{t+1}\right)\left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2. \tag{2.10}$$

Obviously, the decay rate in (2.10) depends on the behaviour of the function Φ_s near 0. Thus, in order to determine the explicit decay rate we need to have a sharp description of the function Φ_s, which depends on s and on the energies E and E_- and thus, on the weights $(c_n^2)_n$ of (2.9) in an essential way. These weights depend on the topology of the network and the number theoretical properties of the lengths of the strings entering in it.

This approach allows getting in a systematic way decay rates for the energy of smooth solutions of the damped system as a consequence of the observability properties of the undamped one.

The analysis in this chapter is limited to networks of strings with damping in one single end but the same methods, combined with the tools developped in [38], can be applied in other situations as, for instance :

a) networks of strings with damping in several end points;

b) networks with damping on end points and internal nodes;

c) networks of beams.

The chapter is organized as follows. In the second section, we show how to pass from the observability inequality for the conservative problem (2.7) with Dirichlet boundary conditions at all exterior nodes to the weighted observability inequality (2.9) for (2.1). In section 2.3 we give an interpolation inequality which is a variant of the one by Bégout and Soria [19] and we apply it to obtain the explicit decay estimate of the energy. Finally we end up discussing some illustrative examples in section 2.4.

2.2 The weighted observability inequality

In this section, we prove that we can obtain a weighted observability estimate for u solution of (2.1) directly from (2.8). First of all, we recall some results about system (2.7) with Dirichlet boundary conditions at all exterior nodes.

2.2.1 Preliminaries about the Dirichlet problem

Recall that if we suppose that $(\psi^{(0)}, \psi^{(1)}) \in \tilde{H}$, then problem (2.7) admits a unique solution

$$\psi \in C(0, T; \tilde{V}) \cap C^1(0, T; L^2(\mathcal{R})).$$

Denote by $(\tilde{\lambda}_n^2)_{n\geq 1}$ the sequence of eigenvalues corresponding to problem (2.7) and let $(\varphi_n^D)_{n\geq 1}$ be the corresponding eigenvectors forming an orthonormal basis of $L^2(\mathcal{R})$. We assume now that $(\psi^{(0)}, \psi^{(1)}) \in \tilde{H}$ and we set

$$\psi^{(0)} = \sum_{n\geq 1} \psi_{0,n}\varphi_n^D \text{ and } \psi^{(1)} = \sum_{n\geq 1} \psi_{1,n}\varphi_n^D,$$

where $(\tilde{\lambda}_n\psi_{0,n})_n$, $(\psi_{1,n})_n \in l^2(\mathbb{N}^*)$.

In [38], a weighted observability inequality is derived, motivated by control problems. More precisely, it is shown that, under some conditions on the topology of the network and the lengths of the strings entering in it, for all $T > T_0$ (where $T_0 > 0$ is large enough), there exists a sequence of positive weights $\{c_n^2\}_{n\geq 1}$ and a positive constant C such that for all solution ψ of (2.7) it holds

$$\sum_{n\geq 1} c_n^2(\tilde{\lambda}_n^2\psi_{0,n}^2 + \psi_{1,n}^2) \leq C\int_0^T \left|\frac{\partial\psi_1}{\partial x}(0, t)\right|^2 dt. \tag{2.11}$$

Notice that, for this to be true, it is essential that the time $T_0 > 0$ to be large enough. More precisely, in [38] inequality (2.11) is proved in the case of general networks for $T > 2L$, where L is the total length of the network. In the case of tree-like networks, (2.11) is obtained by the D'Alembert representation formula and the weights c_n^2 are strictly positive for every $n \in \mathbb{N}^*$ if the tree is non-degenerate. In the case of stars, this comes down to impose irrationality conditions on the ratio of each pair of lengths. In the case of trees, the condition of non-degeneracy of the network is a natural extension of the irrationality one for stars and is also generically true (see [33, 36, 38] for more details). In the case of general networks, (2.11) is established by means of the Beurling-Malliavin Theorem under the assumption that all eigenfunctions have a non-trivial Neumann trace at the observation node. This condition is generically satisfied.

We set

$$E_*^D(\psi, 0) := \frac{1}{2}\sum_{n\geq 1} c_n^2(\tilde{\lambda}_n^2\psi_{0,n}^2 + \psi_{1,n}^2) \tag{2.12}$$

the weighted energy for ψ at time 0. Note that $(E_*^D)^{\frac{1}{2}}$ defines a norm on $\tilde{H}$ because the weights c_n^2, according to the results in [38], are assumed to be positive. In the sequel, we assume that the network under consideration is such that the solutions ψ of (2.7) satisfy (2.11).

2.2.2 The weighted observability inequality

In this section, we prove that we can obtain a weighted observability estimate for u solution of (2.1) directly from (2.11). Thus, we assume that the network is such that (2.11) holds and $(u^{(0)}, u^{(1)}) \in H$.

We introduce w solution of (2.7) with initial data $(w^{(0)}, w^{(1)}) = (u^{(0)} - u_1^{(0)}(0)\varphi, u^{(1)})$, where φ is a given smooth function satisfying

$$\begin{cases} \varphi_1(0) = 1 & \\ \varphi_{j_v}(v) = 0 & \forall v \in \mathcal{D} \\ \varphi_j(v) = \varphi_l(v) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int} \\ \displaystyle\sum_{j \in \mathcal{E}_v} \frac{\partial \varphi}{\partial n_j}(v) = 0 & \forall v \in \mathcal{V}_{int}. \end{cases} \tag{2.13}$$

In this manner, the initial data $(w^{(0)}, w^{(1)})$ belong to $\tilde{H}$, i.e. $w^{(0)}$ satisfies the Dirichlet boundary condition at the end 0 of e_1. Therefore, by hypothesis, w satisfies (2.11).

We also consider ϵ, the solution of the following non-homogeneous Dirichlet problem :

$$\begin{cases} \frac{\partial^2 \epsilon_j}{\partial t^2}(x,\, t) - \frac{\partial^2 \epsilon_j}{\partial x^2}(x,\, t) = 0 & \forall x \in (0,\, l_j),\, t > 0,\, \forall j \in \{1, ..., N\}, \\ \epsilon_j(v,\, t) = \epsilon_l(v,\, t) & \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, t > 0, \\ \displaystyle\sum_{j \in \mathcal{E}_v} \frac{\partial \epsilon_j}{\partial n_j}(v,\, t) = 0 & \forall v \in \mathcal{V}_{int}, t > 0, \\ \epsilon_{j_v}(v, t) = 0 & \forall v \in \mathcal{D}, t > 0, \\ \epsilon_1(0,\, t) = u_1(0,\, t) & t > 0, \\ \epsilon(t = 0) = u_1^{(0)}(0)\varphi,\ \frac{\partial \epsilon}{\partial t}(t = 0) = 0. & \end{cases} \tag{2.14}$$

Note that ϵ satisfies a non-homogeneous Dirichlet boundary condition at $x = 0$. Actually it coincides with the value of the solution u_1 of (2.1) at that point. By Remark 2.1.1, we notice that $\partial u_1/\partial t(0,\, .) \in L^2(0,\, T)$, so that the non-homogeneous Dirichlet boundary condition belongs to $H^1(0,\, T)$.

In this way we have the decomposition

$$u = w + \epsilon. \tag{2.15}$$

In this section, we prove the following theorem

Theorem 2.2.1. *Assume that the network is such that the weighted observability inequality (2.11) is satisfied for the conservative system (2.7) with Dirichlet boundary conditions at all exterior nodes. We split up u, solution of (2.1), as (2.15) where w is solution of (2.7) with initial data $(u^{(0)} - u_1^{(0)}(0)\varphi,\, u^{(1)})$ and ϵ is solution of (2.14). We define $E_*(u,\, 0)$ by*

$$E_*(u,\, 0) := E_*^D(w,\, 0) + u_1^{(0)}(0)^2, \tag{2.16}$$

where $E_*^D(w,\, 0)$ is defined by (2.12). Then for all $T > T_0$, there exists $C_T > 0$ such that all solution u of (2.1) satisfies the weighted observability inequality

$$E_*(u,\, 0) \le C_T \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, t) \right)^2 dt, \tag{2.17}$$

provided $(u^{(0)},\, u^{(1)}) \in H$.

Remark 2.2.2. *Note that, according to Theorem 2.2.1, we transform the observability inequality for the Dirichlet problem (2.7) into a similar one for the dissipative one (2.1). Thus, in particular, estimate (2.17) holds under the same assumptions on the network that are needed for the Dirichlet problem and that are discussed in [38]. Some examples will be discussed in Section 4.*

Note also that $(E_*(u,\,0))^{\frac{1}{2}}$ *defines a norm in the space of initial data* $(u^{(0)},\,u^{(1)}) \in H$. *Indeed, when* $E_*(u,\,0)$ *vanishes,* $u_1^{(0)}(0) = 0$. *Thus* $(u^{(0)},\,u^{(1)}) \in \tilde{H}$ *and then* $E_*(u,\,0) = E_*^D(u,\,0)$, *and, by assumption,* $\left(E_*^D(u,\,0)\right)^{\frac{1}{2}}$ *defines a norm in* $\tilde{H}$.

2.2.3 Proof of Theorem 2.2.1

Let $(u^{(0)},\,u^{(1)}) \in H$. We decompose u as in (2.15) where w and ϵ solve (2.7) and (2.14) respectively. Therefore, w is solution of (2.7) with initial data $(w^{(0)},\,w^{(1)}) = (u^{(0)} - u_1^{(0)}(0)\varphi,\,u^{(1)})$, and thus, by hypothesis, it satisfies (2.11).

First, we have the following lemma

Lemma 2.2.3. *For all* $T > 0$ *there exists* $C_T > 0$ *such that the solutions* u *of (2.1) and* ϵ *of (2.14) satisfy the following estimate*

$$\int_0^T \left(\frac{\partial \epsilon_1}{\partial x}(0,\,t)\right)^2 dt \leq C_T \left(\int_0^T \left(\frac{\partial u_1}{\partial t}(0,\,t)\right)^2 dt + \left(u_1^{(0)}(0)\right)^2\right). \tag{2.18}$$

Remark 2.2.4. *Note that (2.18) holds for all networks since, in fact, it holds locally near the boundary of a single string. In this context we apply it along the string containing the extreme* $x = 0$, *getting (2.18).*

Proof: First of all, we easily show that the energy E_ϵ of the solution ϵ (defined as in (2.3)) satisfies

$$\frac{d}{dt}E_\epsilon(t) = -\frac{\partial \epsilon_1}{\partial x}(0,\,t)\frac{\partial u_1}{\partial t}(0,\,t),$$

and then for all $t > 0$

$$E_\epsilon(t) = E_\epsilon(0) - \int_0^t \frac{\partial \epsilon_1}{\partial x}(0,\,s)\frac{\partial u_1}{\partial t}(0,\,s)ds.$$

Therefore, by using Cauchy's inequality, we obtain

$$E_\epsilon(t) \leq E_\epsilon(0) + \frac{1}{4}\int_0^t \left(\frac{\partial \epsilon_1}{\partial x}(0,\,s)\right)^2 ds + \int_0^t \left(\frac{\partial u_1}{\partial t}(0,\,s)\right)^2 ds. \tag{2.19}$$

Secondly, for all $t \in (0,\,T]$, multiplying the wave equation satisfied by ϵ_1 by $(l_1 - x)\frac{\partial \epsilon_1}{\partial x}$ and by integrating on $(0,\,l_1) \times (0,\,t)$, we have

$$\int_0^t \int_0^{l_1} (l_1 - x)\frac{\partial \epsilon_1}{\partial x}\frac{\partial^2 \epsilon_1}{\partial t^2}dxds - \int_0^t \int_0^{l_1} (l_1 - x)\frac{\partial \epsilon_1}{\partial x}\frac{\partial^2 \epsilon_1}{\partial x^2}dxds = 0.$$

By integration by parts, we have

$$\begin{aligned}
\int_0^t \int_0^{l_1} (l_1 - x) \frac{\partial \epsilon_1}{\partial x} \frac{\partial^2 \epsilon_1}{\partial t^2} dx ds &= -\int_0^t \int_0^{l_1} (l_1 - x) \frac{\partial^2 \epsilon_1}{\partial xt} \frac{\partial \epsilon_1}{\partial t} dx ds \\
&\quad + \int_0^{l_1} (l_1 - x) \frac{\partial \epsilon_1}{\partial x}(s,\, x) \frac{\partial \epsilon_1}{\partial t}(s,\, x) dx \Big|_0^t \\
&= -\frac{1}{2} \int_0^t \int_0^{l_1} \left(\frac{\partial \epsilon_1}{\partial t}\right)^2 dx ds + \frac{l_1}{2} \int_0^t \left(\frac{\partial u_1}{\partial t}(0,\, s)\right)^2 ds \\
&\quad + \int_0^{l_1} (l_1 - x) \frac{\partial \epsilon_1}{\partial x}(t,\, x) \frac{\partial \epsilon_1}{\partial t}(t,\, x) dx,
\end{aligned}$$

because $\frac{\partial \epsilon}{\partial t}(t = 0) = 0$, and

$$\begin{aligned}
\int_0^t \int_0^{l_1} (l_1 - x) \frac{\partial \epsilon_1}{\partial x} \frac{\partial^2 \epsilon_1}{\partial x^2} dx ds &= \frac{1}{2} \int_0^t \int_0^{l_1} (l_1 - x) \frac{\partial}{\partial x} \left(\left(\frac{\partial \epsilon_1}{\partial x}\right)^2\right) dx ds \\
&= \frac{1}{2} \int_0^t \int_0^{l_1} \left(\frac{\partial \epsilon_1}{\partial x}\right)^2 dx ds - \frac{l_1}{2} \int_0^t \left(\frac{\partial \epsilon_1}{\partial x}(0,\, s)\right)^2 ds.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
-\frac{1}{2} \int_0^t \int_0^{l_1} \left(\frac{\partial \epsilon_1}{\partial t}\right)^2 dx ds + \frac{l_1}{2} \int_0^t \left(\frac{\partial u_1}{\partial t}(0,\, s)\right)^2 ds + \int_0^{l_1} (l_1 - x) \frac{\partial \epsilon_1}{\partial x}(t,\, x) \frac{\partial \epsilon_1}{\partial t}(t,\, x) dx \\
-\frac{1}{2} \int_0^t \int_0^{l_1} \left(\frac{\partial \epsilon_1}{\partial x}\right)^2 dx ds + \frac{l_1}{2} \int_0^t \left(\frac{\partial \epsilon_1}{\partial x}(0,\, s)\right)^2 ds = 0,
\end{aligned}$$

and thus

$$\int_0^t \left(\frac{\partial \epsilon_1}{\partial x}(0,\, s)\right)^2 ds \leq \frac{2}{l_1} \int_0^t E_\epsilon(s) ds + 2E_\epsilon(t). \tag{2.20}$$

By grouping (2.19) and (2.20), we find

$$E_\epsilon(t) \leq E_\epsilon(0) + \frac{1}{2l_1} \int_0^t E_\epsilon(s) ds + \frac{1}{2} E_\epsilon(t) + \int_0^t \left(\frac{\partial u_1}{\partial t}(0,\, s)\right)^2 ds.$$

That is to say

$$E_\epsilon(t) \leq 2E_\epsilon(0) + \frac{1}{l_1} \int_0^t E_\epsilon(s) ds + 2 \int_0^t \left(\frac{\partial u_1}{\partial t}(0,\, s)\right)^2 ds.$$

By Gronwall's lemma, we obtain that, for all $t \in (0,\, T]$,

$$E_\epsilon(t) \leq C_T \left(E_\epsilon(0) + \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, s)\right)^2 ds\right),$$

where C_T depends on T. By inserting this expression into (2.20), we find

$$\int_0^T \left(\frac{\partial \epsilon_1}{\partial x}(0,\,t)\right)^2 dt \le C_T \left(E_\epsilon(0) + \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\,t)\right)^2 dt \right).$$

But

$$E_\epsilon(0) = \left(u_1^{(0)}(0)\right)^2 \left(\frac{1}{2}\sum_{j=1}^{N} \int_0^{l_j} \left(\frac{\partial \varphi_j}{\partial x}(x)\right)^2 dx\right) = C\left(u_1^{(0)}(0)\right)^2,$$

where C is a constant, since φ is given as in (2.13), independently of the solution under consideration. This proves (2.18). ■

Let us now return to the proof of Theorem 2.2.1.

Since w satisfies (2.11), we have

$$E_*^D(w,\,0) = E_*^D(w,\,t) \le C \int_0^T \left(\frac{\partial w_1}{\partial x}(0,\,t)\right)^2 dt, \tag{2.21}$$

where $E_*^D(w,\,0)$ is defined as in (2.12) for w solution of (2.7) with initial data $(u^{(0)} - u_1^{(0)}(0)\varphi,\, u^{(1)})$.

Moreover, we have

$$\int_0^T \left(\frac{\partial w_1}{\partial x}(0,\,t)\right)^2 dt \le C \int_0^T \left(\left(\frac{\partial u_1}{\partial x}(0,\,t)\right)^2 + \left(\frac{\partial \epsilon_1}{\partial x}(0,\,t)\right)^2 \right) dt.$$

Therefore, with (2.18) and (2.21), we obtain

$$E_*^D(w,\,0) = E_*^D(w,\,t) \le C_T \int_0^T \left(\left(\frac{\partial u_1}{\partial t}(0,\,t)\right)^2 + \left(\frac{\partial u_1}{\partial x}(0,\,t)\right)^2 \right) dt + C_T \left(u_1^{(0)}(0)\right)^2. \tag{2.22}$$

We recall that $E_*(u,\,0)$ is given by (2.16) where $E_*^D(w,\,0)$ is defined by (2.12). Then, (2.22) becomes

$$E_*(u,\,0) \le C_T \int_0^T \left(\left(\frac{\partial u_1}{\partial t}(0,\,t)\right)^2 + \left(\frac{\partial u_1}{\partial x}(0,\,t)\right)^2 \right) dt + C_T \left(u_1^{(0)}(0)\right)^2. \tag{2.23}$$

In fact, we can remove the last term in the right hand side of (2.23).

Lemma 2.2.5. *For T large enough, the solutions u of (2.1) satisfy*

$$E_*(u,\,0) \le C_T \int_0^T \left(\left(\frac{\partial u_1}{\partial t}(0,\,t)\right)^2 + \left(\frac{\partial u_1}{\partial x}(0,\,t)\right)^2 \right) dt,$$

for a positive constant C_T depending on T.

Proof: In view of (2.23) it is sufficient to show that there exists a positive constant $C > 0$ such that

$$\left|u_1^{(0)}(0)\right|^2 \leq C \int_0^T \left(\left(\frac{\partial u_1}{\partial t}(0, t)\right)^2 + \left(\frac{\partial u_1}{\partial x}(0, t)\right)^2\right) dt.$$

We argue by contradiction. If this is not the case, there exists a sequence of solutions u_n such that

$$\left|u_{n,1}^{(0)}(0)\right| = 1, \forall n \in \mathbb{N} \tag{2.24}$$

and

$$\int_0^T \left(\left(\frac{\partial u_{n,1}}{\partial t}(0, t)\right)^2 + \left(\frac{\partial u_{n,1}}{\partial x}(0, t)\right)^2\right) dx \to 0, \text{ as } n \to +\infty. \tag{2.25}$$

In view of this and (2.23), we deduce that $E_*(u_n, 0)$ are uniformly bounded. Passing weakly to the limit in the Hilbert space $\mathcal{H}$, defined as the closure of V with respect to the norm $(E_*(u_n, 0))^{\frac{1}{2}}$, we obtain a limit solution u such that

$$\left|u_1^{(0)}(0)\right| = 1 \tag{2.26}$$

and

$$\int_0^T \left(\left(\frac{\partial u_1}{\partial t}(0, t)\right)^2 + \left(\frac{\partial u_1}{\partial x}(0, t)\right)^2\right) dx = 0. \tag{2.27}$$

The fact that (2.26) holds is a consequence of the compactness of the trace operator from $\mathcal{R}$ to $\mathbb{R}$ (it is in fact an operator of rank one). On the other hand, (2.27) holds as a consequence of the weak lower semicontinuity. But, by unique continuation, it is easy to see, in view of (2.27), that for all $T > 2L$, the whole limit solution u vanishes. Indeed, we obtain by (2.27)

$$\frac{\partial u_1}{\partial t}(0, t) = \frac{\partial u_1}{\partial x}(0, t) = 0,$$

and therefore $\partial u/\partial t$ solves system (2.7) with initial data $(u^{(1)}, \partial^2 u^{(0)}/\partial x^2)$. Then, we can apply (2.11) to obtain

$$u^{(1)} = \frac{\partial^2 u^{(0)}}{\partial x^2} = 0.$$

Consequently, $\partial u/\partial t = 0$ on $\mathcal{R} \times (0, T)$, and thus u is independent of t. Thus $u = 0$ on $\mathcal{R} \times (0, T)$ since it is harmonic on the network and fulfills the Dirichlet boundary condition. This contradicts (2.26). ∎

This lemma proves Theorem 2.2.1, because u is solution of (2.1).

2.3 The stabilization result

2.3.1 An interpolation inequality

In this subsection, we give an interpolation result similar to the one of [19].

Let $m \in [0, 1)$, $0 < s < 1/2$ and assume that

$$\omega : (m, \infty) \to (0, \omega(m)) \text{ is a convex and decreasing function with } \omega(\infty) = 0, \tag{2.28}$$

$$\Phi_s : (0, \omega(m)) \to (0, \infty) \text{ is a concave and increasing function with } \Phi_s(0) = 0, \tag{2.29}$$

$$\forall t \in [1, \infty),\ 1 \leq \Phi_s(\omega(t))t^{2s}, \tag{2.30}$$

$$\text{The function } t \mapsto \frac{1}{t}\Phi_s^{-1}(t) \text{ is nondecreasing on } (0, 1). \tag{2.31}$$

Before stating the needed interpolation inequality, we recall the inverse Jensen's inequality, which is the inverse version of the classical Jensen's inequality (see Lemma 2.4 from [19] and Rudin [99]).

Lemma 2.3.1 (Inverse Jensen's inequality). *Let (Ω, Υ, ν) be a measure space such that $\nu(\Omega) = 1$ and let $-\infty \leq a < b \leq \infty$. Assume that*

1) $\varphi : (a, b) \to \mathbb{R}$ is a concave function,

2) $g \in L^1(\Omega, \Upsilon, \nu)$ is such that for almost every $x \in \Omega$, $g(x) \in (a, b)$.

Then $\varphi(g)_+ \in L^1(\Omega, \Upsilon, \nu)$ and

$$\int_\Omega \varphi(g)d\nu \leq \varphi\left(\int_\Omega g d\nu\right).$$

Under the conditions (2.29)-(2.30), we have the following result which is a generalized Hölder's inequality, a variant of Theorem 2.1 given in [19] :

Theorem 2.3.2. *Let (ω, Φ_s) be as above satisfying (2.29)-(2.30). Then for any $f = (f_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{N}^*)$, $f \neq 0$, we have*

$$1 \leq \Phi_s\left(\frac{\sum\limits_{n\geq 1}|f_n|\,\omega(n)}{\sum\limits_{n\geq 1}|f_n|}\right)\frac{\sum\limits_{n\geq 1}|f_n|\,n^{2s}}{\sum\limits_{n\geq 1}|f_n|}, \tag{2.32}$$

as soon as $(f_n\omega(n))_n \in l^1(\mathbb{N}^*)$ and $(f_n n^{2s})_n \in l^1(\mathbb{N}^*)$.

Proof: The proof is similar to that of Theorem 2.1 of [19]. We give it for the sake of completeness. By (2.30) and Cauchy-Schwarz's inequality, we have

$$\begin{aligned} 1 = \left(\sum_{n\geq 1}\frac{|f_n|}{\sum\limits_{n\geq 1}|f_n|}\right)^2 &\leq \left(\sum_{n\geq 1}\frac{|f_n|}{\sum\limits_{n\geq 1}|f_n|}\Phi_s^{\frac{1}{2}}(\omega(n))n^s\right)^2 \\ &\leq \left(\sum_{n\geq 1}\frac{|f_n|}{\sum\limits_{n\geq 1}|f_n|}\Phi_s(\omega(n))\right)\left(\sum_{n\geq 1}\frac{|f_n|}{\sum\limits_{n\geq 1}|f_n|}n^{2s}\right). \end{aligned}$$

Now, we apply Lemma 2.3.1 with $\varphi = \Phi_s$ a concave function, $g = \omega$ and the discrete measure $\nu = \sum_{n\geq 1} \frac{|f_n|}{\sum_{n\geq 1}|f_n|}\delta_n$, by noticing that

$$\left(\sum_{n\geq 1}\frac{|f_n|}{\sum_{n\geq 1}|f_n|}\right) = 1.$$

We then obtain (2.32). ∎

We now give some examples of pairs (ω, Φ_s) satisfying (2.28)-(2.31) :

Example 2.3.3. *1. If*

$$\omega(t) = \frac{c}{t^p},$$

for some $p \geq 1$, we can take Φ_s of the form

$$\Phi_s(t) = \left(\frac{t}{c}\right)^{\frac{2s}{p}}.$$

We can easily prove that (ω, Φ_s) satisfy (2.28)-(2.29) with $m = 0$ and (2.30)-(2.31).

2. If

$$\omega(t) = Ce^{-At}$$

where $A > 2(2s+1)$ and $C > 0$, we can take Φ_s of the form

$$\Phi_s(t) = \left(\frac{A}{\ln\left(\frac{C}{t}\right)}\right)^{2s}.$$

We can easily prove that Φ_s is an increasing function on $(0, \omega(0))$, a concave function on $(0, \omega(1/2))$ (because $A > 2(2s-1)$), that $t \mapsto \frac{1}{t}\Phi_s^{-1}(t)$ is nondecreasing on $(0, 1)$ and that the pair (ω, Φ_s) satisfies (2.30) on $[1, \infty)$. Thus (ω, Φ_s) satisfy (2.28)-(2.31) with $m = 1/2$.

In the application of the interpolation inequality of Theorem 2.3.2 to our stabilization problem the weight ω is determined by the weights $(c_n^2)_n$ in (2.12). However, notice that, in general, the weights c_n^2 may degenerate fast and, consequently, we have to work in a more general context.

Moreover, in principle, $(c_n^2)_n$ does not necessarily satisfy the convexity or the monotony property in (2.28). To ensure that this assumption is satisfied, we introduce the notion of the lower convex envelop of a sequence $(u_n)_n$ satisfying $\liminf_{n\to\infty} u_n = 0$. Roughly, it is the "nearest" convex and decreasing function satisfying $0 < \omega(n) \leq u_n$ for all $n \in \mathbb{N}$. The existence of this function is guaranteed by the following lemma :

Lemma 2.3.4. ([19]) *Let $-\infty < a < b \leq \infty$ and let $\varepsilon : [a, b) \to (0, \infty)$ be a continuous function such that $\liminf_{t\to b} \varepsilon(t) = 0$. Then there exists a convex function $\varphi \in C^1([a, b); \mathbb{R})$ such that $0 < \varphi \leq \varepsilon$ and $\varphi' < 0$ on $[a, b)$.*

The proof of this lemma can be found in [19].

Now, we show how to construct the weight ω satisfying (2.28) from $(u_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that $\liminf_{n\to\infty} u_n = 0$. Let $\epsilon \in C([0, \infty); \mathbb{R})$ be such that $0 < \epsilon(n) \leq u_n$, for any $n \in \mathbb{N}$. Let $\varphi \in C([0, \infty); \mathbb{R})$ be a decreasing and convex function such that for any $t \geq 0$, $0 < \varphi(t) \leq \epsilon(t)$ (which exists by the previous lemma) and consider $\mathcal{C} \subset [1, \infty) \times [0, \infty)$ the closure of the convex envelop of the set $\{(n, u_n); n \in \mathbb{N}\}$. Finally, fix arbitrarily $t \geq 1$. Then the set $\mathcal{C}_t = \mathcal{C} \cap (\{t\} \times \mathbb{R})$ is nonempty, closed and the previous lemma ensures that for any $s_t \in \mathbb{R}$ such that $(t, s_t) \in \mathcal{C}_t$,

$$0 < \varphi(t) \leq s_t.$$

So by compactness, we may define the function ω as

$$\forall t \geq 1, \omega(t) = \min \{s_t;\ (t, s_t) \in \mathcal{C}_t\}.$$

Finally we extend ω as a decreasing, continuous and convex function on $[0, 1]$. Therefore, ω satisfies (2.28) with $m = 0$. This function ω is called the lower convex envelop of the sequence $(u_n)_n$.

In the sequel, assuming that the weights in (2.11) are such that $\liminf\limits_{n\to\infty} c_n = 0$ and $c_n \neq 0$ for all $n \in \mathbb{N}^*$, we choose the function ω as follows

$$\omega \text{ is the lower convex envelop of the sequence } \left(1,\ (c_n^2)_{n\geq 1}\right). \tag{2.33}$$

2.3.2 The main results

In this subsection, we assume that the network is such that the weighted observability inequality (2.11) holds, for every solution of problem (2.7). Therefore, by Theorem 2.2.1, u satisfies (2.17). Let us make some remarks :

Remark 2.3.5. *1. If for all $n \in \mathbb{N}^*$, we have $c_n^2 > 0$, then the energy E_u tends to 0 as $t \to \infty$.*

2. If there exists a positive constant c such that for all $n \in \mathbb{N}^*$, $c_n^2 \geq c > 0$, then the energy of u solution of (2.1) is exponentially decreasing. Unfortunately this does not hold in general except, for example, for the simplest network which consists simply of a single string and for trees with all but one damped ends (see section 1.7.3), but never for non-trivial networks with one single damped end.

The first point can be proved by the La Salle's invariance principle and the second one by a classical energy method combining the observability inequality (2.17) and the energy dissipation law (2.4) (see for instance chapter 1 or [89]).

Before stating the main result of this chapter, let us give a technical lemma which will be used in the sequel. For that, we need to define X_s the interpolation space between $D(\mathcal{A})$ and $D(\mathcal{A}^0)$:

$$X_s := \left[D(\mathcal{A}),\ D(\mathcal{A}^0)\right]_{1-s},$$

where $0 < s < 1/2$. Note that if $s = 0$ then $X_0 = D(\mathcal{A}^0) = V \times L^2(\mathcal{R})$ and if $s = 1$ then $X_1 = D(\mathcal{A})$. More precisely we can identify the space X_s :

Lemma 2.3.6. *For* $0 < s < 1/2$,

$$X_s = \left(V \cap \prod_j H^{1+s}(0, l_j)\right) \times \prod_j H^s(0, l_j).$$

Proof: First $D(\mathcal{A})$ is a dense subset of $V \times L^2(\mathcal{R})$ and $\left(V \cap \prod_{j=1}^N H^2(0, l_j)\right) \times V$ is a dense subset of $V \times L^2(\mathcal{R})$ with $D(\mathcal{A}) \subset \left(V \cap \prod_{j=1}^N H^2(0, l_j)\right) \times V$. Then $[D(\mathcal{A}), V \times L^2(\mathcal{R})]_{1-s}$ is a subset of

$$\left[\left(V \cap \prod_{j=1}^N H^2(0, l_j)\right) \times V, V \times L^2(\mathcal{R})\right]_{1-s}$$

with the same norms.

But

$$\left[\left(V \cap \prod_{j=1}^N H^2(0, l_j)\right) \times V, V \times L^2(\mathcal{R})\right]_{1-s} = \left(V \cap \prod_{j=1}^N H^{1+s}(0, l_j)\right) \times \prod_{j=1}^N H^s(0, l_j),$$

and thus, if $D(\mathcal{A})$ is dense in $\left(V \cap \prod_{j=1}^N H^{1+s}(0, l_j)\right) \times \prod_{j=1}^N H^s(0, l_j)$, then, by a classical result of [104],

$$X_s = \left[\left(V \cap \prod_{j=1}^N H^2(0, l_j)\right) \times V, V \times L^2(\mathcal{R})\right]_{1-s} = \left(V \cap \prod_{j=1}^N H^{1+s}(0, l_j)\right) \times \prod_{j=1}^N H^s(0, l_j).$$

To verify that $D(\mathcal{A})$ is dense in $\left(V \cap \prod_{j=1}^N H^{1+s}(0, l_j)\right) \times \prod_{j=1}^N H^s(0, l_j)$, let $(f, g)^T \in \left(V \cap \prod_{j=1}^N H^{1+s}(0, l_j)\right) \times \prod_{j=1}^N H^s(0, l_j)$ be orthogonal to all elements of $D(\mathcal{A})$, namely

$$0 = \left\langle \begin{pmatrix} u \\ w \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \sum_{j=1}^N \int_0^{l_j} \left(\frac{\partial u_j}{\partial x}\frac{\partial f_j}{\partial x} + w_j g_j\right) dx,$$

for all $(u, w)^\top \in D(\mathcal{A})$. We first take $u = 0$ and $w \in \prod_{j=1}^N \mathcal{D}(0, l_j)$, where $\mathcal{D}(0, l_j)$ is the space of C^∞ functions with compact support in $(0, l_j)$. As $(0, w) \in D(\mathcal{A})$, we get

$$\sum_{j=1}^N \int_0^{l_j} w_j g_j dx = 0.$$

Since $\prod_{j=1}^N \mathcal{D}(0, l_j)$ is dense in $\prod_{j=1}^N H^s(0, l_j)$ (because $0 < s < 1/2$), we deduce that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = \sum_{j=1}^N \int_0^{l_j} \frac{\partial u_j}{\partial x}\frac{\partial f_j}{\partial x} dx, \ \forall (u, w) \in D(\mathcal{A}).$$

By restricting ourselves to $w = 0$ we obtain

$$\sum_{j=1}^{N} \int_0^{l_j} \frac{\partial u_j}{\partial x} \frac{\partial f_j}{\partial x} dx = 0, \ \forall (u, 0) \in D(\mathcal{A}).$$

But we easily check that $(u, 0) \in D(\mathcal{A})$ if and only if $u \in D(\Delta)$, where

$$D(\Delta) := \{u \in V \cap \prod_{j=1}^{N} H^2(0, l_j) \ : \ \sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v) = 0, \ \forall v \in \mathcal{V}_{int} \, ; \ \frac{\partial u_1}{\partial x}(0) = 0\}.$$

We can verify that $D(\Delta)$ is dense in $V \cap \prod_{j=1}^{N} H^{1+s}(0, l_j)$. Indeed, let $\xi \in V \cap \prod_{j=1}^{N} H^{1+s}(0, l_j)$. For $j \in \{1, ..., N\}$, we take

$$\hat{\xi}_j = \xi_j - \xi_j(0)\eta_j - \xi_j(l_j)\tilde{\eta}_j,$$

where η_j, $\tilde{\eta}_j$ are C^∞ functions satisfying

$$\eta_j(0) = 1 \text{ near } 0, \quad \eta_j(l_j) = 0 \text{ near } l_j \quad \text{and} \quad \tilde{\eta}_j(0) = 0 \text{ near } 0, \quad \tilde{\eta}_j(l_j) = 1 \text{ near } l_j.$$

Then $\hat{\xi}_j \in H_0^{1+s}(0, l_j)$. As $\mathcal{D}(0, l_j)$ is dense into $H_0^{1+s}(0, l_j)$, there exits a sequence $\hat{\xi}_{n,j} \in \mathcal{D}(0, l_j)$ such that $\hat{\xi}_{n,j} \to \hat{\xi}_j$ in $H_0^{1+s}(0, l_j)$ when $n \to \infty$. Setting

$$\xi_{n,j} := \hat{\xi}_{n,j} + \xi_j(0)\eta_j + \xi_j(l_j)\tilde{\eta}_j,$$

we see that $\xi_n = (\xi_{n,j})_{j=1,...,N} \in D(\Delta)$ and $\xi_n \to \xi$ in $\prod_{j=1}^{N} H^{1+s}(0, l_j)$.

As $D(\Delta)$ is dense in $V \cap \prod_{j=1}^{N} H^{1+s}(0, l_j)$, we conclude that $f = 0$, which proves that $D(\mathcal{A})$ is dense in $\left(V \cap \prod_{j=1}^{N} H^{1+s}(0, l_j)\right) \times \prod_{j=1}^{N} H^s(0, l_j)$ and finishes the proof. ∎

Then we have the following lemma :

Lemma 2.3.7. *Assume that $(u^{(0)}, u^{(1)})$ belongs to X_s, where $0 < s < 1/2$, and $(w^{(0)}, w^{(1)}) = (u^{(0)} - u_1^{(0)}(0)\varphi, u^{(1)})$ where φ is a given smooth function satisfying (2.13). Then there exists a positive constant C such that*

$$\left\| \left(w^{(0)}, w^{(1)}\right) \right\|_{D(\mathcal{A}_D^s)}^2 + \left| u_1^{(0)}(0) \right|^2 \leq C \left\| \left(u^{(0)}, u^{(1)}\right) \right\|_{X_s}^2,$$

where $D(\mathcal{A}_D^s)$ is the domain of the operator $\mathcal{A}^s$ with Dirichlet boundary conditions at all exterior nodes.

Proof: First, we know that there exists $C > 0$ such that

$$\left\| \left(w^{(0)}, w^{(1)}\right) \right\|_{D(\mathcal{A}_D^s)}^2 \leq C \left\| \left(w^{(0)}, w^{(1)}\right) \right\|_{\left(\tilde{V} \cap \prod_{j=1}^{N} H^{1+s}(0, l_j)\right) \times \prod_{j=1}^{N} H^s(0, l_j)}^2$$

by interpolation (see [104] for example).

This estimate leads to the existence of a positive constant C such that

$$\left\| \left(w^{(0)},\, w^{(1)}\right) \right\|^2_{D(\mathcal{A}^s_D)} \leq C \left\| \left(w^{(0)},\, w^{(1)}\right) \right\|^2_{\left(V \cap \prod_{j=1}^N H^{1+s}(0,l_j)\right) \times \prod_{j=1}^N H^s(0,l_j)}, \tag{2.34}$$

because $w^{(0)} \in V \cap \tilde{V}$.

The Sobolev's injection theorem and the fact that $(w^{(0)},\, w^{(1)}) = (u^{(0)} - u_1^{(0)}(0)\varphi,\, u^{(1)})$ imply that there exists $C > 0$ such that

$$\left\| w^{(0)} \right\|^2_{V \cap \prod_{j=1}^N H^{1+s}(0,l_j)} + \left| u_1^{(0)}(0) \right|^2 \leq C \left\| u^{(0)} \right\|^2_{\prod_{j=1}^N H^{1+s}(0,l_j)}. \tag{2.35}$$

By definition of $w^{(1)}$, we have also

$$\left\| w^{(1)} \right\|^2_{\prod_{j=1}^N H^s(0,l_j)} = \left\| u^{(1)} \right\|^2_{\prod_{j=1}^N H^s(0,l_j)}. \tag{2.36}$$

Moreover, by the continuous injection of

$$\left[D(\mathcal{A}),\, D(\mathcal{A}^0)\right]_{1-s} = X_s$$

in

$$\left[\prod_{j=1}^N H^2(0,\, l_j) \times \prod_{j=1}^N H^1(0,\, l_j),\, \prod_{j=1}^N H^1(0,\, l_j) \times L^2(\mathcal{R})\right]_{1-s} = \prod_{j=1}^N H^{1+s}(0,\, l_j) \times \prod_{j=1}^N H^s(0,\, l_j),$$

there exists $C > 0$ such that

$$\left\| \left(u^{(0)},\, u^{(1)}\right) \right\|^2_{\prod_{j=1}^N H^{1+s}(0,l_j) \times \prod_{j=1}^N H^s(0,l_j)} \leq C \left\| \left(u^{(0)},\, u^{(1)}\right) \right\|^2_{X_s}. \tag{2.37}$$

The estimates (2.34)-(2.37) prove this lemma. ■

The main results of the chapter are the following

Theorem 2.3.8. *Assume that the weighted observability inequality (2.11) holds for every solution of (2.7) with* $\liminf_{n\to\infty} c_n = 0$ *and* $c_n \neq 0$ *for all* $n \in \mathbb{N}^*$. *Let* ω *be defined by (2.33). Assume that the initial data* $(u^{(0)},\, u^{(1)})$ *belong to* X_s, *characterized by Lemma 2.3.6, where* $0 < s < 1/2$. *Let* Φ_s *be a function such that the pair* $(\omega,\, \Phi_s)$ *satisfies (2.28)-(2.31). Then there exists a constant* $C > 0$ *such that the corresponding solution* u *of (2.1) verifies*

$$\forall t \geq 0,\; E_u(t) \leq C\Phi_s\left(\frac{1}{t+1}\right) \left\| (u^{(0)},\, u^{(1)}) \right\|^2_{X_s}. \tag{2.38}$$

Remark 2.3.9. *We see that the decay rate of the energy directly depends on the behaviour of the interpolation function* Φ_s *near* 0 *and thus of* ω *and of the weights* c_n^2 *as* $n \to \infty$.

Proof: We split up u as in (2.15) and we decompose $w^{(0)} = u^{(0)} - u_1^{(0)}(0)\varphi$ and $w^{(1)} = u^{(1)}$ as

$$w^{(0)} = u^{(0)} - u_1^{(0)}(0)\varphi = \sum_{n\geq 1} w_{0,n}\varphi_n^D \text{ and } w^{(1)} = u^{(1)} = \sum_{n\geq 1} w_{1,n}\varphi_n^D,$$

where $(\tilde{\lambda}_n w_{0,n})_n$, $(w_{1,n})_n \in l^2(\mathbb{N}^*)$. We write

$$E_-(0) = \sum_{n\geq 1} u_n \omega(n), \tag{2.39}$$

where $(u_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{N}^*; \mathbb{R})$ is defined by

$$\forall n \geq 2,\, u_n = \frac{1}{2}\left(\tilde{\lambda}_{n-1}^2 w_{0,n-1}^2 + w_{1,n-1}^2\right)$$

and

$$u_1 = u_1^{(0)}(0)^2.$$

Observe that, by the construction (2.33) of ω, we have

$$E_-(u,\, 0) \leq E_*(u,\, 0).$$

Then, by (2.4) and (2.17) of Theorem 2.2.1, we have

$$E_u(0) - E_u(T) = \int_0^T \left(\frac{\partial u_1}{\partial t}(0,\, t)\right)^2 dt \geq CE_-(0). \tag{2.40}$$

Assume further that $(u^{(0)},\, u^{(1)}) \in X_s$. We define

$$E_+(0) = \sum_{n\geq 1} u_n n^{2s}. \tag{2.41}$$

It follows from Theorem 2.3.2 (applied to the function $f = u$ and the weight ω) that

$$1 \leq \Phi_s\left(\frac{E_-(0)}{\sum\limits_{n\geq 1} u_n}\right)\frac{E_+(0)}{\sum\limits_{n\geq 1} u_n}. \tag{2.42}$$

By the so-called Weyl's formula (see for instance [2, 20, 82]), we have

$$\tilde{\lambda}_k \sim \frac{k\pi}{L}, \tag{2.43}$$

and thus, there exist $c_1,\, c_2 > 0$ such that, for n large enough, we have

$$c_1\frac{(n+1)\pi}{L} \leq \tilde{\lambda}_n \leq c_2\frac{(n+1)\pi}{L}.$$

Therefore, we have

$$\begin{aligned}
E_+(0) &= \tfrac{1}{2}\left(\tfrac{L}{\pi}\right)^{2s}\sum_{n\geq 1}\left(\tilde{\lambda}_n^2 w_{0,n}^2 + w_{1,n}^2\right)\left(\frac{(n+1)\pi}{L}\right)^{2s} + \left|u_1^{(0)}(0)\right|^2 \\
&\leq C\left(\sum_{n\geq 1}\left(\tilde{\lambda}_n^2 w_{0,n}^2 + w_{1,n}^2\right)\tilde{\lambda}_n^{2s} + \left|u_1^{(0)}(0)\right|^2\right) \\
&\leq C\left(\left\|\left(w^{(0)},\, w^{(1)}\right)\right\|_{D(\mathcal{A}_D^s)}^2 + \left|u_1^{(0)}(0)\right|^2\right).
\end{aligned}$$

Consequently, by Lemma 2.3.7, we obtain that there exists $C > 0$ such that

$$E_+(0) \leq C\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{X_s}^2.$$

Moreover,

$$\sum_{n\geq 1} u_n = \frac{1}{2}\sum_{n\geq 1}\left(\tilde{\lambda}_n^2 w_{0,n}^2 + w_{1,n}^2\right) + u_1^{(0)}(0)^2 = \left\|w^{(0)}\right\|_{\tilde{V}}^2 + u_1^{(0)}(0)^2 + \left\|u^{(1)}\right\|_{L^2(\mathcal{R})}^2.$$

Furthermore, there exists $C' > 0$ such that

$$\left\|u^{(0)}\right\|_V^2 = \left\|w^{(0)} + u_1^{(0)}(0)\varphi\right\|_V^2 \leq C'\left(\left\|w^{(0)}\right\|_V^2 + \left|u_1^{(0)}(0)\right|^2\right) = C'\left(\left\|w^{(0)}\right\|_{\tilde{V}}^2 + \left|u_1^{(0)}(0)\right|^2\right)$$

because φ is given and $w^{(0)} \in \tilde{V}$. Therefore, there exists $C' > 0$ such that

$$\sum_{n\geq 1} u_n \geq C'\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{V\times L^2(\mathcal{R})}^2.$$

Consequently, (2.42) becomes, by the increasing character of Φ_s

$$1 \leq \Phi_s\left(\frac{E_-(0)}{C'\left\|(u^{(0)},\, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^2}\right)\frac{C\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}{C'\left\|(u^{(0)},\, u^{(1)})\right\|_{V\times L^2(\mathcal{R})}^2},$$

which yields

$$E_-(0) \geq C'\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{V\times L^2(\mathcal{R})}^2 \Phi_s^{-1}\left(\frac{\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{V\times L^2(\mathcal{R})}^2}{C\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}\right), \tag{2.44}$$

with C, $C' > 0$. From (2.40) and (2.44), it follows that there exist C, $C' > 0$,

$$E_u(T) \leq E_u(0) - C'\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{V\times L^2(\mathcal{R})}^2 \Phi_s^{-1}\left(\frac{\left\|\left(u^{(0)},\, u^{(1)}\right)\right\|_{V\times L^2(\mathcal{R})}^2}{C\left\|(u^{(0)},\, u^{(1)})\right\|_{X_s}^2}\right), \tag{2.45}$$

for any $(u^{(0)}, u^{(1)}) \in X_s$.

We follow now the proof of Ammari and Tucsnak [11]. We rewrite (2.45) as follows :

$$\begin{aligned} \|(u(T), u_t(T))\|^2_{V\times L^2(\mathcal{R})} \leq & \ \|(u^{(0)}, u^{(1)})\|^2_{V\times L^2(\mathcal{R})} \\ & -C' \left\|(u^{(0)}, u^{(1)})\right\|^2_{V\times L^2(\mathcal{R})} \Phi_s^{-1} \left(\frac{\left\|(u^{(0)}, u^{(1)})\right\|^2_{V\times L^2(\mathcal{R})}}{C\left\|(u^{(0)}, u^{(1)})\right\|^2_{X_s}} \right). \end{aligned}$$

This estimate remains valid in successive time-intervals $[lT, (l+1)T]$. Notice that there exists $C > 0$ such that

$$\forall t \geq 0, \ \|(u(t), u_t(t))\|_{X_s} \leq C \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}, \tag{2.46}$$

for $0 < s < \frac{1}{2}$ by interpolation, because it is true for $s = 0$ and $s = 1$ (see Theorem 5.1 of [72]). By (2.46) and the fact that the energy is decreasing by (2.4) and that Φ_s^{-1} is increasing, we obtain that

$$\begin{aligned} & \|(u((l+1)T), u_t((l+1)T))\|^2_{V\times L^2(\mathcal{R})} \leq \|(u(lT), u_t(lT))\|^2_{V\times L^2(\mathcal{R})} \\ & -C' \|(u(lT), u_t(lT))\|^2_{V\times L^2(\mathcal{R})} \Phi_s^{-1} \left(\frac{\|(u((l+1)T), u_t((l+1)T))\|^2_{V\times L^2(\mathcal{R})}}{C\left\|(u^{(0)}, u^{(1)})\right\|^2_{X_s}} \right), \end{aligned} \tag{2.47}$$

for every $l \in \mathbb{N} \cup \{0\}$.

Our expression (2.47) coincides with (4.16) in Ammari and Tucsnak [11] (with $\mathcal{G} = \Phi_s^{-1}$ and $\theta = 1/2$). The rest of the proof follows as in [11], where (2.31) is used, by noticing that

$$\frac{\|(u((l+1)T), u_t((l+1)T))\|^2_{V\times L^2(\mathcal{R})}}{C\,\|(u^{(0)}, u^{(1)})\|^2_{X_s}} < 1,$$

taking a higher constant C if necessary. Then (2.38) follows. ∎

In addition, using (2.38) and making a particular and explicit choice of the concave function Φ_s, we obtain a more explicit dependence of the weights on the decay rate of the energy :

Theorem 2.3.10. *Assume that the weighted observability inequality (2.11) holds for every solution of (2.7) with $\liminf_{n\to\infty} c_n = 0$ and $c_n \neq 0$ for all $n \in \mathbb{N}^*$. Let ω be defined by (2.33). We set*

$$\forall t > 0, \ \varphi(t) = \frac{\omega(t)}{t^2}.$$

Then there exists a constant $C > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ ($0 < s < 1/2$), the corresponding solution u of (2.1) verifies

$$\forall t \geq 0, \ E_u(t) \leq \frac{C}{\left(\varphi^{-1}\left(\frac{1}{t+1}\right)\right)^{2s}} \left\|(u^{(0)}, u^{(1)})\right\|^2_{X_s}. \tag{2.48}$$

Proof: We set

$$\Phi_s(t) = \frac{1}{\left(\varphi^{-1}(t)\right)^{2s}}$$

where $0 < s < 1/2$. Then the pair (ω, Φ_s) verifies (2.28)-(2.29) on $(0, \infty)$. Indeed, as Φ_s is the composition of the functions $t \mapsto 1/\varphi^{-1}(t)$ and $t \mapsto t^{2s}$ which are increasing and concave functions (by Lemma 2.6 of [19]), Φ_s is an increasing and concave function.

Moreover, we easily check that

$$\frac{1}{\varphi^{-1}(\omega(t))} \geq \frac{1}{t}$$

on $[1, \infty)$. As a consequence, (2.30) holds on $[1, \infty)$.

Finally, we have

$$\frac{1}{t}\Phi_s^{-1}(t) = t^{\frac{1-s}{s}}\omega\left(\frac{1}{t^{\frac{1}{2s}}}\right),$$

which is an increasing function on $(0, \infty)$ because $0 < s < 1/2$, and therefore (ω, Φ_s) satisfy (2.28)-(2.31). We apply now (2.38) of Theorem 2.3.8 with

$$\Phi_s(t) = \left(\frac{1}{\varphi^{-1}(t)}\right)^{2s},$$

to obtain the result. ■

2.4 Examples

In [38] the authors proved observability inequalities of type (2.11) on which our analysis is based. In the case of star-shaped networks, the weights in (2.11) depend on the irrationality properties of the ratios of the lengths of the strings. In the case of tree-shaped networks, the observability inequality is proved under a condition (that is fulfilled generically within the class of tree-shaped networks) that generalizes the condition on the irrationality of the ratios of the lengths of the strings arising in the case of stars. Finally, in the case of general networks, the observability inequality holds under the condition that all eigenfunctions of the network are observable. This last result is a generalization of the previous one for stars and trees.

To illustrate the wide range of applications of the main result of this chapter, in this section, we apply our previous results to some examples of particular networks : a star-shaped network and a particular tree. We obtain the weights $(c_n)_n$ directly by [38] and deduce an explicit decay rate for the corresponding dissipative system.

2.4.1 The star-shaped network with N strings

The star-shaped network with N strings is formed by N strings connected at one point v, which constitutes a particular tree. Recall that the damping term is located on the vertex

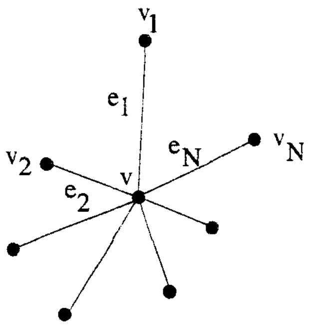

FIG. 2.1 – A star-shaped network with N strings

v_1, the origin of the controlled edge e_1 of length l_1. The remaining $N-1$ exterior nodes are denoted by v_i, $i = 2, ..., N$, the string that contains v_i by e_i and its length by l_i.

In [38], the authors proved the observability inequality (2.11) for the conservative system (2.7) with Dirichlet boundary conditions at all exterior nodes with the following weights

$$c_k = \max_{i=2,...,N} \prod_{j \neq i} \left| \sin(\tilde{\lambda}_k l_j) \right|, \; \forall n \geq 1.$$

First, if the ratio of any two of the lengths of the uncontrolled strings is an irrational number, i.e. $l_i/l_j \notin \mathbb{Q}$ for all $i \neq j$, then for all $k \in \mathbb{N}$, we have $c_k > 0$. In this case, the energy of u solution of the dissipative system (2.1) tends to 0 as $t \to \infty$. This can be easily proved with LaSalle's invariance principle, using the energy as Lyapunov functional, but it does not yield any explicit decay rate.

Thus assume that $l_i/l_j \notin \mathbb{Q}$ for all $i \neq j$. Denote by $\mathcal{S}$ the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and so that its expansion as a continued fraction $[0, a_1, ..., a_n, ...]$ is such that (a_n) is bounded. It is well known that $\mathcal{S}$ is uncountable and that its Lebesgue measure is zero. Roughly speaking, the set $\mathcal{S}$ contains all irrational numbers which are badly approximated by rational numbers. In particular, by the Euler-Lagrange theorem, $\mathcal{S}$ contains all irrational quadratic numbers (i.e. the roots of second order equations with rational coefficients).

We use also a well-known result asserting that, for all $\epsilon > 0$ there exists a set $B_\epsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \backslash B_\epsilon$ is equal to zero, and a constant $C_\epsilon > 0$ for which, if $\xi \in B_\epsilon$, then

$$\left|\left|\left| \xi m \right|\right|\right| \geq \frac{C_\epsilon}{m^{1+\epsilon}},$$

where $|||\eta|||$ is the distance from η to the set $\mathbb{Z}$:

$$\left|\left|\left| \eta \right|\right|\right| = \min_{\eta - x \in \mathbb{Z}} |x| \, .$$

In particular, $\mathcal{S}$ is contained in the sets B_ϵ for every $\epsilon > 0$ (see [38] for more details).

Then, by Corollary A.10 of [38], we have

Lemma 2.4.1. *1. If for all values $i, j = 2, ..., N$, $i \neq j$, the ratios l_i/l_j belong to $\mathcal{S}$, then there exists a constant $c > 0$ such that*

$$c_k \geq \frac{c}{\tilde{\lambda}_k^{N-2}}, \ \forall k \in \mathbb{N}^*.$$

2. If for all values $i, j = 2, ..., N$, $i \neq j$, the ratios l_i/l_j belong to B_ϵ, then there exists a constant $c_\epsilon > 0$ such that

$$c_k \geq \frac{c_\epsilon}{\tilde{\lambda}_k^{N-2+\epsilon}}, \ \forall k \in \mathbb{N}^*.$$

With more restrictive assumptions on the lengths of the uncontrolled strings, we have another bound for c_k. Let us recall a definition of [38].

Definition 2.4.2. *We say that the real numbers $l_1, ..., l_N$ verify the conditions (S) if*

- *$l_1, ..., l_N$ are linearly independent over the field $\mathbb{Q}$ of rational numbers;*
- *the ratios l_i/l_j are algebraic numbers for $i, j = 1, ..., N$.*

In this case, we have the following result (see Corollary A.10 of [38]) :

Lemma 2.4.3. *If the numbers $l_2, ..., l_N$ verify the conditions (S), then for every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that*

$$c_k \geq \frac{c_\epsilon}{\tilde{\lambda}_k^{1+\epsilon}}, \ \forall k \in \mathbb{N}^*.$$

Consequently, we have

Proposition 2.4.4. *1. Assume that for all values $i, j = 2, ..., N$, $i \neq j$, the ratios l_i/l_j belong to $\mathcal{S}$. Then there exists a constant $C > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ $(0 < s < 1/2)$, the corresponding solution u of (2.1) verifies*

$$E_u(t) \leq \frac{C}{(t+1)^{\frac{s}{N-2}}} \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2.$$

2. Assume that for all values $i, j = 2, ..., N$, $i \neq j$, the ratios l_i/l_j belong to B_ϵ for $\epsilon > 0$. Then there exists a constant $C_\epsilon > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ $(0 < s < 1/2)$, the corresponding solution u of (2.1) verifies

$$E_u(t) \leq \frac{C}{(t+1)^{\frac{s}{N-2+\epsilon}}} \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2.$$

3. Assume that the numbers $l_2, ..., l_N$ verify the conditions (S). Let $\epsilon > 0$. Then there exists a constant $C_\epsilon > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ $(0 < s < 1/2)$, the corresponding solution u of (2.1) verifies

$$E_u(t) \leq \frac{C_\epsilon}{(t+1)^{\frac{s}{1+\epsilon}}} \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2.$$

Proof: We set $\alpha = N-2$ and $c_\alpha = c$ in the first case, $\alpha = N-2+\epsilon$ and $c_\alpha = c_\epsilon$ in the second one and $\alpha = 1+\epsilon$ and $c_\alpha = c_\epsilon$ in the third one. For $\alpha \geq 1$, we take

$$\omega(t) = d_\alpha^2 \left(\frac{1}{t}\right)^{2\alpha},$$

where d_α verifies $d_\alpha^2 \leq 1$ and $c_\alpha^2 \tilde{\lambda}_{n-1}^{-2\alpha} \geq d_\alpha^2 n^{-2\alpha}$ for all $n \in \mathbb{N}^*$, which is possible because of Weyl's formula (2.43).

We are in the situation 1 of Example 2.3.3 with $p = 2\alpha \geq 1$. Therefore, we take

$$\Phi_s(t) = \left(\frac{t}{d_\alpha^2}\right)^{\frac{s}{\alpha}}$$

Then (ω, Φ_s) satisfy (2.28)-(2.31) and we apply Theorem 2.3.8 to finish the proof. ■

Finally, if the lengths $l_2, ..., l_N$ verify $l_i/l_j \notin \mathbb{Q}$ for all $i \neq j$, but the conditions 1, 2 and 3 of Proposition 2.4.4 are not verified, it can be proved that c_n verify

$$c_n \geq \psi(n) > 0, \; \forall n \in \mathbb{N}^*, \tag{2.49}$$

where ψ is a positive convex and decreasing function which can be smaller than $c\tilde{\lambda}_n^{-\alpha}$, with $\alpha > 0$. Indeed, we know by Appendix A of [38], that if we set

$$a(\lambda) = \sum_{i=2}^{N} \prod_{j \neq i} |\sin(\lambda l_j)|,$$

we have

$$a(\lambda) \geq C \min_{i=2,\dots,N} \prod_{j \neq i} \left|\left|\left| \frac{l_i}{l_j} m^i(\lambda) \right|\right|\right|,$$

where

$$m^i(\lambda) = \mathbf{E}\left(\frac{l_i}{\pi}\lambda\right),$$

$\mathbf{E}(\eta)$ being the closest integer number to η : $|\eta - \mathbf{E}(\eta)| = |||\eta|||$. Therefore, it is sufficient to bound by below $|||\frac{l_i}{l_j} m|||$ for all $m \in \mathbb{N}$ and $i \neq j$ to get a lower bound of c_n.

Moreover, we know that Liouville's numbers ξ are such that for all $n \in \mathbb{N}$, there exists $q \in \mathbb{N}$, $q > 1$ such that

$$0 < \left|\left|\left| \xi q \right|\right|\right| < \frac{1}{q^{n+1}}.$$

That is why, when l_i/l_j is a Liouville's number (see [29]), the function ψ in (2.49) is smaller than any negative power of $\tilde{\lambda}_n$ (for example of the order of e^{-n}), and thus the decay rate of the energy of u is slower than polynomial. Indeed, for instance, it is possible to construct real numbers ξ of the form

$$\xi = \sum_{k \in \mathbb{N}} 10^{-a_k},$$

where $(a_k)_k$ is an increasing sequence of natural numbers, which are approximated by rational ones faster than any given positive increasing function ρ :

$$\Big|\Big|\Big|10^{a_p}\xi\Big|\Big|\Big|\rho(10^{a_p}) < \epsilon, \ \forall p \in \mathbb{N}$$

for $\epsilon > 0$ (see [38], p.66 for more details).

Let us construct an irrational number x which is approximated by rational ones at an exponential speed. The aim is to find an irrational x_a such that

$$\liminf_{q\to\infty}\Big|\Big|\Big|x_a q\Big|\Big|\Big|^{\frac{1}{q}} = e^{-a}, \text{ where } a \text{ is a positive constant.} \tag{2.50}$$

This construction uses the theory of continued fractions (see [94]). We recall here some results about the continued fractions which are used in the sequel and we refer to [29,54,66] for more details. Let $[a_0, a_1, ..., a_n, ...]$ be the expansion as a continued fraction of $x \in \mathbb{R}\backslash\mathbb{Q}$. We set $x_n = [a_0, a_1, ..., a_n] = \frac{p_n}{q_n}$ where p_n, q_n are integers. The integers p_n, q_n are relatively prime and satisfy the following relations :

$$p_{-1} = 1, \ p_0 = a_0, \ p_{n+1} = a_{n+1}p_n + p_{n-1} \tag{2.51}$$

and

$$q_{-1} = 0, \ q_0 = 1, \ q_{n+1} = a_{n+1}q_n + q_{n-1}. \tag{2.52}$$

We have the following theorem which is a result of best approximation (see [29,54,66,94]) :

Theorem 2.4.5. *Let $x \notin \mathbb{Q}$. Then, with the notation introduced before, we have*
(i) $q_n \geq 2^{(n-1)/2}$
(ii) $|||q_n x||| = |q_n x - p_n|$ *and* $\frac{1}{2q_{n+1}} \leq |||q_n x||| \leq \frac{1}{q_{n+1}}$,
(iii) If $0 < q < q_{n+1}$, *then* $|||qx||| \geq |||q_n x|||$.

First, we show

$$\liminf\Big|\Big|\Big|xq\Big|\Big|\Big|^{\frac{1}{q}} = \liminf a_{n+1}^{-\frac{1}{q_n}}. \tag{2.53}$$

Set $l(x) = \liminf |||xq|||^{\frac{1}{q}}$ and $\lambda(x) = \liminf a_{n+1}^{-\frac{1}{q_n}}$. By the relation (2.52), we have

$$a_{n+1}q_n \leq q_{n+1} \leq 2a_{n+1}q_n.$$

Then, by (ii) of Theorem 2.4.5, we obtain

$$\frac{\ln|||q_n x|||}{q_n} \leq \frac{\ln(1/q_{n+1})}{q_n} \leq \frac{\ln(1/a_{n+1}) + \ln(1/q_n)}{q_n},$$

and thus

$$\liminf \frac{\ln|||qx|||}{q} \leq \liminf \frac{\ln(1/a_{n+1})}{q_n},$$

which leads to $l(x) \leq \lambda(x)$. For an arbitrary $q \geq 1$, let n be an integer such that $q_n \leq q < q_{n+1}$. As $\ln |||qx||| \leq 0$ and by (ii) of Theorem 2.4.5, we have

$$\frac{\ln |||qx|||}{q} \geq \frac{\ln |||qx|||}{q_n} \geq \frac{\ln(1/2q_{n+1})}{q_n} \geq \frac{\ln(1/a_{n+1}) + \ln(1/4q_n)}{q_n},$$

which leads to

$$\liminf \frac{\ln |||qx|||}{q} \geq \liminf \frac{\ln(1/a_{n+1})}{q_n},$$

and to $l(x) \geq \lambda(x)$. Therefore, (2.53) holds.

Thus finding $x_a \in \mathbb{R}\backslash\mathbb{Q}$ such that (2.50) is equivalent to finding $x_a \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$\liminf a_{n+1}^{-\frac{1}{q_n}} = e^{-a}.$$

We construct a such x_a by induction. Set $a_0 = 0$, which determines p_0 and q_0. Then, if $a_0, \ldots, a_n$, $p_0, \ldots, p_n$, $q_0, \ldots, q_n$ are found, we choose a_{n+1} by

$$a_{n+1} = \mathbf{E}(e^{aq_n}).$$

Then, p_{n+1} and q_{n+1} are imposed by the relations (2.51) and (2.52).

Therefore, for all $a > 0$, we have found an irrational number x_a such that there exists a positive constant δ such that,

$$\left|\left|\left|qx_a\right|\right|\right| \geq \delta e^{-aq}, \ \forall q \in \mathbb{N}. \tag{2.54}$$

Consequently, if l_i/l_j are reals of the form $x_{a_{i,j}}$ for $a_{i,j} > 0$, which verify (2.54), then there exists a positive constant C such that for all $k \in \mathbb{N}$,

$$c_k \geq Ce^{-b(N-2)\tilde{\lambda}_k \frac{l}{\pi}},$$

where $b = \max_{i \neq j}(a_{i,j}) > 0$ and $l = \max_j(l_j)$. By the Weyl's formula, there exists $c_2 > 0$ such that $\tilde{\lambda}_k \leq c_2 k\pi/L$, and thus, we obtain

$$c_k^2 \geq Ce^{-2c_2 b(N-2)k} = Ce^{-Ak},$$

where $A = 2c_2 b(N-2)$.

Consequently, we have

Proposition 2.4.6. *Assume that for all values i, $j = 2, \ldots, N$, $i \neq j$, the ratios l_i/l_j are reals of the form $x_{a_{i,j}}$ for $a_{i,j} > 0$ which verify (2.54). Then there exist constants C, $C' > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ ($0 < s < 1/2$), the corresponding solution u of (2.1) verifies*

$$\forall t \geq 0, \ E_u(t) \leq \frac{C'}{(\ln(C(1+t)))^{2s}} \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2. \tag{2.55}$$

Proof: We apply Theorem 2.3.8 with

$$\omega(t) = Ce^{-At},$$

for $C > 0$ and $A > 2(2s+1)$, and

$$\Phi_s(t) = \left(\frac{A}{\ln\left(\frac{C}{t}\right)}\right)^{2s}.$$

We are in the situation 2 of Example 2.3.3 and we simply apply (2.38) to obtain (2.55). ■

2.4.2 A non star-shaped tree

Now let us consider a tree, which is not star-shaped, having the simple structure in figure 2.2.

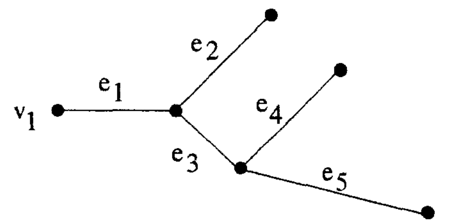

FIG. 2.2 – A non star-shaped tree

Recall that the damping term is on the vertex v_1, the origin of the damped edge e_1 of length l_1. We will assume, in addition, that $l_4 = l_2$.

Recall that in [38], the authors proved the observability inequality (2.11) for the conservative system (2.7) with Dirichlet boundary conditions at all exterior nodes with the weights c_k given by

$$c_k = \max\left\{\left|d_5(\tilde{\lambda}_k)\right|, \left|d_4(\tilde{\lambda}_k)\right|, \left|d_2(\tilde{\lambda}_k)\right|\right\},$$

where

$$d_5(\lambda) = -\sin(\lambda l_2)\sin(\lambda l_4),\ d_4(\lambda) = -\sin(\lambda l_2)\sin(\lambda l_5),$$

$$d_2(\lambda) = -(\cos(\lambda l_3)\sin(\lambda l_5)\sin(\lambda l_4) + \sin(\lambda l_3)\cos(\lambda l_5)\sin(\lambda l_4) + \sin(\lambda l_3)\sin(\lambda l_5)\cos(\lambda l_4)).$$

First, if $c_k = 0$, then $\left|d_5(\tilde{\lambda}_k)\right| = \left|d_4(\tilde{\lambda}_k)\right| = \left|d_2(\tilde{\lambda}_k)\right| = 0$. As $l_4 = l_2$, we obtain $\sin(\lambda l_4) = 0$ and $\sin(\lambda l_3)\sin(\lambda l_5) = 0$. If $\sin(\lambda l_3) = 0$ (resp. $\sin(\lambda l_5) = 0$), then, necessarily

l_3/l_2 (resp. l_5/l_2) is a rational number. Consequently, $c_k \neq 0$ if l_3/l_2 and l_5/l_2 are irrational numbers and then, in this case, the energy of u solution of (2.1) decays to 0.

Secondly, applying Appendix A of [38], we know that there exists a positive constant c such that for every $k \in \mathbb{N}^*$,

$$c_k \geq \frac{c}{\tilde{\lambda}_k^\alpha}$$

if one of the following three conditions holds

• the ratios l_5/l_2, l_3/l_2 and l_3/l_5 belong to $\mathcal{S}$ and $\alpha > 4$

or

• the ratios l_5/l_2, l_3/l_2 and l_3/l_5 belong to some B_ϵ and $\alpha > 4 + \epsilon$

or

• the numbers l_3, l_5, l_2 satisfy the conditions (S) and $\alpha > 2 + \epsilon$.

Consequently, by using (2.38) of Theorem 2.3.8 and the first statement of Example 2.3.3, there exists a constant $C > 0$ such that for any initial data $(u^{(0)}, u^{(1)}) \in X_s$ $(0 < s < 1/2)$, the corresponding solution u verifies

$$E_u(t) \leq \frac{C_\epsilon}{(t+1)^{\frac{s}{\alpha}}} \left\|(u^{(0)}, u^{(1)})\right\|_{X_s}^2 .$$

Finally, if these previous conditions do not hold, we can apply Theorem 2.3.8 to obtain a decay rate of the energy of u with an admissible pair (ω, Φ_s), or Theorem 2.3.10 to obtain a more explicit decay rate of the energy.

As we have seen in these examples, our method to obtain decay rates of the dissipative system (2.1) is of systematic application :

First, we find the weights (c_n^2) of the observability inequality (2.8) for the conservative problem (2.7) with Dirichlet boundary conditions at all exterior nodes.

Then we take, for the weight ω, the lower convex envelop of the sequence $(1, (c_n^2)_n)$.

Finally we choose Φ_s such that (ω, Φ_s) satisfy (2.28)-(2.31) to obtain the decay rate of the energy, given by

$$\Phi_s\left(\frac{1}{1+t}\right).$$

Note that the study of the observability inequality (2.8) for (2.7) and the weights (c_n^2) have been already done in some works (see [38]). Consequently we can use directly these results and it is not necessary to show another observability inequality. In addition, notice that the weighted observability inequality (2.17) holds under the same assumptions on the network that are needed for the Dirichlet problem (2.7). This method allows to treat an important class of networks.

Moreover we can extend this general principle to networks of strings with damping in several exterior vertices, or with damping in interior nodes.

Chapitre 3

Stabilization of second order evolution equations with unbounded feedback with delay

3.1 Introduction

Time-delay often appears in many biological, electrical engineering systems and mechanical applications [1, 51, 101], and in many cases, in particular for distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system, see e.g. [39–41, 52, 75, 85, 89, 98, 110]. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

We further remark that some techniques developed recently in [85] and in chapter 1 in order to obtain some existence results and decay rates have some similarities. We therefore propose to consider an abstract setting as large as possible in order to contain a quite large class of problems with time delay feedbacks. In a second step we prove existence and stability results in this setting under realistic assumptions. Finally in order to show the usefulness of our approach, we give some examples where our abstract framework can be applied. For a similar approach, we refer to the paper in preparation [5]. Without delay such an approach was developed in [11].

Before going on, let us present our abstract framework. Let H be a real Hilbert space with norm and inner product denoted respectively by $\|.\|_H$ and $(.,.)_H$. Let $A : D(A) \rightarrow H$ be a self-adjoint positive operator with a compact inverse in H. Let $V := D(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $D(A^{\frac{1}{2}})'$ the dual space of $D(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Further, for $i = 1, 2$, let U_i be a real Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_{U_i}$ and $(.,.)_{U_i}$ and let $B_i \in \mathcal{L}(U_i, D(A^{\frac{1}{2}})')$.

We consider the system described by

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 u_1(t) + B_2 u_2(t-\tau) = 0, \, t > 0 \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1, \\ u_2(t-\tau) = f^0(t-\tau), \, 0 < t < \tau, \end{cases} \tag{3.1}$$

where $t \in [0, \infty)$ represents the time, τ is a positive constant which represents the delay, $\omega : [0, \infty) \to H$ is the state of the system and $u_1 \in L^2([0, \infty), U_1)$, $u_2 \in L^2([-\tau, \infty), U_2)$ are the input functions. Most of the linear equations modelling the vibrations of elastic structures with distributed control with delay can be written in the form (3.1), where ω stands for the displacement field.

In many problems, coming in particular from elasticity, the input u_i are given in the feedback form $u_i(t) = B_i^* \dot{\omega}(t)$, which corresponds to colocated actuators and sensors. We obtain in this way the closed loop system

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 B_2^* \dot{\omega}(t-\tau) = 0, \, t > 0 \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1, \\ B_2^* \dot{\omega}(t-\tau) = f^0(t-\tau), \, 0 < t < \tau. \end{cases} \tag{3.2}$$

The first natural question is the well posedness of this problem. In section 3.2 we will give a sufficient condition that guarantees that this problem (3.2) is well-posed, where we closely follow the approach developed in [85] for the wave equation. Secondly, we may ask if this system is dissipative. We show in section 3.3 that the condition

$$\exists 0 < \alpha < 1, \, \forall u \in V, \, \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2 \tag{3.3}$$

guarantees the strict decay of the energy; under this condition, using a result from [12] (see also [105]) we pertain a necessary and sufficient condition for the decay to zero of the energy. Note that this last condition is independent of the delay and therefore under the condition (3.3), our system is strongly stable if and only if the same system without delay is strongly stable. Note further that if (3.3) is not satisfied, there exist cases where some instabilities may appear (see [85, 110] and chapter 1 for the wave equation). Hence this assumption seems realistic.

In a third step, again under the condition (3.3) and a certain boundedness assumption from [11] between the resolvent operator of A and of the operators B_1 and B_2, see condition (3.20), we prove that the exponential decay of the system (3.2) follows from a certain observability estimate. Again this observability estimate is independent of the delay term $B_2 B_2^* \dot{\omega}(t-\tau)$ and therefore, under the conditions (3.3) and (3.20), the exponential decay of the system (3.2) follows from the exponential decay of the same system without delay. Nevertheless we give the dependence of the decay with respect to the delay, in particular we show that if the delay increases the decay decreases. This is the content of section 3.4. A similar analysis for the polynomial decay is performed in section 3.5 by weakening the observability estimate. Again we show that if the delay increases the decay decreases. In view of some applications, section 3.6 is devoted to the proof of these two observability estimates by using a frequency domain method and a reduction to some conditions between the eigenvectors of A and the feedback operator B_1^*.

Finally we finish this chapter by considering in section 3.7 different examples where our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

3.2 Well posedness of the problem

We aim to show that problem (3.2) is well-posed. For that purpose, we use semi-group theory and an idea from [85] (see also chapter 1 or [89]). Let us introduce the auxiliary variable $z(\rho, t) = B_2^*\dot{\omega}(t - \tau\rho)$ for $\rho \in (0, 1)$ and $t > 0$. Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$

$$\left\{ \begin{array}{c} \tau\frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0 \\ z(0, t) = B_2^*\dot{\omega}(t) \\ z(\rho, 0) = B_2^*\dot{\omega}(-\tau\rho) = f^0(-\tau\rho). \end{array} \right. \tag{3.4}$$

Therefore, the problem (3.2) is equivalent to

$$\left\{ \begin{array}{c} \ddot{\omega}(t) + A\omega(t) + B_1B_1^*\dot{\omega}(t) + B_2z(1, t) = 0, \, t > 0 \\ \tau\frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0, \, t > 0, \, 0 < \rho < 1 \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1, \, z(\rho, 0) = f^0(-\tau\rho), \, 0 < \rho < 1 \\ z(0, t) = B_2^*\dot{\omega}(t), \, t > 0. \end{array} \right. \tag{3.5}$$

If we introduce

$$U := (\omega, \, \dot{\omega}, \, z)^T,$$

then U satisfies

$$U' = (\dot{\omega}, \, \ddot{\omega}, \, \dot{z})^T = \left(\dot{\omega}, \, -A\omega(t) - B_1B_1^*\dot{\omega}(t) - B_2z(1, \, t), \, -\frac{1}{\tau}\frac{\partial z}{\partial \rho}\right)^T.$$

Consequently the problem (3.2) may be rewritten as the first order evolution equation

$$\left\{ \begin{array}{c} U' = \mathcal{A}U \\ U(0) = (\omega_0, \, \omega_1, f^0(-\tau.)), \end{array} \right. \tag{3.6}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} u \\ -A\omega - B_1B_1^*u - B_2z(1) \\ -\frac{1}{\tau}\frac{\partial z}{\partial \rho} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) := \{(\omega, \, u, \, z) \in V \times V \times H^1((0, 1), U_2); \, z(0) = B_2^*u, \, A\omega + B_1B_1^*u + B_2z(1) \in H\}.$$

Now, we introduce the Hilbert space

$$\mathcal{H} = V \times H \times L^2((0,\,1),\,U_2)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\tilde{\omega}\right)_H + (u,\,\tilde{u})_H + \int_0^1 (z(\rho),\,\tilde{z}(\rho))_{U_2}\,d\rho. \tag{3.7}$$

Let us suppose now that

$$\exists 0 < \alpha \leq 1,\ \forall u \in V,\ \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2\,. \tag{3.8}$$

Under this condition, we will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. For that purpose, we choose a positive real number ξ such that

$$1 \leq \xi \leq \frac{2}{\alpha} - 1. \tag{3.9}$$

This constant exists because $0 < \alpha \leq 1$.

We now introduce the following inner product on $\mathcal{H}$

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} = \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\tilde{\omega}\right)_H + (u,\,\tilde{u})_H + \tau\xi \int_0^1 (z(\rho),\,\tilde{z}(\rho))_{U_2}\,d\rho.$$

This new inner product is clearly equivalent to the usual inner product on $\mathcal{H}$ (3.7).

Theorem 3.2.1. *Under the assumption (3.8), for an initial datum $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C([0,\,+\infty),\,\mathcal{H})$ to system (3.6). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C([0,\,+\infty),\,D(\mathcal{A})) \cap C^1([0,\,+\infty), \mathcal{H}).$$

Proof: By Lumer-Phillips' theorem, it suffices to show that $\mathcal{A}$ is m-dissipative (see Definition 3.3.1 and Theorems 1.4.3 and 1.4.6 of [93]).

We first prove that $\mathcal{A}$ is dissipative. Take $U = (\omega,\,u,\,z)^\top \in D(\mathcal{A})$. Then

$$\begin{aligned}
\langle \mathcal{A}U,\,U\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} u \\ -A\omega - B_1B_1^*u - B_2 z(1) \\ -\frac{1}{\tau}\frac{\partial z}{\partial \rho} \end{pmatrix}, \begin{pmatrix} \omega \\ u \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= \left(A^{\frac{1}{2}}u,\, A^{\frac{1}{2}}\omega\right)_H - (A\omega + B_1B_1^*u + B_2z(1),\,u)_H - \xi\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho),\,z(\rho)\right)_{U_2} d\rho.
\end{aligned}$$

Since $A\omega + B_1B_1^*u + B_2z(1) \in H$, we obtain

$$\begin{aligned}
\langle \mathcal{A}U,\,U\rangle_{\mathcal{H}} &= \left(A^{\frac{1}{2}}u,\, A^{\frac{1}{2}}\omega\right)_H - \langle A\omega,\,u\rangle_{V',V} - \langle B_1B_1^*u,\,u\rangle_{V',V} - \langle B_2z(1),\,u\rangle_{V',V} \\
&\quad -\xi\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho),\,z(\rho)\right)_{U_2} d\rho \\
&= \langle A\omega,\,u\rangle_{V',V} - \langle A\omega,\,u\rangle_{V',V} - \|B_1^*u\|_{U_1}^2 - (z(1),\,B_2^*u)_{U_2} \\
&\quad -\xi\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho),\,z(\rho)\right)_{U_2} d\rho,
\end{aligned}$$

by duality. By intregrating by parts, we obtain

$$\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho),\, z(\rho)\right)_{U_2} d\rho = -\int_0^1 \left(z(\rho),\, \frac{\partial z}{\partial \rho}(\rho)\right)_{U_2} d\rho + (\|z(1)\|_{U_2}^2 - \|z(0)\|_{U_2}^2),$$

and thus

$$\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho),\, z(\rho)\right)_{U_2} d\rho = \frac{1}{2}(\|z(1)\|_{U_2}^2 - \|B_2^* u\|_{U_2}^2).$$

Therefore, by Cauchy-Schwarz's inequality, we find

$$\begin{aligned} \langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} &= -\|B_1^* u\|_{U_1}^2 - (z(1),\, B_2^* u)_{U_2} - \frac{\xi}{2}\|z(1)\|_{U_2}^2 + \frac{\xi}{2}\|B_2^* u\|_{U_2}^2 \\ &\leq -\|B_1^* u\|_{U_1}^2 + (\frac{1}{2} - \frac{\xi}{2})\|z(1)\|_{U_2}^2 + (\frac{1}{2} + \frac{\xi}{2})\|B_2^* u\|_{U_2}^2 . \end{aligned}$$

By (3.8), we obtain

$$\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} \leq (\frac{\alpha}{2} + \frac{\xi\alpha}{2} - 1)\|B_1^* u\|_{U_1}^2 + (\frac{1}{2} - \frac{\xi}{2})\|z(1)\|_{U_2}^2$$

with $\frac{\alpha}{2} + \frac{\xi\alpha}{2} - 1 \leq 0$ and $\frac{1}{2} - \frac{\xi}{2} \leq 0$ because ξ satisfies condition (3.9). This shows that $\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} \leq 0$ and then the dissipativeness of $\mathcal{A}$.

Let us now prove that $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$.

Let $(f,\, g,\, h)^T \in \mathcal{H}$. We look for $U = (\omega,\, u,\, z)^T \in D(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda\omega - u = f \\ \lambda u + A\omega + B_1 B_1^* u + B_2 z(1) = g \\ \lambda z + \frac{1}{\tau}\frac{\partial z}{\partial \rho} = h. \end{cases} \tag{3.10}$$

Suppose that we have found ω with the appropriate regularity. Then, we have

$$u = -f + \lambda\omega \in V.$$

We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1}{\tau}\frac{\partial z}{\partial \rho} = h$$

and the boundary condition $z(0) = B_2^* u = -B_2^* f + \lambda B_2^* \omega$. Therefore z is explicitely given by

$$z(\rho) = \lambda B_2^* \omega e^{-\lambda\tau\rho} - B_2^* f e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau\sigma} h(\sigma) d\sigma.$$

This means that once ω is found with the appropriate properties, we can find z and u. In particular, we have

$$z(1) = \lambda B_2^* \omega e^{-\lambda\tau} - B_2^* f e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 e^{\lambda\tau\sigma} h(\sigma) d\sigma = \lambda B_2^* \omega e^{-\lambda\tau} + z^0, \tag{3.11}$$

where $z^0 = -B_2^* f e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 e^{\lambda\tau\sigma} h(\sigma) d\sigma$ is a fixed element of U_2 depending only on f and h.

It remains to find ω. By (3.10), ω must satisfy

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + B_2 z(1) = g + B_1 B_1^* f + \lambda f,$$

and thus by (3.11),

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau} B_2 B_2^* \omega = g + B_1 B_1^* f + \lambda f - B_2 z^0 =: q,$$

where $q \in V'$. We take then the duality brackets $\langle ., . \rangle_{V',V}$ with $\phi \in V$:

$$\left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau} B_2 B_2^* \omega, \, \phi \right\rangle_{V',V} = \langle q, \, \phi \rangle_{V',V}.$$

Moreover :

$$\begin{aligned} &\left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau} B_2 B_2^* \omega, \, \phi \right\rangle_{V',V} \\ &= \lambda^2 \langle \omega, \, \phi \rangle_{V',V} + \langle A\omega, \, \phi \rangle_{V',V} + \lambda(\langle B_1 B_1^* \omega, \, \phi \rangle_{V',V} + e^{-\lambda\tau} \langle B_2 B_2^* \omega, \, \phi \rangle_{V',V}) \\ &= \lambda^2 (\omega, \, \phi)_H + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \phi\right)_H + \lambda((B_1^* \omega, \, B_1^* \phi)_{U_1} + e^{-\lambda\tau} (B_2^* \omega, \, B_2^* \phi)_{U_2}) \end{aligned}$$

because $\omega \in V \subset H$. Consequently, we arrive at the problem

$$\lambda^2 (\omega, \, \phi)_H + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \phi\right)_H + \lambda((B_1^* \omega, \, B_1^* \phi)_{U_1} + e^{-\lambda\tau} (B_2^* \omega, \, B_2^* \phi)_{U_2}) = \langle q, \, \phi \rangle_{V',V}. \tag{3.12}$$

The left hand side of (3.12) is continuous and coercive on V. Indeed, we have

$$\begin{aligned} &\left|\lambda^2 (\omega, \, \phi)_H + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \phi\right)_H + \lambda((B_1^* \omega, \, B_1^* \phi)_{U_1} + e^{-\lambda\tau} (B_2^* \omega, \, B_2^* \phi)_{U_2})\right| \\ &\leq \lambda^2 \|\omega\|_H \|\phi\|_H + \left\|A^{\frac{1}{2}} \omega\right\|_H \left\|A^{\frac{1}{2}} \phi\right\|_H + \lambda(\|B_1^* \omega\|_{U_1} \|B_1^* \phi\|_{U_1} + e^{-\lambda\tau} \|B_2^* \omega\|_{U_2} \|B_2^* \phi\|_{U_2}) \\ &\leq C\lambda^2 \|\omega\|_V \|\phi\|_H + \left\|A^{\frac{1}{2}}\right\|^2 \|\omega\|_V \|\phi\|_V \\ &+\lambda(\|B_1^*\|_{\mathcal{L}(V,U_1)}^2 \|\omega\|_V \|\phi\|_V + e^{-\lambda\tau} \|B_2^*\|_{\mathcal{L}(V,U_2)}^2 \|\omega\|_V \|\phi\|_V) \\ &\leq C \|\omega\|_V \|\phi\|_V, \end{aligned}$$

and for $\phi = \omega \in V$

$$\begin{aligned} &\lambda^2 \|\omega\|_H^2 + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \omega\right)_H + \lambda(\|B_1^* \omega\|_{U_1}^2 + e^{-\lambda\tau} \|B_2^* \omega\|_{U_2}^2) \\ &\geq \left\|A^{\frac{1}{2}} \omega\right\|_H^2 \geq C \|\omega\|_V^2. \end{aligned}$$

Therefore, this problem (3.12) has a unique solution $\omega \in V$ by Lax-Milgram's lemma. Moreover $A\omega + B_1 B_1^* u + B_2 z(1) = g + \lambda f - \lambda^2 \omega \in H$. In summary, we have found $(\omega, \, u, \, z)^T \in D(\mathcal{A})$ satisfying (3.10). ∎

Remark 3.2.2. *We deduce from Theorem 3.2.1 that $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see [93]).*

3.3 The energy

We now restrict the hypothesis (3.8) to obtain the decay of the energy. Namely, we suppose that (3.3) holds, namely

$$\exists 0 < \alpha < 1,\, \forall u \in V,\, \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2 .$$

Let us choose the following energy (which corresponds to the inner product on $\mathcal{H}$)

$$E(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega\right\|_H^2 + \|\dot{\omega}\|_H^2 + \tau\xi \int_0^1 \|B_2^*\dot{\omega}(t-\tau\rho)\|_{U_2}^2\, d\rho\right), \tag{3.13}$$

where ξ is a positive constant satisfying

$$1 < \xi < \frac{2}{\alpha} - 1, \tag{3.14}$$

that exists because $0 < \alpha < 1$.

3.3.1 Decay of the energy

Proposition 3.3.1. *If (3.3) holds, then for all $(\omega_0,\, \omega_1,\, f^0(-\tau.))^T \in D(\mathcal{A})$, the energy of the corresponding regular solution of (3.2) (i.e. $(\omega,\, \dot{\omega},\, B_2\dot{\omega}(t-\tau\rho))^T \in C([0,\, +\infty),\, D(\mathcal{A})) \cap C^1([0,\, +\infty), \mathcal{H})$) is non-increasing and there exist two positive constants C_1 and C_2 depending only on α and ξ such that*

$$-C_2\left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right) \leq E'(t) \leq -C_1\left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right). \tag{3.15}$$

Proof: Deriving (3.13), we obtain

$$\begin{aligned}
E'(t) &= \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\dot{\omega}\right)_H + (\dot{\omega},\, \ddot{\omega})_H + \tau\xi \int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho \\
&= \langle A\omega,\, \dot{\omega}\rangle_{V',V} - (\dot{\omega},\, A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t-\tau))_H \\
&\quad +\xi\tau \int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho \\
&= \langle A\omega,\, \dot{\omega}\rangle_{V',V} - \langle \dot{\omega},\, A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t-\tau)\rangle_{V,V'} \\
&\quad +\xi\tau \int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho,
\end{aligned}$$

because $A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t-\tau) \in H$. Then

$$\begin{aligned}
E'(t) &= \langle A\omega,\, \dot{\omega}\rangle_{V',V} - \langle \dot{\omega},\, A\omega\rangle_{V,V'} - \langle \dot{\omega},\, B_1B_1^*\dot{\omega}\rangle_{V,V'} - \langle \dot{\omega},\, B_2B_2^*\dot{\omega}(t-\tau)\rangle_{V,V'} \\
&\quad +\xi\tau \int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho \\
&= -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega},\, B_2^*\dot{\omega}(t-\tau))_{U_2} \\
&\quad +\xi\tau \int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho.
\end{aligned}$$

Moreover, recalling that $z(\rho, t) = B_2^*\dot{\omega}(t-\tau\rho)$, we see that

$$\begin{aligned}\int_0^1 (B_2^*\dot{\omega}(t-\tau\rho), B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho &= \int_0^1 \left(z(\rho, t), \frac{\partial z}{\partial t}(\rho, t)\right)_{U_2} d\rho \\ &= -\frac{1}{\tau}\int_0^1 \left(z(\rho, t), \frac{\partial z}{\partial \rho}(\rho, t)\right)_{U_2} d\rho,\end{aligned}$$

because $\frac{\partial z}{\partial \rho}(\rho, t) = -\tau\frac{\partial z}{\partial t}(\rho, t)$. Then, we have

$$\begin{aligned}\int_0^1 (B_2^*\dot{\omega}(t-\tau\rho), B_2^*\ddot{\omega}(t-\tau\rho))_{U_2}\, d\rho &= -\frac{1}{2\tau}\int_0^1 \frac{\partial}{\partial \rho}\|z(\rho, t)\|_{U_2}^2\, d\rho \\ &= -\frac{1}{2\tau}(\|z(1, t)\|_{U_2}^2 - \|z(0, t)\|_{U_2}^2) \\ &= -\frac{1}{2\tau}(\|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2 - \|B_2^*\dot{\omega}(t)\|_{U_2}^2).\end{aligned}$$

Consequently,

$$E'(t) = -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega}, B_2^*\dot{\omega}(t-\tau))_{U_2} - \frac{\xi}{2}\|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2 + \frac{\xi}{2}\|B_2^*\dot{\omega}(t)\|_{U_2}^2.$$

Cauchy-Schwarz's inequality yields

$$E'(t) \leq -\|B_1^*\dot{\omega}\|_{U_1}^2 + (\frac{1}{2}+\frac{\xi}{2})\|B_2^*\dot{\omega}(t)\|_{U_2}^2 + (\frac{1}{2}-\frac{\xi}{2})\|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2$$

and

$$E'(t) \geq -\|B_1^*\dot{\omega}\|_{U_1}^2 + (-\frac{1}{2}+\frac{\xi}{2})\|B_2^*\dot{\omega}(t)\|_{U_2}^2 - (\frac{1}{2}+\frac{\xi}{2})\|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2.$$

Therefore, by (3.3), these estimates leads to

$$E'(t) \leq -C_1\left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right)$$

with

$$C_1 = \min\left\{(1-\frac{\xi\alpha}{2}-\frac{\alpha}{2}), (\frac{\xi}{2}-\frac{1}{2})\right\}$$

which is positive according to the assumption (3.14), and to

$$E'(t) \geq -C_2\left(\|B_1^*\dot{\omega}\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right)$$

with

$$C_2 = \max\left\{1, \frac{\xi+1}{2}\right\}$$

which is also positive. ∎

Remark 3.3.2. Integrating the expression (3.15) between 0 and T, we obtain

$$\int_0^T \left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right) dt \leq \frac{1}{C_1}(E(0) - E(T)) \leq \frac{1}{C_1}E(0).$$

This estimate implies that $B_1^*\dot{\omega}(.) \in L^2((0, T), U_1)$ and $B_2^*\dot{\omega}(. - \tau) \in L^2((0, T), U_2)$. ■

Remark 3.3.3. If (3.3) is not satisfied, there exist cases where instabilities may appear, see chapter 1 and [85,110] for the wave equation. Hence this condition appears to be quite realistic. ■

3.3.2 Decay of the energy to 0

We give a necessary and sufficient condition that guarantees the decay to 0 of the energy.

Proposition 3.3.4. *Assume that (3.3) holds. Then, for all initial data in $\mathcal{H}$, we have*

$$\lim_{t\to\infty} E(t) = 0 \tag{3.16}$$

if and only if for any (non zero) eigenvector $\varphi \in D(A)$ of A, we have

$$B_1^*\varphi \neq 0. \tag{3.17}$$

Remark 3.3.5. Notice that this necessary and sufficient condition is the same than in the case without delay (see [105]) and therefore, the system (3.2) with delay is strongly stable (i.e. the energy tends to zero) if and only if the system without delay (i.e. for $B_2 = 0$) is strongly stable. ■

Proof: $\boxed{\Leftarrow}$ Let us show that (3.17) implies (3.16). For that purpose we closely follow [105].

First, we show that $\mathcal{A}$ has no eigenvalue on the imaginary axis. If it is not the case, let $i\omega$ be an eigenvalue of $\mathcal{A}$ where $\omega \in \mathbb{R}$. Let φ be an eigenvector associated with $i\omega$. Then φ is of the form

$$\varphi = \begin{pmatrix} z \\ i\omega z \\ i\omega e^{-i\omega\tau\rho} B_2^* z \end{pmatrix},$$

with

$$-\omega^2 z + Az + i\omega B_1 B_1^* z + i\omega e^{-i\omega\tau} B_2 B_2^* z = 0. \tag{3.18}$$

It is an immediate consequence of the identity $(i\omega I - \mathcal{A})\varphi = 0$.

First we notice that $\omega \neq 0$ since for $\omega = 0$, the above identity reduces to $Az = 0$ with $z \in D(A)$. Since by hypothesis A is invertible, we get $z = 0$ and therefore 0 is not an eigenvalue of $\mathcal{A}$.

We now take the duality bracket $\langle ., . \rangle_{V',V}$ between (3.18) and $z \in V$:

$$\begin{aligned} 0 &= \langle -\omega^2 z + Az + i\omega B_1 B_1^* z + i\omega e^{-i\omega\tau} B_2 B_2^* z,\ z \rangle_{V',V} \\ &= \langle (-\omega^2 I + A)z,\ z \rangle_{V',V} + i\omega \|B_1^* z\|_{U_1}^2 + i\omega e^{-i\omega\tau} \|B_2^* z\|_{U_2}^2 . \end{aligned}$$

We look at the imaginary part of this expression to obtain

$$\omega\left(\|B_1^*z\|_{U_1}^2 + \cos(\omega\tau)\,\|B_2^*z\|_{U_2}^2\right) = 0,$$

which implies, because $\omega \neq 0$,

$$\|B_1^*z\|_{U_1}^2 + \cos(\omega\tau)\,\|B_2^*z\|_{U_2}^2 = 0.$$

We deduce that

$$0 = \|B_1^*z\|_{U_1}^2 + \cos(\omega\tau)\,\|B_2^*z\|_{U_2}^2 \geq \|B_1^*z\|_{U_1}^2 - \|B_2^*z\|_{U_2}^2 \geq (1-\alpha)\,\|B_1^*z\|_{U_1}^2 \geq 0,$$

by (3.3) with $\alpha < 1$. Consequently

$$\|B_1^*z\|_{U_1} = 0$$

which implies

$$B_1^*z = 0. \tag{3.19}$$

Moreover, by (3.18), (3.19) and (3.3), we have

$$Az = \omega^2 z.$$

Therefore, z is an eigenvector of A of associated eigenvalue ω^2 such that

$$B_1^*z = 0,$$

which contradicts (3.17). Thus, $\mathcal{A}$ has no eigenvalue on the imaginary axis.

Now, we can apply the main theorem of Arendt and Batty [12] : As $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable (because $\sigma(A)$ is countable and what we have done previously) and as $\mathcal{A}$ has no eigenvalue on the imaginary axis, we obtain (3.16).

$\boxed{\Rightarrow}$ Let us show that (3.16) implies (3.17). For that purpose we use a contradiction argument. Suppose that there exists an eigenvector φ of A of associated eigenvalue λ^2 such that

$$B_1^*\varphi = 0.$$

Let us set

$$\omega(.,t) = \varphi\cos(\lambda t).$$

Then ω is solution of (3.2) and satisfies

$$E(t) = E(0)$$

because

$$\|B_1^*\dot{\omega}(t)\|_{U_1}^2 = \lambda^2\sin^2(\lambda t)\,\|B_1^*\varphi\|_{U_1}^2 = 0$$

and

$$\begin{aligned}\|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2 &= \lambda^2\sin^2(\lambda(t-\tau))\,\|B_2^*\varphi\|_{U_2}^2 \\ &\leq \alpha\lambda^2\sin^2(\lambda(t-\tau))\,\|B_1^*\varphi\|_{U_1}^2 = 0,\end{aligned}$$

by (3.3). This means that we have obtained a solution of problem (3.2) with a constant energy, which contradicts (3.16). ∎

3.4 The exponential stability

3.4.1 A priori estimate

In order to obtain the characterization of decay properties of the damped problem via observality inequalities for the conservative problem we will use the following assumption from [11] :
If $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C} \,|\, \Re\lambda = \beta\}$, the function

$$\lambda \in C_\beta \longrightarrow H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U) \text{ is bounded,} \tag{3.20}$$

where $B \in \mathcal{L}(U, V')$ with U a Hilbert space.

We consider the evolution problem

$$\left\{ \begin{array}{c} \ddot{y}(t) + Ay(t) = Bv(t) \\ y(0) = \dot{y}(0) = 0, \end{array} \right. \tag{3.21}$$

and the following conservative system

$$\left\{ \begin{array}{c} \ddot{\phi}(t) + A\phi(t) = 0 \\ \phi(0) = \omega_0,\, \dot{\phi}(0) = \omega_1. \end{array} \right. \tag{3.22}$$

Let us recall the two following results proved in [11] :

Lemma 3.4.1. *Suppose that $v \in L^2(0,T;U)$ and that the solutions ϕ of (3.22) are such that $B^*\phi(.) \in H^1(0,T;U)$ and there exists a constant $C > 0$ such that*

$$\|(B^*\phi)'(.)\|_{L^2(0,T;U)} \leq C \,\|(\omega_0,\, \omega_1)\|_{V\times H}\,,\ \forall(\omega_0,\, \omega_1) \in V \times H.$$

Then the problem (3.21) admits a unique solution having the regularity

$$y \in C(0,T;V) \cap C^1(0,T;H).$$

Proposition 3.4.2. *Suppose that $v \in L^2(0,T;U)$ and that the problem (3.21) admits a unique solution having the regularity*

$$y \in C(0,T;V) \cap C^1(0,T;H).$$

Then hypothesis (3.20) holds if and only if $B^*y(.) \in H^1(0,T;U)$ and there exists a constant $C > 0$ independent of T such that

$$\|(B^*y)'(\cdot)\|_{L^2(0,T;U)} \leq Ce^{\beta T} \,\|v\|_{L^2(0,T;U)}\,.$$

Let $\omega \in C(0,T;V) \cap C^1(0,T;H)$ be the solution of (3.2) with $(\omega_0, \omega_1, f^0(-\tau.))^T \in D(\mathcal{A})$. Then it can be split up in the form

$$\omega = \phi + \psi,$$

where ϕ is solution of the problem without damping (3.22), and ψ satisfies

$$\left\{ \begin{array}{c} \ddot{\psi}(t) + A\psi(t) = -B_1 B_1^* \dot{\omega}(t) - B_2 B_2^* \dot{\omega}(t-\tau) \\ \psi(0) = 0, \, \dot{\psi}(0) = 0. \end{array} \right. \tag{3.23}$$

We now set $B = (B_1 \, B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$. It is easy to verify that $B^* = \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} \in \mathcal{L}(V, U)$. Therefore ψ is solution of

$$\left\{ \begin{array}{l} \ddot{\psi}(t) + A\psi(t) = Bv(t) \\ \psi(0) = 0, \, \dot{\psi}(0) = 0, \end{array} \right. \tag{3.24}$$

where $v(t) = \begin{pmatrix} -B_1^* \dot{\omega}(t) \\ -B_2^* \dot{\omega}(t-\tau) \end{pmatrix}$. In other words, ψ is solution of problem (3.21) with

$$B = (B_1 \, B_2)$$

and by Remark 3.3.2

$$v = \begin{pmatrix} -B_1^* \dot{\omega}(\cdot) \\ -B_2^* \dot{\omega}(\cdot - \tau) \end{pmatrix} \in L^2((0,T), U).$$

Then $\psi = \omega - \phi \in C(0,T;V) \cap C^1(0,T;H)$. Suppose that the hypothesis (3.20) is satisfied for $B = (B_1 \, B_2)$ and $U = U_1 \times U_2$. By applying Proposition 3.4.2, we obtain

$$\int_0^T \|(B^*\psi)'\|_U^2 \, dt \leq Ce^{2\beta T} \int_0^T \|v(t)\|_U^2 \, dt,$$

which is equivalent to

$$\int_0^T (\|(B_1^*\psi)'\|_{U_1}^2 + \|(B_2^*\psi)'\|_{U_2}^2) dt \leq Ce^{2\beta T} \int_0^T (\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2) dt.$$

In particular, we have

$$\int_0^T \|(B_1^*\psi)'\|_{U_1}^2 \, dt \leq Ce^{2\beta T} \int_0^T (\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2) dt.$$

Therefore, since $\omega = \phi + \psi$, we have

$$\begin{aligned} \int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, dt &\leq 2\left(\int_0^T \|(B_1^*\omega)'(t)\|_{U_1}^2 \, dt + \int_0^T \|(B_1^*\psi)'(t)\|_{U_1}^2 \, dt \right) \\ &\leq Ce^{2\beta T} \int_0^T \left(\|(B_1^*\dot{\omega})(t)\|_{U_1}^2 + \|(B_2^*\dot{\omega})(t-\tau)\|_{U_2}^2 \right) dt. \end{aligned}$$

Thus, we have proved the following result :

Lemma 3.4.3. *Suppose that the assumption (3.20) is satisfied for $B = (B_1\, B_2)$, $U = U_1 \times U_2$. Then the solutions ω of (3.2) and ϕ of (3.22) satisfy*

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\, dt \leq Ce^{2\beta T} \int_0^T (\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2)dt,$$

with $C > 0$ independent of T.

3.4.2 The stability result

Theorem 3.4.4. *Assume that the hypotheses (3.3) and (3.20) are verified for $B = (B_1\, B_2)$, $U = U_1 \times U_2$. If there exist a time $T > 0$ and a constant $C > 0$ such that the observability estimate*

$$\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 \leq C \int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\, dt \tag{3.25}$$

holds, where ϕ is solution of (3.22), then the system (3.2) is exponentially stable in the energy space : there exist $C > 0$ independent of τ and $\nu > 0$ such that, for all initial data in $\mathcal{H}$,

$$E(t) \leq CE(0)e^{-\nu t}, \forall t > 0. \tag{3.26}$$

Proof: Let ω be a solution of (3.2) with initial data $(\omega_0,\, \omega_1,\, f^0(-\tau\cdot)) \in D(\mathcal{A})$.

Without loss of generality, we can always assume that (3.25) holds with $T > \tau$ and C independent of τ.

Integrating the inequality (3.15) of Proposition 3.3.1 between 0 and T, we obtain

$$\begin{aligned}
E(0) - E(T) &\geq C \int_0^T \left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right) dt \\
&\geq \frac{C}{2} \int_0^T \left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\right) dt + \frac{C}{2}\int_0^T \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\, dt \\
&\geq Ce^{-2\beta T} \left(\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\, dt + \int_0^T \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\, dt\right) \text{ by Lemma 3.4.3.}
\end{aligned}$$

By assumption (3.25), we obtain

$$E(0) - E(T) \geq Ce^{-2\beta T} \left(\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 + \int_0^T \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\, dt\right),$$

with C independent of τ. Since $T > \tau$, by change of variables, we have

$$\begin{aligned}
\int_0^T \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2\, dt &= \int_{-\tau}^{T-\tau} \|B_2^*\dot{\omega}(t)\|_{U_2}^2\, dt \\
&\geq \int_{-\tau}^0 \|B_2^*\dot{\omega}(t)\|_{U_2}^2\, dt = \tau \int_0^1 \|B_2^*\dot{\omega}(-\tau\rho)\|_{U_2}^2\, d\rho.
\end{aligned}$$

The two previous inequalities and (3.14) directly imply that

$$E(0) - E(T) \geq C'e^{-2\beta T} \left(\left\| A^{\frac{1}{2}} \omega_0 \right\|_H^2 + \|\omega_1\|_H^2 + \xi\tau \int_0^1 \|B_2^* \dot{\omega}(-\tau\rho)\|_{U_2}^2 \, d\rho \right),$$

with $C' = C/(2/\alpha - 1)$. This means that for $T > \tau$, we have

$$E(0) - E(T) \geq C'e^{-2\beta T} E(0).$$

This estimate is equivalent to

$$E(0) - E(T) \geq C'e^{-2\beta T} E(T),$$

because the energy is decreasing, which leads to

$$E(T) \leq \gamma E(0),$$

where $\gamma = \frac{1}{1+C'e^{-2\beta T}} < 1$. Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, ...$ (which is valid because the system is invariant by translation in time), we will get

$$E(mT) \leq \gamma E((m-1)T) \leq ... \leq \gamma^m E(0).$$

Therefore, we have

$$E(mT) \leq e^{-\nu m T} E(0), \; m = 1, 2, ...$$

with $\nu = \frac{1}{T} \ln \frac{1}{\gamma} = \frac{1}{T} \ln(1 + C'e^{-2\beta T}) > 0$ which depends on T and thus on τ (because $T > \tau$). For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$ and by the non-increasing property of the energy, we conclude

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0).$$

Hence the energy decays exponentially with the decay rate $\nu = \frac{1}{T} \ln \frac{1}{\gamma} = \frac{1}{T} \ln(1+C'e^{-2\beta T}) < \frac{1}{\tau} \ln(1 + C'e^{-2\beta\tau})$, because $T > \tau$. Notice that the constant C of (3.26) can be chosen such that $C \geq 1 + C'$ (which is independent of τ) because $\frac{1}{\gamma} = 1 + C'e^{-2\beta T} \leq 1 + C'$.

By density of $D(\mathcal{A})$ into $\mathcal{H}$, we deduce that (3.26) holds for any initial data in $\mathcal{H}$. ∎

Remark 3.4.5. In the previous theorem, we have seen that the decay rate is $\nu = \frac{1}{T} \ln \frac{1}{\gamma} = \frac{1}{T} \ln(1 + C'e^{-2\beta T}) < \frac{1}{\tau} \ln(1 + C'e^{-2\beta\tau})$ because $T > \tau$ and where C' is independent of τ. Therefore, when τ becomes larger, the decay is slower. ∎

Remark 3.4.6. Notice that the sufficient condition (3.25) for the exponential decay of the energy is the same than the case without delay (see [11]). Therefore, if the hypothesis (3.20) holds and if the dissipative system without delay (i.e. with $B_2 = 0$) is exponentially stable, then the system (3.2) is exponentially stable. ∎

3.5 The polynomial stability

In some cases, the decay of the energy is not exponential, but can be polynomial. Our aim here is to give a sufficient condition that yields the explicit decay rate.

The proof of this stability result requires the next technical Lemma proved in [10, Lemma 5.2].

Lemma 3.5.1. *Let $(\varepsilon_k)_k$ be a sequence of positive real numbers satisfying*

$$\varepsilon_{k+1} \leq \varepsilon_k - C\varepsilon_{k+1}^{2+\mu}, \ \forall k \geq 0, \tag{3.27}$$

where $C > 0$ and $\mu > -1$ are constants. Then there exists a positive constant M (depending on μ and C) such that

$$\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{1+\mu}}}, \ \forall k \geq 0,$$

with $M > \left(\frac{4}{(1+\mu)C}\right)^{\frac{1}{1+\mu}}$.

Moreover we recall the following interpolation result.

Lemma 3.5.2. *For $(\omega_0, \omega_1) \in D(A) \times V$, we have*

$$\begin{aligned} \|\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} &\leq C\|\omega_0\|_{D(A)}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})}, \\ \|\omega_1\|_H^{m+1} &\leq C\|\omega_1\|_{D(A^{\frac{1}{2}})}^m \|\omega_1\|_{D(A^{-\frac{m}{2}})}, \end{aligned}$$

where $C > 0$.

Proof: If we denote by $\{\lambda_{k_n}\}_n$ the eigenvalues of $A^{\frac{1}{2}}$ counted without multiplicity, l_n the multiplicity of the eigenvalue λ_{k_n} and $\{\varphi_{k_n+j}\}_{0\leq j\leq l_n-1}$ the orthonormal eigenvectors associated with the eigenvalue λ_{k_n}, this lemma is a direct consequence of the equivalence $\|u\|_{D(A^s)}^2 \sim \sum_{n\geq 1}\sum_{j=0}^{l_n-1}|u_{k_n+j}|^2\lambda_{k_n}^{4s}$ for all $s \in \mathbb{R}$, when $u = \sum_{n\geq 1}\sum_{j=0}^{l_n-1} u_{k_n+j}\varphi_{k_n+j}$ and of Hölder's inequality with $p = 1 + \frac{1}{m}$ and $q = m + 1$. ∎

Since for $(\omega_0, \omega_1, f^0(-\tau.)) \in D(\mathcal{A})$, ω_0 is not necessarily in $D(A)$, we can not apply Lemma 3.5.2 to ω_0, and therefore we need to make the following hypothesis : there exists $C > 0$ such that for all $(\omega_0, \omega_1, z) \in D(\mathcal{A})$, we have

$$\|\omega_0\|_V^{m+1} \leq C\|(\omega_0, \omega_1, z)\|_{D(\mathcal{A})}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})}. \tag{3.28}$$

Theorem 3.5.3. *Let ω be a solution of (3.2) with $(\omega_0, \omega_1, f^0(-\tau\cdot)) \in D(\mathcal{A})$. Assume that the hypotheses (3.3), (3.20) and (3.28) are verified for $B = (B_1\, B_2)$, $U = U_1 \times U_2$. If there exist a positive real number m, a time $T > 0$ and a constant $C > 0$ such that*

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\, dt \geq C\left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2\right) \tag{3.29}$$

holds where ϕ is solution of (3.22), then the energy decays polynomially, i.e. there exists $C > 0$ depending on m and τ such that, for all initial data in $D(\mathcal{A})$,

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{m}}} \left\|(\omega_0, \omega_1, f^0(-\tau\cdot))\right\|^2_{D(\mathcal{A})}, \forall t > 0. \tag{3.30}$$

Proof: Since the hypothesis (3.20) is satisfied for $B = (B_1\, B_2)$, $U = U_1 \times U_2$, by using Lemma 3.4.3, we obtain

$$\int_0^T \left(\|B_1^*\dot{\omega}(t)\|^2_{U_1} + \|B_2^*\dot{\omega}(t-\tau)\|^2_{U_2}\right) dt \geq Ce^{-2\beta T}\left(\|\omega_0\|^2_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{m}{2}})}\right).$$

On the other hand, integrating the inequality (3.15) of Proposition 3.3.1 between 0 and T^* for T^* large enough : $T^* \geq \max(T, \tau)$, we have

$$\begin{aligned} E(0) - E(T^*) &\geq C\int_0^{T^*} (\|B_1^*\dot{\omega}(t)\|^2_{U_1} + \|B_2^*\dot{\omega}(t-\tau)\|^2_{U_2})dt \\ &\geq \frac{C}{2}\int_0^{T^*} (\|B_1^*\dot{\omega}(t)\|^2_{U_1} + \|B_2^*\dot{\omega}(t-\tau)\|^2_{U_2})dt + \frac{C}{2}\int_0^{T^*} \|B_2^*\dot{\omega}(t-\tau)\|^2_{U_2}\, dt \\ &\geq Ce^{-2\beta T^*}\left(\|\omega_0\|^2_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{m}{2}})} + \tau\int_0^1 \|B_2^*\dot{\omega}(-\tau\rho)\|^2_{U_2}\, d\rho\right), \end{aligned}$$

by change of variable (because $T^* > \tau$). Therefore

$$E(T^*) \leq E(0) - K_1 e^{-2\beta T^*}\left(\|\omega_0\|^2_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^2_{D(A^{-\frac{m}{2}})} + \tau\xi\int_0^1 \|B_2^*\dot{\omega}(-\tau\rho)\|^2_{U_2}\, d\rho\right), \tag{3.31}$$

for some $K_1 > 0$ independent of T^* and τ.

Therefore, by (3.28), the previous interpolation result of Lemma 3.5.2 and a convexity inequality, we have :

$$\begin{aligned} \|(\omega_0, \omega_1)\|^{m+1}_{V\times H} &\leq C\left(\|\omega_0\|^{m+1}_V + \|\omega_1\|^{m+1}_H\right) \\ &\leq C\left(\left\|(\omega_0, \omega_1, f^0(-\tau.))\right\|^m_{D(\mathcal{A})}\|\omega_0\|_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|^m_{D(A^{\frac{1}{2}})}\|\omega_1\|_{D(A^{-\frac{m}{2}})}\right) \\ &\leq C\left\|(\omega_0, \omega_1, f^0(-\tau.))\right\|^m_{D(\mathcal{A})}\left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|_{D(A^{-\frac{m}{2}})}\right). \end{aligned}$$

Denoting by $X_{-m} = D(A^{\frac{1-m}{2}}) \times D(A^{-\frac{m}{2}})$, we have shown that

$$\|(\omega_0, \omega_1)\|^2_{X_{-m}} \geq C\frac{\|(\omega_0, \omega_1)\|^{2m+2}_{V\times H}}{\|(\omega_0, \omega_1, f^0(-\tau.))\|^{2m}_{D(\mathcal{A})}}. \tag{3.32}$$

Now introduce the modified energy

$$\tilde{E}(t) = \frac{1}{2}\|U(t)\|^2_{D(\mathcal{A})} = \frac{1}{2}(\|U(t)\|^2_{\mathcal{H}} + \|\mathcal{A}U(t)\|^2_{\mathcal{H}}).$$

As in Proposition 3.3.1, this energy $\tilde{E}$ is decaying.

Combining the estimates (3.31) and (3.32), we obtain

$$E(T^*) \le E(0) - K_2 e^{-2\beta T^*} \left(\frac{\|(\omega_0,\, \omega_1)\|_{V\times H}^{2m+2}}{\tilde{E}(0)^m} + \xi\tau \int_0^1 \|B_2^*\dot{\omega}(-\tau\rho)\|_{U_2}^2\, d\rho \right),$$

for some $K_2 > 0$ independent of T^* and τ, or equivalently

$$E(T^*) \le E(0) - K_2 e^{-2\beta T^*} \left(\frac{\|(\omega_0,\, \omega_1)\|_{V\times H}^{2m+2}}{\tilde{E}(0)^m} + \xi\tau \left\|f^0(-\tau.)\right\|_{L^2((0,1),U_2)}^2 \right). \tag{3.33}$$

Using the trivial estimate

$$\begin{aligned}
(\xi\tau)^{m+1} \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^{2m+2} &= \xi\tau \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^2 (\xi\tau)^m \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^{2m} \\
&\le \tau\xi \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^2 \tilde{E}(0)^m
\end{aligned}$$

the above inequality (3.33) becomes

$$\begin{aligned}
E(T^*) &\le E(0) - K_2 e^{-2\beta T^*} \left(\frac{\|(\omega_0,\, \omega_1)\|_{V\times H}^{2m+2} + (\xi\tau)^{m+1} \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^{2m+2}}{\tilde{E}(0)^m} \right) \\
&\le E(0) - K' e^{-2\beta T^*} \frac{E(0)^{m+1}}{\tilde{E}(0)^m},
\end{aligned}$$

with $K' > 0$ independent of T^* and τ. Since the energy of our system is decaying, we obtain

$$E(T^*) \le E(0) - K' e^{-2\beta T^*} \frac{E(T^*)^{m+1}}{\tilde{E}(0)^m}. \tag{3.34}$$

We now follow the method used in [10]. The estimate (3.34) being valid on the intervals $[kT^*,\, (k+1)T^*]$, for any $k \ge 0$, we have

$$E((k+1)T^*) \le E(kT^*) - K' e^{-2\beta T^*} \frac{E((k+1)T^*)^{m+1}}{\tilde{E}(kT^*)^m}. \tag{3.35}$$

Setting

$$\varepsilon_k = \frac{E(kT^*)}{\tilde{E}(0)},$$

and dividing (3.35) by $\tilde{E}(0)$, we obtain

$$\varepsilon_{k+1} \le \varepsilon_k - K' e^{-2\beta T^*} \varepsilon_{k+1}^{m+1}, \tag{3.36}$$

because $\tilde{E}(kT^*) \le \tilde{E}(0)$. By Lemma 3.5.1 with $\mu = m - 1 > -1$ (since $m > 0$), there exists a constant $M' > 0$ (depending on m and $K' e^{-2\beta T^*}$, and verifying $M' > \left(\frac{4e^{2\beta T^*}}{mK'}\right)^{\frac{1}{m}}$) such that

$$\varepsilon_k \le \frac{M'}{(1+k)^{\frac{1}{m}}},\ \forall k \ge 0,$$

or equivalently

$$E(kT^*) \leq \frac{M'}{(1+k)^{\frac{1}{m}}} \tilde{E}(0).$$

This estimate and again the decay of the energy lead to the estimate (3.30), where $C = M'(1+T^*)^{\frac{1}{m}}$. ■

Remark 3.5.4. Since the proof of the above theorem reveals that $C = M'(1+T^*)^{\frac{1}{m}}$ with $M' > \left(\frac{4e^{2\beta T^*}}{mK'}\right)^{\frac{1}{m}}$ and $T^* > \tau$, the constant C depends on τ and when τ becomes larger, the decay becomes slower. ■

3.6 Checking the observability inequalities

In this section, we show how to obtain the observability inequalities used in Theorems 3.4.4 and 3.5.3. Our method is based on the generalized gap condition. Before giving spectral conditions to obtain exponential or polynomial decay, we recall some results about Ingham's inequality.

3.6.1 Preliminaries about Ingham's inequality

Let $\{\lambda_k\}_{k\geq 1}$ be the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities (i.e. we repeat the eigenvalues according to their multiplicities). We further rewrite the sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ as follows :

$$\lambda_{k_1} < \lambda_{k_2} < \ldots < \lambda_{k_i} < \ldots$$

where $k_1 = 1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. For all $i \in \mathbb{N}^*$, let l_i be the multiplicity of the eigenvalue λ_{k_i}, i.e.

$$\lambda_{k_i-1} < \lambda_{k_i} = \lambda_{k_i+1} = \ldots = \lambda_{k_i+l_i-1} < \lambda_{k_i+l_i} = \lambda_{k_{i+1}}.$$

We have $k_1 = 1$, $k_2 = l_1$, $k_3 = l_1 + l_2$, etc. Let $\{\varphi_{k_i+j}\}_{0\leq j\leq l_i-1}$ be the orthonormal eigenvectors associated with the eigenvalue λ_{k_i}. We assume that the following generalized gap condition holds :

$$\exists M \in \mathbb{N}^*, \exists \gamma_0 > 0, \forall k \geq 1, \lambda_{k+M} - \lambda_k \geq M\gamma_0. \tag{3.37}$$

Fix a positive real number $\gamma_0' \leq \gamma_0$ and denote by A_k, $k = 1, \ldots, M$ the set of natural numbers k_m satisfying (see for instance [13])

$$\begin{cases} \lambda_{k_m} - \lambda_{k_{m-1}} \geq \gamma_0' \\ \lambda_{k_n} - \lambda_{k_{n-1}} < \gamma_0' & \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{k_{m+k}} - \lambda_{k_{m+k-1}} \geq \gamma_0'. \end{cases}$$

Then one easily checks that the sets $A_k + j$, $j = 0, ..., k-1$, $k = 1, ..., M$ form a partition of $\mathbb{N}^*$. Notice that some sets A_k may be empty because, for the generalized gap condition, the choice of M takes into account multiple eigenvalues.

Now for $k_m \in A_k$, we recall that the finite differences $e_{m+j}(t)$, $j = 0, \cdots, k-1$, corresponding to the exponential functions $e^{i\lambda_{k_{m+j}} t}, j = 0, \cdots, k-1$ are given by

$$e_{m+j}(t) = \sum_{p=m}^{m+j} \prod_{\substack{q=m \\ q \neq p}}^{m+j} (\lambda_{k_p} - \lambda_{k_q})^{-1} e^{i\lambda_{k_p} t}.$$

Write for shortness, $e_{-n}(t)$ the same finite differences functions corresponding to $-\lambda_{k_n}$.

Now we are ready to recall the next inequality of Ingham's type, see for instance Theorem 1.5 of [13] :

Theorem 3.6.1. *If the sequence $(\lambda_n)_{n\geq 1}$ satisfies (3.37), then for all sequence $(a_n)_{n\in\mathbb{Z}^*}$ (where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$), the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} a_n e_n(t),$$

satisfies the estimates

$$\int_0^T |f(t)|^2 dt \sim \sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{3.38}$$

for $T > \frac{2\pi}{\gamma_0}$.

Going back to the original functions $e^{i\lambda_{k_n} t}$, the above equivalence (3.38) means that, for $T > \frac{2\pi}{\gamma_0}$, the function (from now on $\lambda_{-k_n} = -\lambda_{k_n}$)

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n} t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 dt \sim \sum_{k=1}^{M} \sum_{|k_n|\in A_k} \|B_{k_n}^{-1} C_{k_n}\|_2^2, \tag{3.39}$$

where $\|\cdot\|_2$ means the Euclidean norm of the vector, for $k_n \in A_k$ the vector C_{k_n} is given by

$$C_{k_n} = (\alpha_n, \cdots, \alpha_{n+k-1})^T,$$

and the $k \times k$ matrix B_{k_n} allows to pass from the coefficients a_{k_n} to α_{k_n}, namely

$$C_{k_n} = B_{k_n} \cdot (a_n, \cdots, a_{n+k-1})^T,$$

and is given by $B_{k_n} = (B_{k_n,ij})_{1\le i,j\le k}$ the matrix of size $k\times k$ such that

$$B_{k_n,ij} = \begin{cases} \displaystyle\prod_{\substack{q=n \\ q\neq n+i-1}}^{n+j-1} (\lambda_{k_{n+i-1}} - \lambda_{k_q})^{-1} & \text{if } i\le j,\ (i,\,j)\neq(1,\,1), \\ 1 & \text{if } (i,\,j)=(1,\,1), \\ 0 & \text{if } i>j. \end{cases}$$

More explicitly, we have

$$B_{k_n} = \begin{pmatrix} 1 & \frac{1}{\lambda_{k_n}-\lambda_{k_{n+1}}} & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})(\lambda_{k_n}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})\cdots(\lambda_{k_n}-\lambda_{k_{n+k-1}})} \\ 0 & \frac{1}{\lambda_{k_{n+1}}-\lambda_{k_n}} & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})(\lambda_{k_{n+1}}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+1}}-\lambda_{k_{n+k-1}})} \\ 0 & 0 & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})(\lambda_{k_{n+2}}-\lambda_{k_{n+1}})} & \cdots & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})\cdots(\lambda_{k_{n+2}}-\lambda_{k_{n+k-1}})} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(\lambda_{k_{n+k-1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+k-1}}-\lambda_{k_{n+k-2}})} \end{pmatrix}.$$

We proceed similarly for $n\le -1$, the indices being decreasing from n to $n-k+1$.

Remark 3.6.2. If the standard gap condition

$$\exists\gamma_0>0,\ \forall n\ge 1,\ \lambda_{k_{n+1}}-\lambda_{k_n}\ge\gamma_0 \tag{3.40}$$

holds, then $A_1=\mathbb{Z}^*$ and $B_1=1$ and in that case $f(t)=\sum_{n\in\mathbb{Z}^*}\alpha_n e^{i\lambda_{k_n}t}$ satisfy Ingham's inequality (see [57]) :

$$\int_0^T |f(t)|^2dt \sim \sum_{n\in\mathbb{Z}^*}|\alpha_n|^2, \tag{3.41}$$

for $T>\frac{2\pi}{\gamma_0}$. ∎

Now, let U be a separable Hilbert space (in the sequel, U will be U_1). For a vector $c=\begin{pmatrix} c_1 \\ \vdots \\ c_m\end{pmatrix}$ in U^m, we set $\|.\|_{U,2}$ the norm in U^m defined by

$$\|c\|_{U,2}^2=\sum_{l=1}^m\|c_l\|_U^2\,.$$

Then we obtain the inequality of Ingham's type in U :

Proposition 3.6.3. *If we have the standard gap condition (3.40), then for all sequence $(a_n)_n$ in U, the function*

$$u(t)=\sum_{n\in\mathbb{Z}^*}a_ne^{i\lambda_{k_n}t}$$

satisfies the estimates

$$\int_0^T \|u(t)\|_U^2 \, dt \sim \sum_{n\in\mathbb{Z}^*} \|a_n\|_U^2 ,$$

for $T > \frac{2\pi}{\gamma_0}$.

Proof: Since U is a separable Hilbert space, there exists a Hilbert basis $(\psi_k)_{k\geq 1}$ of U. Therefore, $a_n \in U$ can be written as

$$a_n = \sum_{k=1}^{+\infty} a_n^k \psi_k.$$

We truncate a_n as follows : for $K \in \mathbb{N}^*$, let $a_n^{(K)} = \sum_{k=1}^{K} a_n^k \psi_k$ and set $u_K(t) = \sum_{k=1}^{K} (\sum_{n\in\mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t})\psi_k$. Since $(\psi_k)_{k\geq 1}$ is a Hilbert basis, we have by Fubini's theorem

$$\|u_K(t)\|_U^2 = \sum_{k=1}^{K} \left| \sum_{n\in\mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t} \right|^2 .$$

Thus, by applying Ingham's inequality (3.41), we have

$$\begin{aligned} \int_0^T \|u_K(t)\|_U^2 \, dt &= \sum_{k=1}^{K} \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t} \right|^2 dt \\ &\sim \sum_{k=1}^{K} \sum_{n\in\mathbb{Z}^*} (a_n^k)^2 \\ &\sim \sum_{n\in\mathbb{Z}^*} \sum_{k=1}^{K} (a_n^k)^2. \end{aligned}$$

Therefore

$$\int_0^T \|u_K(t)\|_U^2 \, dt \sim \sum_{n\in\mathbb{Z}^*} \|a_n^{(K)}\|_U^2 .$$

Since $u_K \to u$ and $a_n^{(K)} \to a_n$ when $K \to +\infty$, we obtain the result. ∎

In the same way, we obtain an Ingham's type inequality in a Hilbert space U in the case of the generalized gap condition (3.37).

Corollary 3.6.4. *If the sequence* $(\lambda_n)_{n\geq 1}$ *satisfies (3.37), then for all sequence* $(\alpha_n)_{n\in\mathbb{Z}^*}$ *in* U, *the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n} t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 dt \sim \sum_{k=1}^{M} \sum_{|k_n| \in A_k} \|B_{k_n}^{-1} C_{k_n}\|_{U,2}^2, \tag{3.42}$$

for $T > \frac{2\pi}{\gamma_0}$, *where*

$$C_{k_n} = (\alpha_n, \cdots, \alpha_{n+k-1})^T \in U^k.$$

3.6.2 A first observability inequality

Proposition 3.6.5. *Assume that the generalized gap condition (3.37) holds and that* U_1 *is separable. Let* ϕ *be the solution of (3.22) with* $(\omega_0, \omega_1) \in V \times H$. *Then there exists a time* $T > 0$ *and a constant* $C > 0$ *(depending on* T*) such that (3.25) holds if and only if*

$$\exists \gamma > 0, \forall k = 1, \cdots, M, \forall k_n \in A_k, \forall \xi \in \mathbb{R}^{L_n}, \left\|B_{k_n}^{-1} \Phi_{k_n} \xi\right\|_{U_1,2} \geq \gamma \|\xi\|_2, \tag{3.43}$$

where the matrix Φ_{k_n} *with coefficients in* U_1 *and size* $k \times L_n$, *where* $L_n = \sum_{i=1}^k l_{n+i-1} - 1$, *is given as follow : for all* $i = 1, \cdots, k$, *we set*

$$(\Phi_{k_n})_{ij} = \begin{cases} B_1^* \varphi_{k_{n+i-1}+j-L_{n,i-1}} & \text{if } L_{n,i-1} < j \leq L_{n,i}, \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} L_{n,0} &= 0, \\ L_{n,i} &= \sum_{i'=1}^{i} l_{n+i'-1} - 1, \ \forall i \geq 1. \end{aligned} \tag{3.44}$$

Proof: We first show that (3.43)⇒(3.25). Writting

$$\omega_0 = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} a_{k_i+j} \varphi_{k_i+j}$$

and

$$\omega_1 = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} b_{k_i+j} \varphi_{k_i+j}$$

where $(\lambda_{k_i} a_{k_i+j})_{i,j}, (b_{k_i+j})_{i,j} \in l^2(\mathbb{N}^*)$, then the solution ϕ of problem without damping (3.22) is given by

$$\phi(\cdot, t) = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} \left(a_{k_i+j} \cos(\lambda_{k_i} t) + \frac{b_{k_i+j}}{\lambda_{k_i}} \sin(\lambda_{k_i} t) \right) \varphi_{k_i+j}.$$

Consequently

$$(B_1^*\phi)'(t) = \sum_{i\geq 1}\sum_{j=0}^{l_i-1} \left(-a_{k_i+j}\lambda_{k_i}\sin(\lambda_{k_i}t) + b_{k_i+j}\cos(\lambda_{k_i}t)\right) B_1^*\varphi_{k_i+j}.$$

By grouping the terms corresponding to the same eigenvalue, we get

$$\begin{aligned}(B_1^*\phi)'(t) &= \sum_{i\geq 1}\left(\sum_{j=0}^{l_i-1} - a_{k_i+j}B_1^*\varphi_{k_i+j}\right)\lambda_{k_i}\sin(\lambda_{k_i}t) + \sum_{i\geq 1}\left(\sum_{j=0}^{l_j-1} b_{k_i+j}B_1^*\varphi_{k_i+j}\right)\cos(\lambda_{k_i}t)\\ &= \sum_{n\in\mathbb{Z}^*}\alpha_n e^{i\lambda_{k_n}t},\end{aligned}$$

where

$$\alpha_n = \frac{1}{2}\left(\left(\sum_{j=0}^{l_n-1} b_{k_n+j}B_1^*\varphi_{k_n+j}\right) + i\left(\sum_{j=0}^{l_n-1} a_{k_n+j}B_1^*\varphi_{k_n+j}\right)\lambda_{k_n}\right),\ \forall n\geq 1,$$

$$\alpha_{-n} = \frac{1}{2}\left(\left(\sum_{j=0}^{l_n-1} b_{k_n+j}B_1^*\varphi_{k_n+j}\right) - i\left(\sum_{j=0}^{l_n-1} a_{k_n+j}B_1^*\varphi_{k_n+j}\right)\lambda_{k_n}\right),\ \forall n\geq 1.$$

Integrating the square of the norm of this identity between 0 and $T>0$ and using Ingham's inequality (3.42) in U_1, for T large enough, we get

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\,dt \geq C\sum_{k=1}^{M}\sum_{|k_n|\in A_k}\left\|B_{k_n}^{-1}C_{k_n}\right\|_{U_1,2}^2,$$

where $C_{k_n} = (\alpha_n, ..., \alpha_{n+k-1})^T$ is a vector of U_1^k.

But for all $k_n\in A_k$, setting

$$\begin{aligned}\tilde{A}_{k_n} &= \left(\lambda_{k_n}a_{k_n},\cdots,\lambda_{k_n}a_{k_n+l_n-1},\lambda_{k_{n+1}}a_{k_{n+1}},\cdots,\lambda_{k_{n+1}}a_{k_{n+1}+l_{n+1}-1},\cdots,\right.\\ &\qquad\left.\lambda_{k_{n+k-1}}a_{k_{n+k-1}},\cdots,\lambda_{k_{n+k-1}}a_{k_{n+k-1}+l_{n+k-1}-1}\right)^T,\\ \tilde{B}_{k_n} &= \left(b_{k_n},\cdots,b_{k_n+l_n-1},b_{k_{n+1}},\cdots,b_{k_{n+1}+l_{n+1}-1},\cdots,b_{k_{n+k-1}},\cdots,\right.\\ &\qquad\left.b_{k_{n+k-1}+l_{n+k-1}-1}\right)^T,\end{aligned}$$

we readily check that

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\,dt \geq C\sum_{k=1}^{M}\sum_{|k_n|\in A_k}\left(\left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{A}_{k_n}\right\|_{U_1,2}^2 + \left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{B}_{k_n}\right\|_{U_1,2}^2\right).$$

Hence the assumption (3.43) yields

$$\begin{aligned}\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\,dt &\geq C\sum_{k=1}^{M}\sum_{|k_n|\in A_k}\left(\left\|\tilde{A}_{k_n}\right\|_2^2+\left\|\tilde{B}_{k_n}\right\|_2^2\right)\\ &= C\left(\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2+\|\omega_1\|_H^2\right)\end{aligned}$$

because $(\varphi_{k_n+i})_{n,i}$ is an orthonormal basis associated with the operator $A^{\frac{1}{2}}$.

It remains to show that (3.25)⇒(3.43).

Let $k = 1,\cdots,M$ and $k_n \in A_k$ be fixed. Take $\omega_0 = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1} a_{k_i+j}\varphi_{k_i+j}$ and $\omega_1 = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1} b_{k_i+j}\varphi_{k_i+j}$. Then the solution ϕ of problem (3.22) is given by

$$\phi(\cdot,\,t) = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1}\left(a_{k_i+j}\cos(\lambda_{k_i}t)+\frac{b_{k_i+j}}{\lambda_{k_i}}\sin(\lambda_{k_i}t)\right)\varphi_{k_i+j},$$

and then

$$(B_1^*\phi)'(t) = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1}(-a_{k_i+j}\lambda_{k_i}\sin(\lambda_{k_i}t)+b_{k_i+j}\cos(\lambda_{k_i}t))\,B_1^*\varphi_{k_i+j}.$$

Applying again Ingham's inequality, we get for T large enough and $\tilde{A}_{k_n}$, $\tilde{B}_{k_n}$ define above

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2\,dt \sim \left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{A}_{k_n}\right\|_{U_1,2}^2+\left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{B}_{k_n}\right\|_{U_1,2}^2.$$

By (3.25), we obtain

$$\begin{aligned}\left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{A}_{k_n}\right\|_{U_1,2}^2+\left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{B}_{k_n}\right\|_{U_1,2}^2 &\geq C\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2+\|\omega_1\|_H^2\\ &= C\sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1}(a_{k_i+j}^2\lambda_{k_i}^2+b_{k_i+j}^2),\end{aligned} \quad (3.45)$$

for some $C > 0$. Hence we conclude that

$$\left\|B_{k_n}^{-1}\Phi_{k_n}\xi\right\|_{U_1,2} \geq \gamma\,\|\xi\|_2\,.$$

This ends the proof. ■

Remark 3.6.6. If the standard gap condition (3.40) holds, then $A_1 = \mathbb{N}^*$ and $B_1 = 1$. In this case, the assumption (3.43) becomes

$$\exists\gamma>0,\ \forall k_n\geq 1,\ \forall\xi\in\mathbb{R}^{l_n},\ \|\Phi_{k_n}\xi\|_{U_1}\geq\gamma\,\|\xi\|_2\,.$$

Moreover, if the standard gap condition (3.40) holds and if the eigenvalues are simple, the assumption (3.43) becomes

$$\exists\gamma>0,\ \forall k\geq 1,\ \|B_1^*\varphi_k\|_{U_1}\geq\gamma.$$

■

Remark 3.6.7. The above Proposition 3.6.5 yields a time $T > 0$ and a constant $C > 0$ depending on T such that (3.25) holds but the time T and the constant C do not depend on the delay τ. Hence if the minimal time T is not strictly greater than τ, by choosing $T' > \max\{T, \tau\}$, we still have

$$\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 \leq C\int_0^{T'} \|(B_1^*\phi)'(t)\|_{U_1}^2\, dt,$$

with the same constant C as before and that does not depend on τ. This means that under the generalized gap condition (3.37), the condition (3.43) is equivalent to the sufficient condition of Theorem 3.4.4. ■

3.6.3 A second observability inequality

Proposition 3.6.8. *Assume that the generalized gap condition (3.37) holds and that U_1 is separable. Let ϕ be the solution of (3.22) with $(\omega_0, \omega_1) \in V \times H$. Then for a fixed real number $m > 0$, there exist a time $T > 0$ and a constant $C > 0$ such that (3.29) holds if and only if*

$$\exists\gamma > 0,\ \forall k = 1, \cdots, M,\ \forall k_n \in A_k,\ \forall\xi \in \mathbb{R}^{L_n},\ \left\|B_{k_n}^{-1}\Phi_{k_n}\xi\right\|_{U_1,2} \geq \frac{\gamma}{\lambda_{k_n}^m}\|\xi\|_2. \qquad (3.46)$$

Proof: The proof is similar to the one of Proposition 3.6.5 because

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{\lambda_{k_n}^{2m}}\sum_{j=0}^{l_n-1}(a_{k_n+j}^2\lambda_{k_n}^2 + b_{k_n+j}^2) &= \sum_{n\geq 1}\sum_{j=0}^{l_n-1}(a_{k_n+j}^2\lambda_{k_n}^{2(1-m)} + b_{k_n+j}^2\lambda_{k_n}^{-2m}) \\ &\sim \|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2.\end{aligned}$$

The details are therefore omitted. ■

Remark 3.6.9. If the standard gap condition (3.40) holds, the assumption (3.46) becomes

$$\exists\gamma > 0,\ \forall k_n \geq 1,\ \forall\xi \in \mathbb{R}^{l_n},\ \|\Phi_{k_n}\xi\|_{U_1} \geq \frac{\gamma}{\lambda_{k_n}^m}\|\xi\|_2.$$

Moreover, if the standard gap condition (3.40) holds and if the eigenvalues are simple, the assumption (3.46) becomes

$$\exists\gamma > 0,\ \forall k \geq 1,\ \|B_1^*\varphi_k\|_{U_1} \geq \frac{\gamma}{\lambda_{k_n}^m}.$$

■

A remark similar to Remark 3.6.7 can be made for polynomial stability.

3.7 Examples

We end up this chapter by considering different examples for which our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

3.7.1 A wave equation on 1-d networks with nodal feedbacks

In this section we show that the result obtained in chapter 1 (see also [89]) enter in the framework of this chapter. Obviously we will use the same notation for a network that in chapter 1 that we briefly recall. We denote by $\mathcal{E} = \{e_j;\ 1 \leq j \leq N\}$ the set of edges e_j of length $l_j > 0$ of a given network $\mathcal{R}$ and $\mathcal{V}$ the set of vertices of $\mathcal{R}$. For a function $u : \mathcal{R} \to \mathbb{R}$, we set $u_j = u_{|e_j}$ the restriction of u to e_j. For a fixed vertex v, we set

$$\mathcal{E}_v = \{j \in \{1, ..., N\}\,;\, v \in \bar{e_j}\}.$$

If $card(\mathcal{E}_v) \geq 2$, v is an interior node. Let $\mathcal{V}_{int}$ be the set of interior nodes. If $card(\mathcal{E}_v) = 1$, v is an exterior node. Let $\mathcal{V}_{ext}$ be the set of exterior nodes. For $v \in \mathcal{V}_{ext}$, we set $\mathcal{E}_v = \{j_v\}$.

We now fix a partition of $\mathcal{V}_{ext}$:

$$\mathcal{V}_{ext} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{V}^c_{ext}, \text{ where } \mathcal{D} \neq \emptyset.$$

We actually will impose Dirichlet boundary condition at the nodes of $\mathcal{D}$, Neumann boundary condition at the nodes of $\mathcal{N}$ and finally a feedback boundary condition at the nodes of $\mathcal{V}^c_{ext}$. We further fix a subset $\mathcal{V}^c_{int}$ of $\mathcal{V}_{int}$ where a feedback transmission condition will be imposed. By shortness, we denote by $\mathcal{V}_c$ the set of controlled nodes, namely

$$\mathcal{V}_c = \mathcal{V}^c_{int} \cup \mathcal{V}^c_{ext}.$$

We here consider the following initial and boundary problem :

$$\left\{\begin{array}{ll}
\frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., N\} \\
u_j(v,\,t) = u_l(v,\,t) = u(v,\,t) & t > 0,\, \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int} \\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = -\alpha_1^{(v)} \frac{\partial u}{\partial t}(v,\,t) - \alpha_2^{(v)} \frac{\partial u}{\partial t}(v,\,t-\tau) & t > 0,\, \forall v \in \mathcal{V}_c \\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{V}_{int} \backslash \mathcal{V}^c_{int} \\
u_{j_v}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{D} \\
\frac{\partial u_{j_v}}{\partial n_{j_v}}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{N} \\
u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & \\
\frac{\partial u}{\partial t}(v,\,t-\tau) = f_v^0(t-\tau) & \forall v \in \mathcal{V}_c,\, 0 < t < \tau,
\end{array}\right. \tag{3.47}$$

where $\alpha_i^{(v)} \geq 0$ are fixed non-negative real numbers and the delay τ is positive.

To rewrite this problem in the form (3.2), we introduce

$$H = L^2(\mathcal{R}) = \{u : \mathcal{R} \to \mathbb{R};\ u_j \in L^2(0, l_j),\ \forall j = 1, ..., N\}$$

and the operator

$$A : \begin{cases} D(A) & \to & H \\ (\varphi_j)_j & \mapsto & (-\frac{d^2}{dx^2}\varphi_j)_j \end{cases} \tag{3.48}$$

where

$$D(A) = \{\varphi \in V \cap \prod_{j=1}^{N} H^2(0, l_j)\,;\ \sum_{j \in \mathcal{E}_v} \frac{\partial \varphi_j}{\partial n_j}(v) = 0,\ \forall v \in \mathcal{V}_{int};$$
$$\frac{\partial \varphi_{j_v}}{\partial n_{j_v}}(v) = 0,\ \forall v \in \mathcal{N} \cup \mathcal{V}_{ext}^c\}, \tag{3.49}$$

and

$$V := \{\varphi \in \prod_{j=1}^{N} H^1(0, l_j) \,:\, \varphi_j(v) = \varphi_k(v)\, \forall j,\, k \in \mathcal{E}_v,\ \forall v \in \mathcal{V}_{int}\,;\ \varphi_{j_v}(v) = 0,\ \forall v \in \mathcal{D}\}.$$

The operator A is self-adjoint and positive with a compact inverse in H. Moreover

$$D(A^{\frac{1}{2}}) = V.$$

We now define $U = U_1 = U_2 = \mathbb{R}^{V_c}$, where V_c is the cardinal of $\mathcal{V}_c$, with norm $\|.\|_U = \|.\|_2$ and the operators B_i for $i = 1, 2$ as

$$B_i : \begin{cases} U & \to & D(A^{\frac{1}{2}})' \\ (k_v)_{v \in \mathcal{V}_c} & \mapsto & \sum_{v \in \mathcal{V}_c} \sqrt{\alpha_i^{(v)}}\, k_v \delta_v \end{cases} . \tag{3.50}$$

It is easy to verify that $B_i^*(\varphi) = (\sqrt{\alpha_i^{(v)}}\varphi(v))^T_{v \in \mathcal{V}_c}$ for $\varphi \in D(A^{\frac{1}{2}})$ and thus $B_i B_i^*(\varphi) = \sum_{v \in \mathcal{V}_c} \alpha_i^{(v)} \varphi(v) \delta_v$ for $\varphi \in D(A^{\frac{1}{2}})$. Hence the system (3.47) can be rewritten in the form (3.2).

We notice that (3.8) is here reduced to

$$\exists 0 < \alpha \leq 1,\ \forall \varphi \in V,\ \sum_{v \in \mathcal{V}_c} (\alpha_2^{(v)} \varphi(v))^2 \leq \alpha \sum_{v \in \mathcal{V}_c} (\alpha_1^{(v)} \varphi(v))^2,$$

and therefore, the system (3.47) is well posed for $\alpha_2^{(v)} \leq \alpha_1^{(v)}$ for all $v \in \mathcal{V}_c$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2^{(v)} < \alpha_1^{(v)}$ for all $v \in \mathcal{V}_c$ by Proposition 3.3.1.

By Proposition 6.2 of [38], the generalized gap condition (3.37) holds with $M = N + 1$.

We know, by chapter 1, that the hypothesis (3.20) is satisfied. Moreover, the hypothesis (3.28) is verified because

$$\begin{aligned} \|\omega_0\|_V^{m+1} &\leq C\,\|\omega_0\|_X^m\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})} \\ &\leq C\,\|(\omega_0,\,\omega_1,\,z)\|_{D(\mathcal{A})}^m\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}, \end{aligned}$$

where $X = V \cap (\prod_{j=1}^{N} H^2(0,\,l_j))$, by using Corollary 1.6.4.

Now we define $\Psi_{k_n}(v)$ the matrix of size $k \times L_n$ by : for all $i = 1, ..., k$, we set

$$(\Psi_{k_n}(v))_{ij} = \begin{cases} \varphi_{k_{n+i-1}+j-L_{n,\,i-1}}(v) & \text{if } L_{n,\,i-1} < j < L_{n,\,i}, \\ 0 & \text{else,} \end{cases}$$

where $L_{n,0} = 1$ and $L_{n,\,i} = \sum_{i'=1}^{i}(l_{n+i'-1} - 1)$ for $i \geq 1$. Then, for all $\xi \in \mathbb{R}^{L_n}$, we have

$$\begin{aligned} \left\|B_{k_n}^{-1}\Phi_{k_n}\xi\right\|_{U_1,2}^2 &= \sum_{v\in\mathcal{V}_c}\alpha_1^l\left\|B_{k_n}^{-1}\Psi_{k_n}(v)\xi\right\|_2^2 \\ &= \sum_{v\in\mathcal{V}_c}\alpha_1^l\xi^T\Psi_{k_n}(v)^T B_{k_n}^{-T}B_{k_n}^{-1}\Psi_{k_n}(v)\xi \\ &= \xi^T\left(\sum_{v\in\mathcal{V}_c}\alpha_1^l\Psi_{k_n}(v)^T B_{k_n}^{-T}B_{k_n}^{-1}\Psi_{k_n}(v)\right)\xi. \end{aligned}$$

Therefore by setting

$$\tilde{\mathcal{M}}^{k_n} = \sum_{v\in\mathcal{V}_c}\alpha_1^l\Psi_{k_n}(v)^T B_{k_n}^{-T}B_{k_n}^{-1}\Psi_{k_n}(v),$$

we see that the assumption (3.43) becomes

$$\exists\gamma > 0,\,\forall k \in \{1, ..., M\},\,\forall k_n \in A_k,\,\lambda_{min}(\tilde{\mathcal{M}}^{k_n}) \geq \gamma,$$

and the assumption (3.46) becomes

$$\exists m \in \mathbb{R}_+^*,\,\exists\gamma > 0,\,\forall k \in \{1, ..., M\},\,\forall k_n \in A_k,\,\lambda_{min}(\tilde{\mathcal{M}}^{k_n}) \geq \frac{\gamma}{\lambda_{k_n}^{2m}}$$

which corresponds respectively to the conditions (1.44) and (1.50) from chapter 1, because $\lambda_k \sim \frac{k\pi}{L}$, where $L = \sum_{j=1}^{N} l_j$.

Note that if the standard gap condition (3.40) holds and if all eigenvalues are simple (i.e. $l_k = 1$), then the condition (3.43) becomes

$$\exists\gamma > 0,\,\forall k \geq 1,\,\sum_{v\in\mathcal{V}_c}\alpha_1^{(v)}\left|\varphi_k(v)\right|^2 \geq \gamma, \tag{3.51}$$

while the conditions (3.46) becomes

$$\exists m \in \mathbb{R}_+^*,\ \exists \gamma > 0,\ \forall k \geq 1,\ \sum_{v \in \mathcal{V}_c} \alpha_1^{(v)} \left|\varphi_k(v)\right|^2 \geq \frac{\gamma}{\lambda_k^{2m}}. \tag{3.52}$$

Consequently, we find again all the results from chapter 1 (see for instance the examples treated in section 1.7), here we can even precise the dependence of the decay with respect to the delay τ.

3.7.2 One Euler-Bernoulli beam with interior damping

We consider an Euler-Bernoulli beam of length 1 with interior damping and a delay term at ξ. Two types of boundary conditions will be considered. Without delay, these two problems were analyzed in [10,11], where some decay rates similar to the ones proved below were obtained.

Mixed boundary conditions

We consider the following initial and boundary problem :

$$\left\{\begin{array}{ll} \frac{\partial^2 \omega}{\partial t^2}(x,\, t) + \frac{\partial^4 \omega}{\partial x^4}(x,\, t) + \alpha_1 \frac{\partial \omega}{\partial t}(\xi,\, t)\delta_\xi + \alpha_2 \frac{\partial \omega}{\partial t}(\xi,\, t-\tau)\delta_\xi = 0 & 0 < x < 1,\, t > 0 \\ \omega(0,\, t) = \frac{\partial \omega}{\partial x}(1,\, t) = \frac{\partial^2 \omega}{\partial x^2}(0,\, t) = \frac{\partial^3 \omega}{\partial x^3}(1,\, t) = 0 & t > 0 \\ \omega(x,\, 0) = \omega_0(x),\ \frac{\partial \omega}{\partial t}(x,\, 0) = \omega_1(x) & 0 < x < 1 \\ \frac{\partial \omega}{\partial t}(\xi,\, t-\tau) = f^0(t-\tau) & 0 < t < \tau, \end{array}\right. \tag{3.53}$$

where $\xi \in (0,\, 1)$, $\alpha_1, \alpha_2 > 0$ and $\tau > 0$. To enter into the framework of section 1, we rewrite this problem in the form (3.2). For that purpose, we introduce $H = L^2(0,\, 1)$ and the operator

$$A : D(A) \to H\ :\ \varphi \mapsto \frac{d^4}{dx^4}\varphi \tag{3.54}$$

where $D(A) = \{\varphi \in H^4(0,\, 1)\,;\, \varphi(0) = \frac{\partial \varphi}{\partial x}(1) = \frac{\partial^2 \varphi}{\partial x^2}(0) = \frac{\partial^3 \varphi}{\partial x^3}(1) = 0\}$. The operator A is self-adjoint and positive with a compact inverse in H. We now define $U = U_1 = U_2 = \mathbb{R}$ and the operators B_1 and B_2 as

$$B_i : U \to D(A^{\frac{1}{2}})'\ :\ k \mapsto \sqrt{\alpha_i}\, k\, \delta_\xi, i = 1, 2. \tag{3.55}$$

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi(\xi)$ for $\varphi \in D(A^{\frac{1}{2}})$ and thus $B_i B_i^*(\varphi) = \alpha_i \varphi(\xi)\delta_\xi$ for $\varphi \in D(A^{\frac{1}{2}})$ and $i = 1,\, 2$. Then the system (3.53) can be rewritten in the form (3.2). We notice that (3.8) is equivalent to

$$\exists 0 < \alpha \leq 1,\, \alpha_2 \leq \alpha\alpha_1,$$

and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.3.1.

Let us now state the next well known results about the spectral properties of A.

Proposition 3.7.1. *The eigenvalues of the operator A defined in (3.54) are simple and are given by* $\lambda_k^2 = (\frac{2k+1}{2}\pi)^4$ *of associated eigenvector* $\varphi_k(x) = \sqrt{2}\sin(\frac{2k+1}{2}\pi x)$, *for all* $k \in \mathbb{N}$. *Consequently the standard gap condition (3.40) holds, i.e. there exists a constant* $\gamma_0 > 0$ *such that*

$$\lambda_{k+1} - \lambda_k \geq \gamma_0 > 0,\ \forall k \geq 0,$$

and moreover for all $k \geq 0$, $\|B_1^*\varphi_k\|_{U_1} = \sqrt{2\alpha_1}\left|\sin((k\pi + \frac{\pi}{2})\xi)\right|$.

The hypothesis (3.20) was verified in [10]. Moreover, we have by Lemma 2.9 of [97] :

Lemma 3.7.2. ξ *is a rational number with an irreductible fraction*

$$\xi = \frac{p}{q},\ \textit{where } p \textit{ is odd}$$

if and only if there exists a constant $\gamma > 0$ *such that*

$$\forall k \geq 1,\ \left|\sin\left((k\pi + \frac{\pi}{2})\xi\right)\right| > \gamma.$$

Therefore, by applying Proposition 3.3.4 and Theorem 3.4.4, we obtain the following results :

Proposition 3.7.3. *Assume that* $\alpha_2 < \alpha_1$. *Then*

(i) The energy of system (3.53) decays to 0 *if and only if*

$$\xi \neq \frac{2m}{2k+1},\ m, k \in \mathbb{N}.$$

(ii) The energy of system (3.53) decays exponentially if ξ *is a rational number with an irreductible fraction*

$$\xi = \frac{p}{q},\ \textit{where } p \textit{ is odd.}$$

Remark 3.7.4. As mentioned before, in the case $\alpha_2 = 0$ we recover the results from [10]. ■

Other boundary conditions

We here consider the following initial and boundary problem :

$$\begin{cases} \frac{\partial^2\omega}{\partial t^2}(x,t) + \frac{\partial^4\omega}{\partial x^4}(x,t) + \alpha_1\frac{\partial\omega}{\partial t}(\xi,t)\delta_\xi + \alpha_2\frac{\partial\omega}{\partial t}(\xi,t-\tau)\delta_\xi = 0 & 0 < x < 1,\ t > 0 \\ \omega(0,t) = \omega(1,t) = \frac{\partial^2\omega}{\partial x^2}(0,t) = \frac{\partial^2\omega}{\partial x^2}(1,t) = 0 & t > 0 \\ \omega(x,0) = \omega^0(x),\ \frac{\partial\omega}{\partial t}(x,0) = \omega^1(x) & 0 < x < 1 \\ \frac{\partial\omega}{\partial t}(\xi,t-\tau) = f^0(t-\tau) & 0 < t < \tau, \end{cases} \tag{3.56}$$

where $\xi \in (0,1)$, $\alpha_1, \alpha_2 > 0$ and $\tau > 0$. This system (3.56) is not exponentially stable if $\alpha_2 = 0$ as shown in [11]. Hence we only consider the polynomial decay of system (3.56).

As before we rewrite this problem in the form (3.2) by introducing $H = L^2(0, 1)$ and the operator

$$A : D(A) \to H : \varphi \mapsto \frac{d^4}{dx^4}\varphi \tag{3.57}$$

with $D(A) = \{\varphi \in H^4(0, 1) \cap V;\ \frac{\partial^2\varphi}{\partial x^2}(0) = \frac{\partial^2\varphi}{\partial x^2}(1) = 0\}$, $V = H^2(0, 1) \cap H_0^1(0, 1)$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U = U_1 = U_2 = \mathbb{R}$ and the operators B_1, B_2 by (3.55).

Then the system (3.56) can be rewritten in the form (3.2) and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.3.1.

The spectral properties of A are well known and can be summarized as follows :

Proposition 3.7.5. *The eigenvalues of the operator A defined in (3.57) are simple and given by $\lambda_k^2 = k^4\pi^4$ of eigenvector $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, for all $k \in \mathbb{N}^*$. Therefore there exists a constant $\gamma_0 > 0$ such that the standard gap condition (3.40) is verified and moreover*

$$\|B_1^*\varphi_k\|_{U_1} = \sqrt{2\alpha_1}\,|\sin(k\pi\xi)|\,.$$

The hypothesis (3.20) was verified in [11]. Let us prove that the condition (3.28) is satisfied.

Lemma 3.7.6. *Let $m \in \mathbb{R}_+^*$. Then there exists $C > 0$ such that for all $\omega_0 \in X = \left\{u \in V : u\,\big|_{(0,\xi)} \in H^4(0, \xi),\ u\,\big|_{(\xi,1)} \in H^4(\xi, 1),\ \frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial x^2}(1) = 0\right\}$, we have*

$$\|\omega_0\|_V^{m+1} \leq C\,\|\omega_0\|_X^m\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}\,,$$

where the natural norm in X is given by $\|u\|_X^2 = \|u\|_{H^4(0,\xi)}^2 + \|u\|_{H^4(\xi,1)}^2$.

Proof: Let us fix a cut-off function $\eta \in \mathcal{D}(0, 1)$ such that $\eta = 1$ in a neighbourhood of ξ, $\eta = 0$ on $\left[\frac{2}{3} + \frac{\xi}{3},\ 1\right]$ and $\eta = 0$ on $\left[0,\ \frac{\xi}{3}\right]$. Since $(1-\eta)\omega_0 \in D(A)$, by Lemma 3.5.2, we have

$$\|(1-\eta)\omega_0\|_V^{m+1} \leq C\,\|(1-\eta)\omega_0\|_X^m\,\|(1-\eta)\omega_0\|_{D(A^{\frac{1-m}{2}})}\,.$$

Since

$$\begin{aligned}\|(1-\eta)\omega_0\|_{D(A^{\frac{1-m}{2}})} &= \sup_{\varphi\in D(A^{\frac{m-1}{2}})} \frac{((1-\eta)\omega_0,\varphi)}{\|\varphi\|_{D(A^{\frac{m-1}{2}})}}\\ &= \sup_{\varphi\in D(A^{\frac{m-1}{2}})} \frac{(\omega_0,(1-\eta)\varphi)}{\|\varphi\|_{D(A^{\frac{m-1}{2}})}}\\ &\leq C\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}\,,\end{aligned}$$

for some $C > 0$ (depending on η) we get

$$\|(1-\eta)\omega_0\|_V^{m+1} \leq C\,\|\omega_0\|_X^m\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}\,. \tag{3.58}$$

In a second step, we set

$$\begin{array}{ll} \omega_1(x) = \eta(\xi - x)\omega_0(\xi - x) & \text{if } 0 < x < l_1 := \xi \\ \omega_2(x) = \eta(x + \xi)\omega_0(x + \xi) & \text{if } 0 < x < l_2 := 1 - \xi. \end{array}$$

For any $j = 1, 2$, we introduce the following extension of ω_j :

$$\begin{array}{rll} (E\omega)_j(x) &= \omega_j(x) & \text{if } x \in (0, l_j), \\ (E\omega)_{-j}(x) &= \sum_{i=0}^{n-1} \nu_i \omega_j(-2^i x) & \text{if } x \in (-2^{-(n-1)} l_j, 0), \end{array}$$

where ω_j is extended by zero outside its support and the real numbers ν_i are the unique solution of the system

$$\left\{ \begin{array}{l} \sum_{i=0}^{n-1} \nu_i = 1 \\ -\sum_{i=0}^{n-1} 2^i \nu_i = 1 \\ \sum_{i=0}^{n-1} 2^{2i} \nu_i = 1 \\ -\sum_{i=0}^{n-1} 2^{3i} \nu_i = 1 \\ \sum_{i=0}^{n-1} 2^{-2ki} \nu_i = 1, \forall k = 1, \cdots, n-4, \end{array} \right.$$

and finally $n \in \mathbb{N}^*$ is choosen large enough such that $n \geq m + 3$.

We obtain an extension of ω to a function $E\omega$, which belongs to $\mathcal{D}(\tilde{A})$ (due to the four first properties of the ν_i), where $\tilde{A}$ is the positive operator $\frac{d^4}{dx^4}$ on the star shaped network $\tilde{\mathcal{S}} = \cup_{j=1,2}(0, l_j) \bigcup \cup_{j=1,2}(-2^{-(n-1)} l_j, 0)$, with interior vertex ξ (identified to 0) and Dirichlet boundary conditions at all other vertices.

Therefore, we can apply the interpolation lemma 3.5.2 to $\tilde{A}$ and then write

$$\|E\omega\|_{D(\tilde{A}^{\frac{1}{2}})}^{m+1} \leq C \|E\omega\|_{D(\tilde{A})}^{m} \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})}.$$

But we easily check that

$$\|\eta\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} \leq \|E\omega\|_{D(\tilde{A}^{\frac{1}{2}})}^{m+1}.$$

Consequently, we have

$$\|\eta\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} \leq C \|E\omega\|_{D(\tilde{A})}^{m} \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})}.$$

Moreover, since E is an extension operator, we have

$$\|E\omega\|_{D(\tilde{A})} \leq K \|\omega_0\|_X,$$

and thus

$$\|\eta\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} \leq C \|\omega_0\|_X^m \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})}.$$

To estimate the last factor, we use a duality argument. We write

$$\|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} = \sup_{\varphi \in D(\tilde{A}^{\frac{m-1}{2}})} \frac{|(E\omega, \varphi)|}{\|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}}.$$

For $\varphi \in D(\tilde{A}^{\frac{m-1}{2}})$, we have

$$\int_{\tilde{S}} E\omega\varphi = \sum_{j=1,2}\int_0^{l_j} \omega_j(x)\varphi_j(x)\,dx + \sum_{j=1,2}\int_{-2^{-(n-1)}l_j}^{0} (E\omega)_{-j}(x)\varphi_{-j}(x)\,dx.$$

By changes of variables, we obtain

$$\int_{\tilde{S}} E\omega\varphi = \sum_{j=1,2}\int_0^{l_j} \omega_j(x)(F\varphi)_j(x)\,dx,$$

where

$$(F\varphi)_j(x) = \varphi_j(x) + \chi_j(x)\sum_{i=0}^{n-1}\nu_i 2^{-i}\varphi_j(-2^{-i}x), \forall x \in (0, l_j),$$

the cut-off function χ_j being fixed such that $\chi_j \equiv 1$ on $[0, 2l_j/3]$ and $\chi_j \equiv 0$ on $[5l_j/6, l_j]$ (reminding that $\omega_j(x) \equiv 0$ for $x > 2l_j/3$). Now we notice that the conditions on ν_i guarantees that $F\varphi$ belongs to $D(A^{\frac{m-1}{2}})$ and by Leibniz's rule we have

$$\|F\varphi\|_{D(A^{\frac{m-1}{2}})} \leq C\,\|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}.$$

Therefore

$$\int_{\tilde{S}} E\omega\varphi \leq C\,\|\omega\|_{D(A^{\frac{1-m}{2}})}\,\|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}.$$

By duality, we conclude that

$$\|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} \leq C\,\|\omega\|_{D(A^{\frac{1-m}{2}})}.$$

Consequently, with the previous inequalities, we obtain

$$\|\eta\omega_0\|^{m+1}_{D(A^{\frac{1}{2}})} \leq C\,\|\omega\|^m_X\,\|\omega\|_{D(A^{\frac{1-m}{2}})}. \tag{3.59}$$

The conclusion follows from (3.58) and (3.59). ■

This lemma leads to (3.28). Indeed for $(\omega_0,\, \omega_1,\, z) \in D(\mathcal{A})$, we have

$$\begin{aligned}\|\omega_0\|_V^{m+1} &\leq C\,\|\omega_0\|_X^m\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}\\ &\leq C\,\|(\omega_0,\, \omega_1,\, z)\|^m_{D(\mathcal{A})}\,\|\omega_0\|_{D(A^{\frac{1-m}{2}})}.\end{aligned}$$

Now, we denote by $\mathcal{S}$ the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0,\, a_1, \ldots,\, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then the sequence (a_n) is bounded. It is well known that $\mathcal{S}$ is uncountable and that its Lebesgue measure is zero. Roughly speaking, the set $\mathcal{S}$ contains all irrational numbers which are badly approximated by rational numbers. In particular, by the Euler-Lagrange theorem, $\mathcal{S}$ contains all irrational quadratic numbers (i.e. the roots of a second order equation with rational coefficients). By a classical result, we have :

Lemma 3.7.7. *If $s \in \mathcal{S}$, then there exists a positive constant γ such that*

$$|\sin(k\pi s)| \geq \frac{\gamma}{k}, \; \forall k \geq 1.$$

Therefore, by applying Proposition 3.3.4 and Theorem 3.5.3 with $m = \frac{1}{2}$, we obtain the next results :

Proposition 3.7.8. *Assume that $\alpha_2 < \alpha_1$. Then*
(i) The energy of system (3.56) decays to 0 if and only if ξ is irrational.
(ii) The energy of system (3.56) decays polynomially like $\frac{1}{(1+t)^2}$ if ξ belongs to $\mathcal{S}$.

Remark 3.7.9. Again in the case $\alpha_2 = 0$ we recover the results from [10, 11]. ■

3.7.3 Examples with distributed damping terms

A non homogeneous string with distributed damping terms (1-d)

We consider the following initial and boundary problem :

$$\begin{cases} \frac{\partial^2\omega}{\partial t^2}(x,\, t) - \frac{\partial^2\omega}{\partial x^2}(x,\, t) + \alpha_1 \frac{\partial\omega}{\partial t}(x,\, t)\chi_{|I_1} + \alpha_2 \frac{\partial\omega}{\partial t}(x,\, t-\tau)\chi_{|I_2} = 0 & \text{in } (0,\, 1) \times (0,\, \infty) \\ \omega(0,\, t) = \omega(1,\, t) = 0 & t > 0 \\ \omega(x,\, 0) = \omega_0(x),\, \frac{\partial\omega}{\partial t}(x,\, 0) = \omega_1(x) & \text{in } (0,\, 1) \\ \frac{\partial\omega}{\partial t}(x,\, t-\tau) = f^0(x,\, t-\tau) & \text{in } I_2 \times (0,\, \tau), \end{cases} \tag{3.60}$$

where here and below $\chi_{|I}$ denotes the characteristic function of the set I. In the remainder of this subsection we assume that $\alpha_1,\, \alpha_2 > 0$, $\tau > 0$ and

$$I_2 \subset I_1 \subset [0,\, 1].$$

Later we will need that

$$\exists \delta \in [0,\, 1] \text{ and } \epsilon > 0 : \; [\delta,\, \delta + \epsilon] \subset I_1. \tag{3.61}$$

We rewrite this problem in the form (3.2). For that purpose, we introduce $H = L^2(0,\, 1)$ and the operator

$$A : D(A) \to H \; : \varphi \mapsto -\frac{d^2}{dx^2}\varphi \tag{3.62}$$

where $D(A) = H_0^1(0,\, 1) \cap H^2(0,\, 1)$ and $V = D(A^{\frac{1}{2}}) = H_0^1(0,\, 1)$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U_i = L^2(I_i)$ and the operators B_i as

$$B_i : U_i \to H \subset V' \; : \; k \mapsto \sqrt{\alpha_i}\, \tilde{k}\, \chi_{|I_i} \tag{3.63}$$

where $\tilde{k}$ is the extension of k by zero outside I_i (which defines an element of $L^2(0,\, 1)$).

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|I_i}$ for $\varphi \in V$ and thus $B_iB_i^*(\varphi) = \alpha_i\varphi_{|I_i}\chi_{|I_i} = \alpha_i\varphi\chi_{|I_i}$ for $\varphi \in V$ and $i = 1, 2$. Then the system (3.60) can be rewritten in the form (3.2). Moreover

$$\|B_i^*\varphi\|_{U_i}^2 = \alpha_i \int_{I_i} |\varphi|^2\, dx.$$

Therefore, we notice that (3.8) is equivalent to

$$\exists 0 < \alpha \leq 1,\ \alpha_2 \int_{I_2} |\varphi|^2\, dx \leq \alpha\alpha_1 \int_{I_1} |\varphi|^2\, dx,$$

and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.3.1.

Proposition 3.7.10. *The eigenvalues of the operator A defined in (3.62) are simple and given by $\lambda_k^2 = (k\pi)^2$ of associated eigenvector $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, for all $k \in \mathbb{N}^*$. Hence there exists a constant $\gamma_0 > 0$ such that the standard gap condition (3.40) holds. Moreover if (3.61) holds, then there exists $\gamma > 0$ such that $\forall k \geq 1$, $\|B_1^*\varphi_k\|_{U_1} \geq \gamma$.*

Proof: It is well known that the eigenvectors of the operator A are $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$ of eigenvalue $(k\pi)^2$, $k \geq 1$ of multiplicity 1. Hence, the standard gap condition (3.40) is verified.

From the definition of B_1^* we have

$$\begin{array}{rcl}\|B_1^*\varphi_k\|_{U_1}^2 & = & \alpha_1 \int_{I_1} |\varphi_k(x)|^2\, dx \\ & \geq & 2\alpha_1 \int_\delta^{\delta+\epsilon} |\sin(k\pi x)|^2\, dx = \alpha_1 \left[x - \frac{\sin(2k\pi x)}{2k\pi}\right]_\delta^{\delta+\epsilon} \\ & \geq & \alpha_1\left(\epsilon - \frac{1}{k\pi}\right) \geq \alpha_1\frac{\epsilon}{2},\end{array}$$

for $k \geq \mathbf{E}(\frac{2}{\epsilon\pi}) + 1 =: k_\epsilon$, where $\mathbf{E}(x)$ is the entire part of x. This leads to the conclusion because for $k \in \{1, ..., k_\epsilon\}$, we have

$$\int_\delta^{\delta+\epsilon} |\sin(k\pi x)|^2\, dx > 0.$$

■

Lemma 3.7.11. *The operators A and $B = (B_1B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy assumption (3.20).*

Proof: Let $i \in \{1, 2\}$ and $\varphi \in L^2(I_i)$. It can be easily checked that $v = (\lambda^2 + A)^{-1}B_i\varphi$ satisfies

$$\left\{\begin{array}{l}\lambda^2 v - \frac{d^2v}{dx^2} = \sqrt{\alpha_i}\tilde{\varphi}\chi_{|I_i} \\ v(0) = v(1) = 0.\end{array}\right. \tag{3.64}$$

Since $v \in L^2(0, 1)$, v can be written as

$$v = \sum_{k=1}^{\infty} c_k\varphi_k.$$

By replacing v in (3.64), c_k must satisfy

$$c_k = \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} \left(\tilde{\varphi},\, \varphi_k\right),$$

and therefore

$$v = \sum_{k=1}^{\infty} \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} \left(\tilde{\varphi},\, \varphi_k\right) \varphi_k.$$

Moreover

$$\|\lambda v\|^2_{L^2(0,\,1)} = \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \alpha_i \left|(\tilde{\varphi},\, \varphi_k)\right|^2.$$

Now, we set $z = \frac{\lambda}{\lambda^2+\lambda_k^2}$ and, if $\lambda = \beta + iy$, with $y \in \mathbb{R}$, then

$$|z|^2 = \frac{\beta^2 + y^2}{(\beta^2 - y^2 + \lambda_k^2)^2 + 4\beta^2 y^2} \le C(\beta),$$

where $C(\beta)$ is a positive constant depending only on β. Indeed, if $y^2 \le \frac{\beta^2+\lambda_k^2}{2}$, then

$$|z|^2 \le \frac{\beta^2 + y^2}{\left(\frac{\beta^2+\lambda_k^2}{2}\right)^2} \le \frac{\beta^2 + \frac{\beta^2+\lambda_k^2}{2}}{\left(\frac{\beta^2+\lambda_k^2}{2}\right)^2}$$

which is bounded uniformly in k, and if $y^2 \ge \frac{\beta^2+\lambda_k^2}{2}$, then

$$|z|^2 \le \frac{\beta^2 + y^2}{4\beta^2 y^2},$$

which is a decreasing function with respect to y and thus

$$|z|^2 \le \frac{\beta^2 + \frac{\beta^2+\lambda_k^2}{2}}{4\beta^2 \left(\frac{\beta^2+\lambda_k^2}{2}\right)},$$

which is again uniformly bounded in k. Therefore, we have

$$\|\lambda v\|^2_{L^2(0,\,1)} \le C(\beta)\alpha_i \sum_{k=1}^{\infty} \left|(\tilde{\varphi},\, \varphi_k)\right|^2 \le C(\beta)\alpha_i \|\varphi\|^2_{L^2(I_i)},$$

which leads to

$$\left\|\lambda B_j^* v\right\|^2_{L^2(I_j)} \le \alpha_j \|\lambda v\|^2_{L^2(0,\,1)} \le C(\beta)\alpha_i\alpha_j \|\varphi\|^2_{L^2(I_i)},\ \forall j \in \{1,\, 2\}.$$

Consequently, the operator $\lambda \to \lambda B_j^*(\lambda I + A)^{-1} B_i$ is bounded on C_β and the lemma is proved. ■

Therefore, by applying Theorem 3.4.4, we obtain :

Proposition 3.7.12. *If $\alpha_2 < \alpha_1$ and (3.61) holds, then the energy of system (3.60) decays exponentially.*

The wave equation with distributed damping terms

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_D and Γ_N, i.e. $\Gamma = \Gamma_D \cup \Gamma_N$, with $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$. Let

$$O_2 \subset O_1 \subset \Omega$$

such that O_1 is an open neighborhood of Γ_N (i.e. $\Gamma_N \subset \partial O_1$). Moreover, we assume that $x_0 \in \mathbb{R}^n$ is such that

$$(x - x_0) \cdot \nu(x) \leq 0, \forall x \in \Gamma_D. \tag{3.65}$$

We consider the following initial and boundary problem :

$$\left\{ \begin{array}{lr} \frac{\partial^2 \omega}{\partial t^2}(x,\, t) - \Delta\omega(x,\, t) + \alpha_1 \frac{\partial \omega}{\partial t}(x,\, t)\chi_{|O_1} + \alpha_2 \frac{\partial \omega}{\partial t}(x,\, t-\tau)\chi_{|O_2} = 0 & \text{in } \Omega \times (0,\, \infty), \\ \omega(x,\, t) = 0 & \text{on } \Gamma_D \times (0,\, \infty), \\ \frac{\partial \omega}{\partial \nu}(x,\, t) = 0 & \text{on } \Gamma_N \times (0,\, \infty), \\ \omega(x,\, 0) = \omega_0(x),\, \frac{\partial \omega}{\partial t}(x,\, 0) = \omega_1(x) & \text{in } \Omega, \\ \frac{\partial \omega}{\partial t}(x,\, t-\tau) = f^0(x,\, t-\tau) & \text{in } O_2 \times (0,\, \tau), \end{array} \right. \tag{3.66}$$

where $\frac{\partial \omega}{\partial \nu}$ is the normal derivative of ω and $\alpha_1,\, \alpha_2 > 0,\, \tau > 0$. In order to reformulate this problem in the form (3.2), we introduce $H = L^2(\Omega)$ and the operator

$$A : D(A) \longrightarrow H \; : \; \varphi \mapsto -\Delta\varphi \tag{3.67}$$

where $D(A) = \{\varphi \in V \cap H^2(\Omega) : \frac{\partial \varphi}{\partial \nu}(x) = 0 \text{ on } \Gamma_N\}$ and $V = D(A^{\frac{1}{2}}) = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D\}$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U_i = L^2(O_i)$ and the operators B_i as

$$B_i : U_i \rightarrow H \subset V' \; : \; k \mapsto \sqrt{\alpha_i}\, \tilde{k}\, \chi_{|O_i} \tag{3.68}$$

where $\tilde{k}$ is the extension of k by zero outside O_i (which defines an element of $L^2(\Omega)$).

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|O_i}$ for $\varphi \in V$ and thus $B_i B_i^*(\varphi) = \alpha_i \varphi_{|O_i}\chi_{|O_i}$ for $\varphi \in V$ and $i = 1,\, 2$. Then the system (3.66) can be rewritten in the form (3.2). Moreover

$$\|B_i^*\varphi\|_{U_i}^2 = \alpha_i \int_{O_i} |\varphi|^2 \, dx,$$

and therefore (3.8) is equivalent to

$$\exists 0 < \alpha \leq 1,\, \alpha_2 \int_{O_2} |\varphi|^2 \, dx \leq \alpha\alpha_1 \int_{O_1} |\varphi|^2 \, dx.$$

Consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.3.1.

To obtain the exponential decay of problem (3.66), we simply check the observability inequality (3.25) and the hypothesis (3.20), which are the aim of the two following propositions.

Proposition 3.7.13. *There exists a time T_0 such that for all times $T > T_0$ there exists a positive constant C (depending on T) for which the observability inequality (3.25) holds for any regular solution of problem (3.22).*

Proof: This proposition is proved in [85] and [68]. ∎

Proposition 3.7.14. *The operators A and $B = (B_1 B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy the assumption (3.20).*

Proof: Let $i \in \{1, 2\}$ and $\varphi \in L^2(O_i)$. It can be easily checked that $v = (\lambda^2 + A)^{-1} B_i \varphi$ satisfies

$$\begin{cases} \lambda^2 v - \Delta v = \sqrt{\alpha_i}\tilde{\varphi}\chi_{|O_i} & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_D, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.69}$$

Since $v \in L^2(\Omega)$, it can be written as

$$v = \sum_{k=1}^{\infty} c_k \varphi_k.$$

By replacing v in (3.69), c_k must satisfy

$$c_k = \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} (\tilde{\varphi}, \varphi_k),$$

and therefore

$$v = \sum_{k=1}^{\infty} \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} (\tilde{\varphi}, \varphi_k) \varphi_k.$$

Moreover, by Parseval's identity, we have

$$\|\lambda v\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \alpha_i \left| (\tilde{\varphi}, \varphi_k) \right|^2.$$

Now, for $\lambda = \beta + iy$, with $y \in \mathbb{R}$, we have checked in the previous subsection that

$$\left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \leq C(\beta),$$

where $C(\beta)$ is a positive constant depending only on β. Therefore, we have

$$\|\lambda v\|^2_{L^2(\Omega)} \leq C(\beta)\alpha_i \sum_{k=1}^{\infty} \left| (\tilde{\varphi}, \varphi_k) \right|^2 \leq C(\beta)\alpha_i \|\varphi\|^2_{L^2(O_i)},$$

which leads to

$$\left\|\lambda B_j^* v\right\|^2_{L^2(O_j)} \leq \alpha_j \|\lambda v\|^2_{L^2(\Omega)} \leq C(\beta)\alpha_i\alpha_j \|\varphi\|^2_{L^2(O_i)}, \forall j \in \{1, 2\}.$$

Consequently, the operator $\lambda \longrightarrow \lambda B_j^* (\lambda I + A)^{-1} B_i$ is bounded on C_β and the lemma is proved. ∎

Therefore, by applying Theorem 3.4.4, we obtain :

Proposition 3.7.15. *If $\alpha_2 < \alpha_1$, then the energy of system (3.66) decays exponentially.*

Remark 3.7.16. This result is a generalization of [85] because in [85] the authors supposed that $O_2 = O_1$. ■

A beam with distributed damping terms

We consider the following initial and boundary problem :

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x,\, t) + \frac{\partial^4 \omega}{\partial x^4}(x,\, t) + \alpha_1 \frac{\partial \omega}{\partial t}(x,\, t)\chi_{|I_1} + \alpha_2 \frac{\partial \omega}{\partial t}(x,\, t-\tau)\chi_{|I_2} = 0 & \text{in } (0,\, 1) \times (0,\, \infty), \\ \omega(0,\, t) = \omega(1,\, t) = \frac{\partial^2 \omega}{\partial x^2}(0,\, t) = \frac{\partial^2 \omega}{\partial x^2}(1,\, t) = 0 & t > 0, \\ \omega(x,\, 0) = \omega_0(x),\, \frac{\partial \omega}{\partial t}(x,\, 0) = \omega_1(x) & \text{in } \Omega, \\ \frac{\partial \omega}{\partial t}(x,\, t-\tau) = f^0(x,\, t-\tau) & \text{in } I_2 \times (0,\, \tau), \end{cases} \tag{3.70}$$

where $\alpha_1,\, \alpha_2 > 0,\, \tau > 0$ and where

$$I_2 \subset I_1 \subset [0,\, 1].$$

We rewrite this problem in the form (3.2) by introducing $H = L^2(0,\, 1)$, the operator A by (3.57) and the operators B_1 and B_2 by (3.63). Hence this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.3.1.

By the results of the previous subsections, we know that the standard gap condition holds and if (3.61) holds that the eigenvector $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$ of A associated with $\lambda_k^2 = k^4\pi^4$ satisfies

$$\|B_1^* \varphi_k\|_{U_1} \geq \gamma,$$

for some $\gamma > 0$.

Lemma 3.7.17. *The operators A and $B = (B_1 B_2) \in \mathcal{L}(U,\, V')$ where $U = U_1 \times U_2$ satisfy assumption (3.20).*

Proof: The proof is the same than in the previous subsection. ■

Therefore, by applying Proposition 3.3.4 and Theorem 3.4.4, we obtain :

Proposition 3.7.18. *If $\alpha_2 < \alpha_1$ and (3.61) holds, then the energy of system (3.70) decays exponentially.*

A wave equation on 1-d networks with internal damping terms

In this last subsection we consider again a wave equation on a given network $\mathcal{R}$ but we suppose that the feedbacks are located in the edges. Namely with the notation from section 3.7.1 we consider the problem :

$$
\begin{cases}
\frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) + \alpha_1^{(j)}\frac{\partial u_j}{\partial t}(x,\,t)\chi_{|I_1^{(j)}} & \\
+\alpha_2^{(j)}\frac{\partial u_j}{\partial t}(x,\,t-\tau)\chi_{|I_2^{(j)}} = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1,\cdots,N\}, \\
u_j(v,\,t) = u_l(v,\,t) = u(v,\,t) & t > 0,\, \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{int}, \\
\sum\limits_{j\in\mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{V}\backslash\mathcal{D}, \\
u_{j_v}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{D}, \\
u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)}, & \\
\frac{\partial u_j}{\partial t}(x,\,t-\tau) = f_j^0(x,\,t-\tau) & \text{in } I_2^{(j)} \times (0,\,\tau),\, \forall j \in \{1,\cdots,N\},
\end{cases}
\tag{3.71}
$$

where $\alpha_i^{(j)}$ are fixed non-negative real numbers, the delay τ is positive and the intervals $I_i^{(j)}$ satisfy

$$I_2^{(j)} \subset I_1^{(j)} \subset e_j.$$

As before we can rewrite this problem in the form (3.2), by introducing $H = L^2(\mathcal{R})$, the operator A defined by (3.48), $U_i = L^2(\cup_{j=1}^N I_i^{(j)})$ and the operators B_i for $i = 1,\,2$ as

$$
B_i : \begin{cases} U & \longrightarrow & D(A^{\frac{1}{2}})' \\ k & \longmapsto & \sum\limits_{j=1}^{N} \sqrt{\alpha_i^{(j)}}\tilde{k}\chi_{|I_i^{(j)}} \end{cases}
\tag{3.72}
$$

where $\tilde{k}$ means the extension of k by zero outside $\cup_{j=1}^N I_i^{(j)}$. It is easy to verify that

$$B_i^*(\varphi) = \sum_{j=1}^{N} \sqrt{\alpha_i^{(j)}}\varphi_{|I_i^{(j)}}\chi_{|I_i^{(j)}}$$

for $\varphi \in D(A^{\frac{1}{2}})$. Hence the system (3.71) can be rewritten in the form (3.2).

As before it is easy to see that the system (3.47) is well posed for $\alpha_2^{(j)} \leq \alpha_1^{(j)}$ for all $j = 1,\cdots,N$ by Theorem 3.2.1, and the energy is decreasing for $\alpha_2^{(j)} < \alpha_1^{(j)}$ for all $j = 1,\cdots,N$ by Proposition 3.3.1.

As mentioned before by Proposition 6.2 from [38], the generalized gap condition (3.37) holds with $M = N+1$, but in this general setting, the conditions (3.43) or (3.46) seem difficult to check. Hence, for the sake of simplicity, if we suppose here that the standard gap condition (3.40) holds and that all eigenvalues are simple (i.e. $l_k = 1$), then the condition (3.43) becomes

$$\exists \gamma > 0,\, \forall k \geq 1, \sum_{j=1}^{N} \alpha_1^{(j)} \int_{I_1^{(j)}} \varphi_k(x)^2\, dx \geq \gamma, \tag{3.73}$$

while the conditions (3.46) becomes

$$\exists m \in \mathbb{R}_+^*,\, \exists \gamma > 0,\, \forall k \geq 1, \sum_{j=1}^{N} \alpha_1^{(j)} \int_{I_1^{(j)}} \varphi_k(x)^2\, dx \geq \frac{\gamma}{\lambda_k^{2m}}. \tag{3.74}$$

For instance using an argument like in Proposition 3.7.10, we easily see that (3.73) (resp. (3.74)) holds if (3.51) (resp. (3.52)) holds and if $\cup_{j=1}^{N} I_1^{(j)}$ contains a neighborhood of the set of control points $\mathcal{V}_c$.

In the same manner, we have :

Proposition 3.7.19. *If there exists $\varepsilon > 0$ and for all $j \in \{1, \cdots, N\}$, there exists $\delta_j \in (0, l_j)$ such that*

$$[\delta_j, \delta_j + \varepsilon] \subset I_1^{(j)} \quad \forall j \in \{1, \cdots, N\},$$

then (3.73) holds.

Proof: For any $k \in \mathbb{N}^*$, let φ_k be the eigenvector of A associated with the eigenvalue λ_k^2. Then its restriction $\varphi_{k,j}$ to the edge e_j can be written in the form

$$\varphi_{k,j}(x) = c_{k,j} \cos(\lambda_k x) + d_{k,j} \sin(\lambda_k x) \quad \forall x \in (0, l_j),$$

for some real numbers $c_{k,j}$ and $d_{k,j}$. Hence the normalization of φ_k yields

$$1 = \sum_{j=1}^{N} \int_0^{l_j} |\varphi_{k,j}(x)|^2 \, dx \leq 2(\max_j l_j) \sum_{j=1}^{N} (c_{k,j}^2 + d_{k,j}^2). \tag{3.75}$$

On the other hand by the above expression of $\varphi_{k,j}$ and direct calculations, we see that

$$\begin{aligned}
\int_{I_1^{(j)}} \varphi_k(x)^2 \, dx &\geq \int_{\delta_j}^{\delta_j + \varepsilon} \varphi_k(x)^2 \, dx \\
&\geq (c_{k,j}^2 + d_{k,j}^2)(\frac{\varepsilon}{2} - \frac{1}{2\lambda_k}) - \frac{|c_{k,j} d_{k,j}|}{\lambda_k} \\
&\geq (c_{k,j}^2 + d_{k,j}^2)(\frac{\varepsilon}{2} - \frac{1}{\lambda_k}).
\end{aligned}$$

Therefore for k large enough such that

$$\frac{\varepsilon}{2} - \frac{1}{\lambda_k} \geq \frac{\varepsilon}{4},$$

which is equivalent to $k \geq k_\varepsilon$, for some $k_\varepsilon \in \mathbb{N}^*$, we deduce that

$$\int_{I_1^{(j)}} \varphi_k(x)^2 \, dx \geq \frac{\varepsilon}{4}(c_{k,j}^2 + d_{k,j}^2).$$

By summing this estimate on j and using the normalization estimate (3.75), we obtain (3.73) for $k \geq k_\varepsilon$. The proof is complete since for $k \leq k_\varepsilon$,

$$\sum_{j=1}^{N} \alpha_1^{(j)} \int_{I_1^{(j)}} \varphi_k(x)^2 \, dx > 0.$$

■

The analysis of the condition (3.73) in some particular cases reveals that the condition of the above Proposition is far from being optimal but in its full generality we cannot easily obtain a weaker condition.

As in the previous subsection one easily shows (see the proof of Lemma 3.7.11) that the operators A and $B = (B_1 B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy the assumption (3.20).

In conclusion applying either Theorem 3.4.4 or Theorem 3.5.3, we obtain :

Proposition 3.7.20. *Assume that $\alpha_2^{(j)} < \alpha_1^{(j)}$ for all $j = 1, \cdots, N$ and that the standard gap condition (3.40) holds and that all eigenvalues are simple (i.e. $l_k = 1$). Then*

(i) The energy of system (3.71) decays exponentially if (3.73) holds.

(ii) The energy of system (3.71) decays like $t^{-\frac{1}{m}}$ if (3.74) holds.

Chapitre 4

Stability of the heat and of the wave equations with boundary time-varying delays

4.1 Introduction

Time-delay often appears in many biological, electrical engineering systems and mechanical applications, and in many cases, delay is a source of instability [52]. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. [39, 75, 85, 98]). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

There are only a few works on Lyapunov-based technique for Partial Differential Equations (PDEs) with delay. Most of these works analyze the case of *constant delays*. Thus, stability conditions and exponential bounds were derived for some scalar heat and wave equations with constant delays and with Dirichlet boundary conditions without delay in [107, 108]. Stability and instability conditions for the wave equations with constant delay can be found in [85] and in chapter 1. The stability of linear parabolic systems with constant coefficients and internal constant delays has been studied in [55] in the frequency domain.

Recently the stability of PDEs with *time-varying delays* was analyzed in [27, 45, 92] via Lyapunov method. In the case of linear systems in the Hilbert space, the conditions of [27,45,92] assume that the operator acting on the delayed state is bounded, which means that this condition can not be applied to boundary delays. These conditions were applied to PDEs without delays in the boundary conditions (to $2D$ Navier-Stokes and to a scalar heat equations in [27], to a scalar heat and to a scalar wave equations in [45, 92]).

In the present chapter we analyze exponential stability of the heat and wave equations with time-varying boundary delay. Our main novel contribution is an extension of previous results from [85, 89] and chapter 1 to time-varying delays. This extension is not straightforward due to the loss of translation-invariance. In the constant delay case the exponential stability was proved in [85] and in chapter 1 by using the observability inequality which can not be applicable in the time-varying case (since the system is not invariant by translation). Hence we introduce new Lyapunov functionals with exponential terms and an additional term for the wave equation, which take into account the dependence of the delay with respect to time. For the treatment of others problems with Lyapunov technique see [45,86,92]. Note further that to the best of our knowledge the heat equation with boundary delay has not been treated in the literature. Contrary [85, 89] (and chapter 1), the existence results do not follow from standard semi-group theory because the spatial operator depends on time due to the time-varying delay. Therefore we use the variable norm technique of Kato [59, 60]. Finally for each problem we give explicit sufficient conditions that guarantee the exponential decay and for the first time we characterize the optimal decay rate that can be explicitly computed once the data are given.

The chapter is mainly decomposed in two parts treating the heat equation (section 4.2) and the wave equation (section 4.3). In the first subsection, we set the problem under consideration and prove existence results by using semigroup theory. In the second subsection we find sufficient conditions for the strict decay of the energy and finally in the last subsection we show that these conditions yield an exponential decay.

4.2 Exponential stability of the delayed heat equation

First, we consider the system described by

$$\left\{\begin{array}{ll} u_t(x,t) - a u_{xx}(x,t) = 0, & 0 < x < \pi,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = -\mu_0 u(\pi,t) - \mu_1 u(\pi, t-\tau(t)), & t > 0, \\ u(x,0) = u^0(x), & 0 < x < \pi, \\ u(\pi, t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{array}\right. \tag{4.1}$$

with the constant parameter $a > 0$ and where $\mu_0, \mu_1 \geq 0$ are fixed nonnegative real numbers, the time-varying delay $\tau(t)$ satisfies

$$\dot{\tau}(t) < 1,\ \forall t > 0, \tag{4.2}$$

and

$$\exists M > 0 :\ 0 < \tau_0 \leq \tau(t) \leq M,\ \forall t > 0. \tag{4.3}$$

Moreover, we assume that

$$\tau \in W^{2,\infty}([0,T]),\ \forall T > 0. \tag{4.4}$$

The boundary-value problem (4.1) describes the propagation of heat in a homogeneous one-dimensional rod with a fixed temperature at the left end. Here a stands for the heat conduction coefficient, $u(x,t)$ is the value of the temperature field of the plant at time moment t and location x along the rod. In the sequel, the state dependence on time t and spatial variable x is suppressed whenever possible.

4.2.1 Well posedness of the problem

We aim to show that problem (4.1) is well-posed. For that purpose, we use semi-group theory and adapt the ideas from [85].

We introduce the Hilbert space

$$V = \{\phi \in H^1(0,\, \pi)\ :\ \phi(0) = 0\}.$$

We transform our system (4.1) as follows. Let us introduce the auxiliary variable $z(\rho, t) = u(\pi,\, t - \tau(t)\rho)$ for $\rho \in (0,\, 1)$ and $t > 0$. Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$

$$\begin{cases} \tau(t) z_t(\rho,\, t) + (1 - \dot{\tau}(t)\rho) z_\rho(\rho,\, t) = 0, \\ z(0,\, t) = u(\pi,\, t), \\ z(\rho,\, 0) = f^0(-\tau(0)\rho). \end{cases} \tag{4.5}$$

Therefore, the problem (4.1) is equivalent to

$$\begin{cases} u_t(x,\, t) - a u_{xx}(x,\, t) = 0, & 0 < x < \pi,\, t > 0, \\ \tau(t) z_t(\rho,\, t) + (1 - \dot{\tau}(t)\rho) z_\rho(\rho,\, t) = 0, & 0 < \rho < 1,\, t > 0, \\ u(0,\, t) = 0,\, u_x(\pi,\, t) = -\mu_0 u(\pi,\, t) - \mu_1 z(1,\, t), & t > 0, \\ z(0,\, t) = u(\pi,\, t), & t > 0, \\ u(x,\, 0) = u^0(x), & 0 < x < \pi, \\ z(\rho,\, 0) = f^0(-\tau(0)\rho), & 0 < \rho < 1. \end{cases} \tag{4.6}$$

If we introduce

$$U := (u,\, z)^\top,$$

then U satisfies

$$U_t = (u_t,\, z_t)^\top = \left(a u_{xx},\, \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \right)^\top.$$

Consequently the problem (4.1) may be rewritten as the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}(t) U \\ U(0) = (u^0,\, f^0(-\tau(0)))^\top = U_0, \end{cases} \tag{4.7}$$

where the time dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} a u_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \{(u,\, z) \in (V \cap H^2(0,\, \pi)) \times H^1(0,\, 1)\ : \\ z(0) = u(\pi),\, u_x(\pi) = -\mu_0 u(\pi) - \mu_1 z(1)\}.$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\ \forall t > 0. \tag{4.8}$$

Now, we introduce the Hilbert space

$$H = L^2(0, \pi) \times L^2(0, 1)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^\pi u\tilde{u}dx + \int_0^1 z(\rho)\tilde{z}(\rho)d\rho.$$

A general theory for equations of type (4.7) has been developed using semigroup theory [59, 60, 93]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{\mathcal{A}, H, Y\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$, for some fixed $T > 0$ and $Y = \mathcal{D}(\mathcal{A}(0))$, forms a CD-system (or constant domain system, see [59, 60]). More precisely, the following theorem give the existence and uniqueness results and is proved in Theorem 1.9 of [59] (see also Theorem 2.13 of [60] or [3])

Theorem 4.2.1. *Assume that*

(i) $Y = \mathcal{D}(\mathcal{A}(0))$ is a dense subset of H,

(ii) (4.8) holds,

(iii) for all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on H and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies $\|S_t(s)u\|_H \leq Ce^{ms}\|u\|_H$, for all $u \in H$ and $s \geq 0$),

(iv) $\partial_t\mathcal{A}$ belongs to $L^\infty_*([0, T], B(Y, H))$, the space of equivalent classes of essentially bounded, strongly measure functions from $[0, T]$ into the set $B(Y, H)$ of bounded operators from Y into H.

Then, problem (4.7) has a unique solution $U \in C([0, T], Y) \cap C^1([0, T], H)$ for any initial datum in Y.

Lemma 4.2.2. *$D(\mathcal{A}(0))$ is dense in H.*

Proof: Let $(f, h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A}(0))$, namely

$$0 = \left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} f \\ h \end{pmatrix} \right\rangle = \int_0^\pi uf dx + \int_0^1 z(\rho)h(\rho)d\rho,$$

for all $(u, z)^\top \in D(\mathcal{A}(0))$.

We first take $u = 0$ and $z \in \mathcal{D}(0, 1)$. Since $(0, z) \in D(\mathcal{A}(0))$, we get

$$\int_0^1 z(\rho)h(\rho)d\rho = 0.$$

Since $\mathcal{D}(0, 1)$ is dense in $L^2(0, 1)$, we deduce that $h = 0$.

In the same manner, by taking $z = 0$ and $u \in \mathcal{D}(0, \pi)$ we see that $f = 0$. ∎

Let us suppose now that the speed of the delay satisfies

$$\dot{\tau}(t) \leq d < 1, \forall t > 0 \tag{4.9}$$

and that μ_0, μ_1 satisfy

$$\mu_1^2 \leq (1-d)\mu_0^2. \tag{4.10}$$

Under these conditions, we will show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in H and using the variable norm technique of Kato from [59], that problem (4.6) (and then (4.1)) has a unique solution.

For that purpose, we introduce the following time-dependent inner product on H

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^\pi u\tilde{u}dx + q\tau(t)\int_0^1 z(\rho)\tilde{z}(\rho)d\rho,$$

where q is a positive constant chosen later on, with associated norm denoted by $\|.\|_t$.

Theorem 4.2.3. *For an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (4.7). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then*

$$U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

Proof: We first prove that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t,\, s \in [0, T] \tag{4.11}$$

where $\phi = (u, z)^\top$ and c is a positive constant. For all s, $t \in [0, T]$, we have

$$\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^\pi u^2dx + q\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^1 z(\rho)^2d\rho.$$

We notice that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. Indeed,

$$\tau(t) = \tau(s) + \dot{\tau}(a)(t-s), \quad \text{where } a \in (s,\, t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\dot{\tau}(a)|}{\tau(s)}|t-s|.$$

By (4.4), $\dot{\tau}$ is bounded and therefore,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0}|t-s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

by (4.3), which proves (4.11).

Now we calculate $\langle \mathcal{A}(t)U, U\rangle_t$ for a t fixed. Take $U = (u, z)^\top \in D(\mathcal{A}(t))$. Then

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &= \left\langle \begin{pmatrix} au_{xx} \\ \frac{\dot{\tau}(t)\rho-1}{\tau(t)}z_\rho \end{pmatrix}, \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t \\
&= a\int_0^\pi u_{xx}udx - q\int_0^1 z_\rho(\rho)z(\rho)(1-\dot{\tau}(t)\rho)d\rho.
\end{aligned}$$

By integrating by parts in space in the first term of this right hand side, we have

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &= -a\int_0^\pi u_x^2dx + a[uu_x]_0^\pi - q\int_0^1 z_\rho(\rho)z(\rho)(1-\dot\tau(t)\rho)d\rho \\
&= -a\int_0^\pi u_x^2dx - a\mu_0 u(\pi, t)^2 - a\mu_1 u(\pi, t)u(\pi, t-\tau(t)) \\
&\quad -q\int_0^1 z_\rho(\rho)z(\rho)(1-\dot\tau(t)\rho)d\rho.
\end{aligned}$$

Moreover, we have by integrating by parts in ρ :

$$\begin{aligned}
\int_0^1 z_\rho(\rho)z(\rho)(1-\dot\tau(t)\rho)d\rho &= \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}(z(\rho)^2)(1-\dot\tau(t)\rho)d\rho \\
&= \frac{\dot\tau(t)}{2}\int_0^1 z(\rho)^2d\rho + \frac{1}{2}u^2(\pi, t-\tau(t))(1-\dot\tau(t)) - \frac{1}{2}u^2(\pi, t).
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \mathcal{A}(t)U, U\rangle_t &= -a\int_0^\pi u_x^2dx - \frac{q\dot\tau(t)}{2}\int_0^1 z(\rho)^2d\rho - a\mu_0 u(\pi, t)^2 \\
&\quad -a\mu_1 u(\pi, t)u(\pi, t-\tau(t)) - \frac{q}{2}u(\pi, t-\tau(t))^2(1-\dot\tau(t)) + \frac{q}{2}u(\pi, t)^2 \\
&\leq -a\int_0^\pi u_x^2dx + (\frac{q}{2} - a\mu_0)u^2(\pi, t) - a\mu_1 u(\pi, t)u(\pi, t-\tau(t)) \\
&\quad -\frac{q}{2}u(\pi, t-\tau(t))^2(1-d) + \frac{q\,|\dot\tau(t)|}{2\tau(t)}\tau(t)\int_0^1 z(\rho)^2d\rho.
\end{aligned}$$

We can see that this inequality can be written

$$\langle \mathcal{A}(t)U, U\rangle_t \leq -a\int_0^\pi u_x^2dx + (u(\pi, t), u(\pi, t-\tau(t)))\Psi_q(u(\pi, t), u(\pi, t-\tau(t)))^\top + \kappa(t)\langle U, U\rangle_t,$$

where

$$\kappa(t) = \frac{(\dot\tau(t)^2+1)^{\frac{1}{2}}}{2\tau(t)} \tag{4.12}$$

and where Ψ_q is the 2×2 matrix defined by

$$\Psi_q = \frac{1}{2}\begin{pmatrix} q-2a\mu_0 & -a\mu_1 \\ -a\mu_1 & -q(1-d) \end{pmatrix}. \tag{4.13}$$

Since $-q(1-d) < 0$, we notice that the matrix Ψ_q is negative (in the sense that $X\Psi_q X^\top \leq 0$, for all $X = (x_1, x_2) \in \mathbb{R}^2$) if and only if

$$q^2 - 2a\mu_0 q + \frac{a^2\mu_1^2}{1-d} \leq 0. \tag{4.14}$$

The discriminant of this second order polynomial (in q) is

$$\Delta = 4a^2 \left(\mu_0^2 - \frac{\mu_1^2}{1-d} \right),$$

which is non negative if and only if (4.10) holds. Therefore, the matrix Ψ_q is negative for some $q > 0$ if and only if (4.10) is satisfied. Hence, we choose q satisfying (4.14) or equivalently such that

$$a\mu_0 - a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}} \leq q \leq a\mu_0 + a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}}.$$

Such a choice of q yields

$$\langle \mathcal{A}(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq 0, \tag{4.15}$$

which proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle \cdot, \cdot \rangle_t$.

Moreover $\kappa'(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)(\dot{\tau}(t)+1)^{\frac{1}{2}}} - \frac{\dot{\tau}(t)(\dot{\tau}(t)^2+1)^{\frac{1}{2}}}{2\tau(t)^2}$ is bounded on $[0, T]$ for all $T > 0$ (by (4.4)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ \frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with $\frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau(t)^2}$ bounded on $[0, T]$ by (4.4). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), H)), \tag{4.16}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0, T]$ into $B(D(\mathcal{A}(0)), H)$.

Let us prove that $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$.

Let $(f, h)^T \in H$. We look for $U = (u, z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda u - a u_{xx} = f \\ \lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \tag{4.17}$$

Suppose that we have found u with the appropriate regularity. We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h$$

and the boundary condition $z(0) = u(\pi)$. Therefore z is explicitly given by

$$z(\rho) = u(\pi)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma,$$

if $\dot\tau(t) = 0$, and

$$z(\rho) = u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)} + e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)}\int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma,$$

otherwise. This means that once u is found with the appropriate properties, we can find z. In particular, we have if $\dot\tau(t) = 0$,

$$z(1) = u(\pi)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma = u(\pi)e^{-\lambda\tau(t)} + z^0,$$

where $z^0 = \tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma$ is a fixed real number depending only on h, and if $\dot\tau(t) \neq 0$

$$\begin{aligned} z(1) &= u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))} + e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma \\ &= u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))} + z^0, \end{aligned}$$

where $z^0 = e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma$ depends only on h.

It remains to find u. By (4.17), u must satisfy

$$\lambda u - au_{xx} = f.$$

Multiplying this identity by a test function ϕ, integrating in space and using integration by parts, we obtain

$$\int_0^\pi (\lambda u - au_{xx})\phi dx = \int_0^\pi (\lambda u\phi + au_x\phi_x)dx - au_x(\pi)\phi(\pi) + au_x(0)\phi(0).$$

But using the fact that $(u, z)^\top$ must belong to $D(\mathcal{A}(t))$, we have

$$\int_0^\pi (\lambda u - au_{xx})\phi dx = \int_0^\pi (\lambda u\phi + au_x\phi_x)dx + a\mu_0 u(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi).$$

Therefore

$$\int_0^\pi (\lambda u\phi + au_x\phi_x)dx + a\mu_0 u(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi) = \int_0^\pi f\phi dx.$$

Using the above expression for $z(1)$, we arrive at the problem

$$\int_0^\pi (\lambda u\phi + au_x\phi_x)dx + a(\mu_0 + \mu_1 e^{-\lambda\tau(t)})u(\pi)\phi(\pi) = \int_0^\pi f\phi dx - a\mu_1 z^0\phi(\pi),\ \forall\phi\in V \quad (4.18)$$

if $\dot{\tau}(t) = 0$, or otherwise

$$\int_0^\pi (\lambda u\phi + au_x\phi_x)dx + a(\mu_0 + \mu_1 e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))})u(\pi)\phi(\pi) = \int_0^\pi f\phi dx - a\mu_1 z^0\phi(\pi),\ \forall\phi \in V. \tag{4.19}$$

These problems have a unique solution $u \in V$ by Lax-Milgram's lemma, because the left-hand side of (4.18) or (4.19) is coercive on V.

If we consider $\phi \in \mathcal{D}(0, \pi) \subset V$, then u satisfies

$$\lambda u - au_{xx} = f \text{ in } \mathcal{D}'(0, \pi).$$

This directly implies that $u \in H^2(0, \pi)$ and then $u \in V \cap H^2(0, \pi)$. Coming back to (4.18) and by integrating by parts, we find, for $\dot{\tau}(t) = 0$,

$$a\left(u_x(\pi) + (\mu_0 + \mu_1 e^{-\lambda\tau(t)})u(\pi)\right)\phi(\pi) = -a\mu_1 z^0\phi(\pi),$$

and then

$$\begin{aligned} u_x(\pi) &= -(\mu_0 + \mu_1 e^{-\lambda\tau(t)})u(\pi) - \mu_1 z^0 \\ &= -\mu_0 u(\pi) - \mu_1(e^{-\lambda\tau(t)}u(\pi) + z^0) \\ &= -\mu_0 u(\pi) - \mu_1 z(1). \end{aligned}$$

We find the same result if $\dot{\tau}(t) \neq 0$.

In summary we have found $(u, z)^\top \in D(\mathcal{A}(t))$ satisfying (4.17) and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Since $\kappa(t) > 0$, we directly deduce that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \tag{4.20}$$

for some $\lambda > 0$ and $t > 0$.

Then, (4.11), (4.15) and (4.20) imply that the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is a stable family of generators in H with stability constants independent of t, by Proposition 1.1 from [59]. Therefore, the assumptions (i)-(iv) of Theorem 4.2.1 are verified by (4.8), (4.11), (4.15), (4.16), (4.20) and Lemma 4.2.2, and thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0. \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), H)$ and $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H)$ if $U_0 \in D(\mathcal{A}(0))$. Setting

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$, we remark that it is a solution of (4.7) because

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}_t(t) \\ &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\beta(t)}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. ∎

4.2.2 The decay of the energy

We here suppose that

$$\mu_1^2 < (1-d)\mu_0^2. \tag{4.21}$$

Let us choose the following energy

$$E(t) = \frac{1}{2}\int_0^\pi u^2(x,\, t)dx + \frac{q\tau(t)}{2}\int_0^1 u^2(\pi,\, t-\tau(t)\rho)d\rho, \tag{4.22}$$

where q is a positive constant chosen later.

Proposition 4.2.4. *Let (4.9) and (4.21) be satisfied. Then for all regular solution of problem (4.1), the energy is decreasing and satisfies*

$$E'(t) \leq -a\int_0^\pi u_x^2(x,\, t)dx + (u(\pi,\, t),\, u(\pi,\, t-\tau(t)))\, \Psi_q\, (u(\pi,\, t),\, u(\pi,\, t-\tau(t)))^\top < 0, \tag{4.23}$$

where Ψ_q is the matrix defined in (4.13).

Proof: Differentiating (4.22) and by (4.1), we obtain

$$\begin{aligned} E'(t) &= \int_0^\pi u u_t dx + \frac{q\dot\tau(t)}{2}\int_0^1 u^2(\pi,\, t-\tau(t)\rho)d\rho \\ &\quad + q\tau(t)\int_0^1 u(\pi,\, t-\tau(t)\rho)u_t(\pi,\, t-\tau(t)\rho)(1-\dot\tau(t)\rho)d\rho \\ &= a\int_0^\pi u u_{xx} dx + \frac{q\dot\tau(t)}{2}\int_0^1 u^2(\pi,\, t-\tau(t)\rho)d\rho \\ &\quad + q\tau(t)\int_0^1 u(\pi,\, t-\tau(t)\rho)u_t(\pi,\, t-\tau(t)\rho)(1-\dot\tau(t)\rho)d\rho. \end{aligned}$$

By integrating by parts in space, we find

$$\begin{aligned} a\int_0^\pi u u_{xx} dx &= -a\int_0^\pi u_x^2 dx + a[u(x,\, t)u_x(x,\, t)]_0^\pi \\ &= -a\int_0^\pi u_x^2 dx - a\mu_0 u^2(\pi,\, t) - a\mu_1 u(\pi,\, t)u(\pi,\, t-\tau(t)). \end{aligned}$$

Setting $z(\rho, t) = u(\pi, t-\tau(t)\rho)$, we see that $z_\rho(\rho, t) = -\tau(t)u_t(\pi,\, t-\tau(t)\rho)$, and by integrating by parts in ρ, we get

$$\begin{aligned} \int_0^1 u(\pi,\, t-\tau(t)\rho)u_t(\pi,\, t-\tau(t)\rho)(1-\dot\tau(t)\rho)d\rho &= -\tfrac{1}{\tau(t)}\int_0^1 z(\rho,\, t)z_\rho(\rho,\, t)(1-\dot\tau(t)\rho)d\rho \\ &= -\tfrac{1}{2\tau(t)}\int_0^1 \partial_\rho(z(\rho,\, t)^2)(1-\dot\tau(t)\rho)d\rho \\ &= \tfrac{1}{2\tau(t)}\int_0^1 z(\rho,\, t)^2(-\dot\tau(t))d\rho \\ &\quad -\tfrac{1}{2\tau(t)}[z^2(\rho,\, t)(1-\dot\tau(t)\rho)]_0^1 \\ &= -\tfrac{\dot\tau(t)}{2\tau(t)}\int_0^1 u^2(\pi,\, t-\tau(t)\rho)d\rho \\ &\quad -\tfrac{1-\dot\tau(t)}{2\tau(t)}u^2(\pi,\, t-\tau(t)) + \tfrac{1}{2\tau(t)}u^2(\pi,\, t). \end{aligned}$$

Therefore, we obtain

$$E'(t) = -a\int_0^\pi u_x^2dx - (a\mu_0 - \frac{q}{2})u^2(\pi, t) - a\mu_1 u(\pi, t)u(\pi, t-\tau(t)) - \frac{q}{2}(1-\dot{\tau}(t))u^2(\pi, t-\tau(t)).$$

We can see that this inequality can be written as

$$E'(t) \leq -a\int_0^\pi u_x^2dx + (u(\pi, t), u(\pi, t-\tau(t)))\Psi_q(u(\pi, t), u(\pi, t-\tau(t)))^\top.$$

Since $-q(1-d) < 0$, Ψ_q is negative definite if and only if

$$q^2 - 2a\mu_0 q + \frac{a^2\mu_1^2}{1-d} < 0. \tag{4.24}$$

The discriminant of this second order polynomial is $\Delta = 4a^2\left(\mu_0^2 - \frac{\mu_1^2}{1-d}\right)$, which is positive if and only if (4.21) holds. Therefore, the matrix Ψ_q is negative definite for some $q > 0$ if and only if (4.21) is satisfied, and in that case, we choose q such that

$$a\mu_0 - a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}} < q < a\mu_0 + a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}}, \tag{4.25}$$

which concludes the proof. ■

4.2.3 Exponential stability

In this section, we prove the exponential stability of the heat equation (4.1) by using the following Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma\mathcal{E}_2(t), \tag{4.26}$$

where $\gamma > 0$ is a parameter that will be fixed small enough later on, E is the standard energy defined by (4.22) and $\mathcal{E}_2$ is defined by

$$\mathcal{E}_2(t) = q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u^2(\pi, s)ds = q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u^2(\pi, t-\tau(t)\rho)d\rho, \tag{4.27}$$

where $\delta > 0$ is a fixed positive real number.

Remark 4.2.5. *Let us notice that the energies E and $\mathcal{E}$ are equivalent, since*

$$E(t) \leq \mathcal{E}(t) \leq (2\gamma + 1)E(t).$$

The result about the decay of the energy E is the following one :

Theorem 4.2.6. *Let (4.3), (4.9) and (4.21) be satisfied. Then the energy E decays exponentially, more precisely there exist two positive constants α and C such that*

$$E(t) \leq Ce^{-\alpha t}E(0), \forall t > 0.$$

Proof: First, we differentiate $\mathcal{E}_2$ to have

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t)\int_0^1 (-2\delta\dot{\tau}(t)\rho)e^{-2\delta\tau(t)\rho}u^2(\pi, t-\tau(t)\rho)d\rho + J,$$

where

$$J = 2q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u(\pi, t-\tau(t)\rho)u_t(\pi, t-\tau(t)\rho)(1-\dot{\tau}(t)\rho)d\rho.$$

Moreover, by noticing one more time that $z(\rho, t) = u(\pi, t-\tau(t)\rho)$ and by integrating by parts in ρ, we have

$$\begin{aligned} J &= -q\int_0^1 e^{-2\delta\tau(t)\rho}\frac{\partial}{\partial\rho}(z(\rho, t)^2)(1-\dot{\tau}(t)\rho)d\rho \\ &= q\int_0^1 z^2(\rho, t)(-2\delta\tau(t)(1-\dot{\tau}(t)\rho) - \dot{\tau}(t))e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}z^2(1, t)(1-\dot{\tau}(t)) + qz^2(0, t) \\ &= q\int_0^1 (-2\delta\tau(t)(1-\dot{\tau}(t)\rho) - \dot{\tau}(t))u^2(\pi, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u^2(\pi, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi, t). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &= \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\int_0^1 (-2\delta\tau(t) - \dot{\tau}(t))u^2(\pi, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u^2(\pi, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi, t) \\ &= -2\delta\mathcal{E}_2(t) - qe^{-2\delta\tau(t)}u^2(\pi, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi, t). \end{aligned}$$

Since $\dot{\tau}(t) < 1$ (see (4.2)), we obtain

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -2\delta\mathcal{E}_2(t) + qu^2(\pi, t). \tag{4.28}$$

Consequently, gathering (4.23), (4.26) and (4.28), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leq -(a\mu_0 - \frac{q}{2} - q\gamma)u^2(\pi, t) - \frac{q}{2}(1-d)u^2(\pi, t-\tau(t)) \\ &\quad -a\mu_1 u(\pi, t)u(\pi, t-\tau(t)) - a\int_0^\pi u_x^2(x, t)dx - 2\gamma\delta\mathcal{E}_2(t), \end{aligned}$$

or equivalently

$$\frac{d}{dt}\mathcal{E}(t) \leq (u(\pi, t), u(\pi, t-\tau(t)))\tilde{\Psi}_q(u(\pi, t), u(\pi, t-\tau(t)))^\top - a\int_0^\pi u_x^2(x, t)dx - 2\gamma\delta\mathcal{E}_2(t), \tag{4.29}$$

where $\tilde{\Psi}_q$ is the 2×2 matrix defined by

$$\tilde{\Psi}_q = \frac{1}{2} \begin{pmatrix} q(1+2\gamma) - 2a\mu_0 & -a\mu_1 \\ -a\mu_1 & -q(1-d) \end{pmatrix} = \Psi_q + \gamma \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Now fix $q > 0$ such that Ψ_q is negative definite (consequence of the assumption (4.21)). By a perturbation argument, we deduce that for $\gamma > 0$ small enough, $\tilde{\Psi}_q$ is negative. More precisely, we take $\gamma = \frac{-\lambda}{q}$, when λ is the greatest negative eigenvalue of Ψ_q, or equivalently $X\Psi_q X^\top \leq \lambda |X|^2$, for all $X \in \mathbb{R}^2$. We can easily check that

$$\lambda = \frac{1}{4}\left(-2a\mu_0 + dq + \sqrt{4a^2(\mu_0^2 + \mu_1^2) + 4a(d-2)\mu_0 q + (d-2)^2 q^2}\right) < 0. \tag{4.30}$$

Therefore, for $\gamma = \frac{-\lambda}{q}$, we find

$$\frac{d}{dt}\mathcal{E}(t) \leq -2\delta\gamma\mathcal{E}_2(t) - a\int_0^\pi u_x^2(x,\, t)dx. \tag{4.31}$$

Since $u(0,\, t) = 0$ for all $t > 0$, by the min-max principle, we have

$$\int_0^\pi u^2(x,\, t)dx \leq 4\int_0^\pi u_x^2(x,\, t)dx,$$

because the first eigenvalue of the Laplace operator with Dirichlet boundary condition at 0 and Neumann boundary condition at π is $-\frac{1}{4}$. Therefore

$$-a\int_0^\pi u_x^2(x,\, t)dx \leq -\frac{a}{4}\int_0^\pi u^2(x,\, t)dx.$$

This estimate in (4.31) and by the definition (4.27) of $\mathcal{E}_2$, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{a}{4}\int_0^\pi u^2(x,\, t)dx - 2q\delta\gamma\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u^2(\pi,\, t - \tau(t)\rho)d\rho.$$

Since $\tau(t) \leq M$ (see (4.3)) and in view of the definition (4.22) of $E(t)$, there exists a constant $\gamma' > 0$ (depending on γ and δ, namely $\gamma' \leq \min(\frac{a}{2},\, 4\delta\gamma e^{-2\delta M})$) such that

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma' E(t).$$

By applying Remark 4.2.5, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{\gamma'}{2\gamma+1}\mathcal{E}(t).$$

This implies that

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\alpha t},$$

with

$$\alpha = \frac{\gamma'}{2\gamma+1} \leq \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\ -4\lambda\delta e^{-2\delta M}\right).$$

Remark 4.2.5 leads to

$$E(t) \leq \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\alpha t} \leq (2\gamma+1)E(0)e^{-\alpha t}.$$

∎

Remark 4.2.7. *In the proof of Theorem 4.2.6, we notice that we have explicitely calculated the decay rate of the energy, given by*

$$\alpha = \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\ -4\lambda\delta e^{-2\delta M}\right),$$

where λ is given by (4.30), q by (4.25) and δ is a positive real number. Therefore, we can choose δ so that the decay of the energy is as quick as possible. For that purpose, we notice that the function $\delta \longrightarrow -4\lambda\delta e^{-2\delta M}$ admits a maximum at $\delta = \frac{1}{2M}$ and that this maximum is $\frac{-2\lambda}{Me}$. Thus the larger decay rate of the energy is given by

$$\alpha_{max} = \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\ \frac{-2\lambda}{Me}\right).$$

Obviously, this quantity can be calculated if the data μ_0, μ_1 and τ are given.

4.3 Exponential stability of the delayed wave equation

We now consider the system described by

$$\left\{\begin{array}{lr} u_{tt}(x,\,t) - a u_{xx}(x,\,t) = 0, & 0 < x < \pi,\ t > 0, \\ u(0,\,t) = 0, & t > 0, \\ u_x(\pi,\,t) = -\mu_0 u_t(\pi,\,t) - \mu_1 u_t(\pi,\,t-\tau(t)), & t > 0, \\ u(x,\,0) = u^0(x),\ u_t(x,\,0) = u^1(x), & 0 < x < \pi, \\ u_t(\pi,\,t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{array}\right. \tag{4.32}$$

whith the constant parameter $a > 0$ and where μ_0, $\mu_1 \geq 0$ are fixed nonnegative real numbers, the time-varying delay $\tau(t)$ still satisfies (4.2), (4.3) and (4.4).

The boundary-value problem (4.32) describes the oscillations of a homogeneous string fixed at 0 and with a feedback law at π.

4.3.1 Well posedness of the problem

We aim to show that problem (4.32) is well-posed. For that purpose, we use the same ideas than before.

We transform our system (4.32) as follows. Let us introduce the auxiliary variable $z(\rho, t) = u_t(\pi, t - \tau(t)\rho)$ for $\rho \in (0, 1)$ and $t > 0$. Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$ (compare with (4.5))

$$\begin{cases} \tau(t)z_t(\rho, t) + (1 - \dot{\tau}(t)\rho)z_\rho(\rho, t) = 0, \\ z(0, t) = u_t(\pi, t), \\ z(\rho, 0) = f^0(-\tau(0)\rho). \end{cases} \tag{4.33}$$

Therefore, the problem (4.32) is equivalent to

$$\begin{cases} u_{tt}(x, t) - au_{xx}(x, t) = 0, & 0 < x < \pi,\ t > 0, \\ \tau(t)z_t(\rho, t) + (1 - \dot{\tau}(t)\rho)z_\rho(\rho, t) = 0, & 0 < \rho < 1,\ t > 0, \\ u(0, t) = 0,\ u_x(\pi, t) = -\mu_0 u_t(\pi, t) - \mu_1 z(1, t), & t > 0, \\ z(0, t) = u_t(\pi, t), & t > 0, \\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x), & 0 < x < \pi, \\ z(\rho, 0) = f^0(-\tau(0)\rho), & 0 < \rho < 1. \end{cases} \tag{4.34}$$

If we introduce

$$U := (u,\ u_t,\ z)^\top,$$

then U satisfies

$$U_t = (u_t,\ u_{tt},\ z_t)^\top = \left(u_t,\ au_{xx},\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho\right)^\top.$$

Consequently the problem (4.32) may be rewritten as the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}(t)U \\ U(0) = (u^0,\ u^1, f^0(-\tau(0).))^\top = U_0, \end{cases} \tag{4.35}$$

where the time dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ \omega \\ z \end{pmatrix} = \begin{pmatrix} \omega \\ au_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \{(u,\ \omega,\ z) \in (V \cap H^2(0,\ \pi)) \times V \times H^1(0,\ 1) : \\ z(0) = \omega(\pi),\ u_x(\pi) = -\mu_0\omega(\pi) - \mu_1 z(1)\},$$

where we recall that

$$V = \{\phi \in H^1(0,\ \pi) : \phi(0) = 0\}.$$

Again, we notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \forall t > 0. \tag{4.36}$$

Now, we introduce the Hilbert space

$$H = V \times L^2(0, \pi) \times L^2(0, 1)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{\omega} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^\pi (au_x\tilde{u}_x + \omega\tilde{\omega})dx + \int_0^1 z(\rho)\tilde{z}(\rho)d\rho.$$

Lemma 4.3.1. *$D(\mathcal{A}(0))$ is dense in H.*

Proof: The proof is the same as the one of Lemma 1.2.1, we give it for the sake of completeness. Let $(f, g, h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A}(0))$, namely

$$0 = \left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle = \int_0^\pi (au_x f_x + \omega g)dx + \int_0^1 z(\rho)h(\rho)d\rho,$$

for all $(u, \omega, z)^\top \in D(\mathcal{A}(0))$.

We first take $u = 0$ and $w = 0$ and $z \in \mathcal{D}(0, 1)$. As $(0, 0, z) \in D(\mathcal{A}(0))$, we get

$$\int_0^1 z(\rho)h(\rho)d\rho = 0.$$

Since $\mathcal{D}(0, 1)$ is dense in $L^2(0, 1)$, we deduce that $h = 0$.

In the same manner, by taking $u = 0$, $z = 0$ and $\omega \in \mathcal{D}(0, \pi)$ we see that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = a\int_0^\pi u_x f_x dx, \forall (u, \omega, z) \in D(\mathcal{A}(0)).$$

By restricting ourselves to $\omega = 0$ and $z = 0$, we obtain

$$\int_0^1 u_x f_x dx = 0, \forall (u, 0, 0) \in D(\mathcal{A}(0)).$$

But we easily check that $(u, 0, 0) \in D(\mathcal{A}(0))$ if and only if $u \in D(\Delta) = \{v \in H^2(0,\pi) : v(0) = 0, v'(1) = 0\}$, the domain of the Laplace operator with mixed boundary conditions. Since it is well known that $D(\Delta)$ is dense in V (equipped with the inner product $< ., . >_V$), we conclude that $f = 0$. ■

As before we suppose that the speed of the delay satisfies (4.9) and (4.10). Under these conditions, we will show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in H and the unique solvability of problem (4.35).

For that purpose, we introduce the following time-dependent inner product on H

$$\left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{\omega} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^\pi (au_x\tilde{u}_x + \omega\tilde{\omega})dx + q\tau(t)\int_0^1 z(\rho)\tilde{z}(\rho)d\rho,$$

where q is a positive constant chosen such that Ψ_q is negative (guaranteed by the assumptions (4.9) and (4.10)), with associated norm denoted by $\|.\|_t$.

Theorem 4.3.2. *For an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (4.35). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then*

$$U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

Proof: We first notice that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t, s \in [0, T] \tag{4.37}$$

where $\phi = (u, \omega, z)^\top$ and c is a positive constant. Indeed, for all $s, t \in [0, T]$, we have

$$\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^\pi (au_x^2 + \omega^2)dx + q\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^1 z(\rho)^2 d\rho,$$

and we conclude as in the proof of Theorem 4.2.3.

Now we calculate $\langle \mathcal{A}(t)U, U\rangle_t$ for a $t > 0$ fixed. For an arbitrary $U = (u, \omega, z)^\top \in D(\mathcal{A}(t))$, we have

$$\begin{aligned} \langle \mathcal{A}(t)U, U\rangle_t &= \left\langle \begin{pmatrix} \omega \\ au_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ \omega \\ z \end{pmatrix} \right\rangle_t \\ &= \int_0^\pi (a\omega_x u_x + au_{xx}\omega)dx - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot{\tau}(t)\rho)d\rho. \end{aligned}$$

By integrating by parts in space, we have

$$\begin{aligned} \langle \mathcal{A}(t)U, U\rangle_t &= a[\omega u_x]_0^\pi - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot{\tau}(t)\rho)d\rho \\ &= -a\mu_0 z(0)^2 - a\mu_1 z(0)z(1) - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot{\tau}(t)\rho)d\rho. \end{aligned}$$

Moreover, we have by integrating by parts in ρ :

$$\begin{aligned}\int_0^1 z_\rho(\rho)z(\rho)(1-\dot{\tau}(t)\rho)d\rho &= \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}(z(\rho)^2)(1-\dot{\tau}(t)\rho)d\rho \\ &= \frac{\dot{\tau}(t)}{2}\int_0^1 z(\rho)^2 d\rho + \frac{1}{2}z(1)^2(1-\dot{\tau}(t)) - \frac{1}{2}z^2(0).\end{aligned}$$

These two identities yield

$$\langle \mathcal{A}(t)U, U\rangle_t = -a\mu_0 z(0)^2 - a\mu_1 z(0)z(1) - \frac{q}{2}z(1)^2(1-\dot{\tau}(t)) + \frac{q}{2}z^2(0) - \frac{q\dot{\tau}(t)}{2}\int_0^1 z(\rho)^2 d\rho.$$

We can see, that this identity implies that

$$\langle \mathcal{A}(t)U, U\rangle_t \leq (z(0), z(1))\Psi_q(z(0), z(1))^\top + \kappa(t)\langle U, U\rangle_t,$$

where Ψ_q is the matrix defined by (4.13) and $\kappa(t)$ is given by (4.12). Since we have chosen q such that the matrix Ψ_q is negative, we have

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0, \tag{4.38}$$

which proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle\cdot,\cdot\rangle_t$.

As in the proof of Theorem 4.2.3, we see that (4.4) implies that

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0,T], B(D(\mathcal{A}(0)), H)). \tag{4.39}$$

Let us finally prove that $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$.

Let $(f, g, h)^T \in H$. We look for $U = (u, \omega, z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\begin{pmatrix} u \\ \omega \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda u - \omega = f \\ \lambda\omega - au_{xx} = g \\ \lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)}z_\rho = h. \end{cases} \tag{4.40}$$

Suppose that we have found u with the appropriate regularity. Then, we have

$$\omega = -f + \lambda u \in V.$$

We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)}z_\rho = h$$

and the boundary condition $z(0)=\omega(\pi)=-f(\pi)+\lambda u(\pi)$. Therefore z is explicitly given by

$$z(\rho)=\lambda u(\pi)e^{-\lambda\tau(t)\rho}-f(\pi)e^{-\lambda\tau(t)\rho}+\tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma,$$

if $\dot\tau(t)=0$, and

$$\begin{aligned} z(\rho) &= \lambda u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)}-f(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)} \\ &\quad +e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)}\int_0^\rho\frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma,\end{aligned}$$

otherwise. This means that once u is found with the appropriate properties, we can find z and ω. In particular, we have if $\dot\tau(t)=0$,

$$z(1)=\lambda u(\pi)e^{-\lambda\tau(t)}-f(\pi)e^{-\lambda\tau(t)}+\tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma=\lambda u(\pi)e^{-\lambda\tau(t)}+z^0,$$

where $z^0=-f(\pi)e^{-\lambda\tau(t)}+\tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma$ is a fixed real number depending only on f and h, and if $\dot\tau(t)\neq 0$

$$\begin{aligned} z(1) &= \lambda u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}-f(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))} \\ &\quad +e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\int_0^1\frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma \\ &= \lambda u(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}+z^0,\end{aligned}$$

where

$$z^0=-f(\pi)e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}+e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t))}\int_0^1\frac{h(\sigma)\tau(t)}{1-\dot\tau(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\sigma)}d\sigma$$

depends only on f and h.

It remains to find u. By (4.40), u must satisfy

$$\lambda^2u-au_{xx}=g+\lambda f.$$

Multiplying this identity by a test function ϕ, integrating in space and using integration by parts, we obtain

$$\int_0^\pi(\lambda^2u-au_{xx})\phi dx=\int_0^\pi(\lambda^2u\phi+au_x\phi_x)dx-au_x(\pi)\phi(\pi)+au_x(0)\phi(0).$$

But using the fact that $(u,\,\omega,\,z)^\top$ must belong to $D(\mathcal{A}(t))$, we have

$$\int_0^\pi(\lambda^2u-au_{xx})\phi dx=\int_0^\pi(\lambda^2u\phi+au_x\phi_x)dx+a\mu_0\omega(\pi)\phi(\pi)+a\mu_1z(1)\phi(\pi).$$

Therefore

$$\int_0^\pi (\lambda^2 u\phi + au_x\phi_x)dx + a\mu_0\omega(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi) = \int_0^\pi (g+\lambda f)\phi dx.$$

Using the above expression for $z(1)$ and $\omega = \lambda u - f$, we arrive at the problem

$$\int_0^\pi (\lambda^2 u\phi + au_x\phi_x)dx + a(\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi)\phi(\pi) = \int_0^\pi (g+\lambda f)\phi dx + a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi), \ \forall\phi \in V \tag{4.41}$$

if $\dot\tau(t) = 0$, or otherwise

$$\begin{aligned} &\int_0^\pi (\lambda^2 u\phi + au_x\phi_x)dx + a(\mu_0 + \mu_1 e^{\lambda\frac{\tau(t)}{\dot\tau(t)}\ln(1-\dot\tau(t)\rho)})\lambda u(\pi)\phi(\pi) \\ &\quad = \int_0^\pi (g+\lambda f)\phi dx + a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi), \forall\phi \in V. \end{aligned} \tag{4.42}$$

These problems have a unique solution $u \in V$ by Lax-Milgram's lemma, because the left-hand side of (4.41) or (4.42) is coercive on V.

If we consider $\phi \in \mathcal{D}(0, \pi) \subset V$, then u satisfies

$$\lambda^2 u - au_{xx} = g + \lambda f \text{ in } \mathcal{D}'(0, \pi).$$

This directly implies that $u \in H^2(0, \pi)$ and then $u \in V \cap H^2(0, \pi)$. Coming back to (4.41) and by integrating by parts, we find, for $\dot\tau(t) = 0$,

$$a\left(u_x(\pi) + (\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi)\right)\phi(\pi) = a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi),$$

and then

$$\begin{aligned} u_x(\pi) &= -(\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi) - (\mu_1 z^0 - \mu_0 f(\pi)) \\ &= -\mu_0(\lambda u(\pi) - f(\pi)) - \mu_1(e^{-\lambda\tau(t)}\lambda u(\pi) + z^0) \\ &= -\mu_0\omega(\pi) - \mu_1 z(1). \end{aligned}$$

We find the same result if $\dot\tau(t) \neq 0$.

In summary we have found $(u, \omega, z)^\top \in D(\mathcal{A}(t))$ satisfying (4.40) and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \tag{4.43}$$

for some $\lambda > 0$ and $t > 0$.

Then, (4.37), (4.38) and (4.43) imply that the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is a stable family of generators in H with stability constants independent of t, by Proposition 1.1 from [59]. Therefore, the assumptions (i)-(iv) of Theorem 4.2.1 are verified by (4.36), (4.37), (4.38), (4.39), (4.43) and Lemma 4.3.1, and thus, the problem

$$\begin{cases} \tilde U_t = \tilde{\mathcal{A}}(t)\tilde U \\ \tilde U(0) = U_0. \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), H)$ and $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H)$ if $U_0 \in D(\mathcal{A}(0))$. As before, the requested solution of (4.35) is given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$. ■

4.3.2 The decay of the energy

As for the heat equation, we restrict the hypothesis (4.10) to obtain the decay of the energy. Namely we suppose that (4.21) holds.

Let us choose the following energy (which corresponds to the time-dependent inner product in H)

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2(x,\, t) + au_x^2(x,\, t)\right) dx + \frac{q\tau(t)}{2}\int_0^1 u_t^2(\pi,\, t - \tau(t)\rho)d\rho, \tag{4.44}$$

where q is a positive constant chosen such that Ψ_q is negative definite (possible if (4.9) and (4.21) hold).

Proposition 4.3.3. *Let the assumptions (4.9) and (4.21) be satisfied. Then for all regular solution of problem (4.32), the energy is decreasing and verifies*

$$E'(t) \leq (u_t(\pi,\, t),\, u_t(\pi,\, t - \tau(t)))\, \Psi_q\, (u_t(\pi,\, t),\, u_t(\pi,\, t - \tau(t)))^\top < 0, \tag{4.45}$$

where Ψ_q is the matrix defined in (4.13).

Remark 4.3.4. *In the case where the delay is constant in time, i.e. $\tau(t) = \tau$ for all $t > 0$ and thus $d = 0$, we recover the results from [85, 89] and chapter 1. Indeed in [85, 89], the energy is decreasing under the assumption $\mu_1 < \mu_0$.*

Proof: Differentiating (4.44) and integrating by parts in space, we obtain

$$\begin{aligned}
E'(t) &= \int_0^\pi (u_t u_{tt} + au_x u_{xt})dx + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi,\, t - \tau(t)\rho)d\rho \\
&\quad + q\tau(t)\int_0^1 u_t(\pi,\, t - \tau(t)\rho)u_{tt}(\pi,\, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho \\
&= \int_0^\pi u_t(u_{tt} - au_{xx})dx + a[u_x u_t]_0^\pi + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi,\, t - \tau(t)\rho)d\rho \\
&\quad + q\tau(t)\int_0^1 u_t(\pi,\, t - \tau(t)\rho)u_{tt}(\pi,\, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho \\
&= au_x(\pi,\, t)u_t(\pi,\, t) + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi,\, t - \tau(t)\rho)d\rho \\
&\quad + q\tau(t)\int_0^1 u_t(\pi,\, t - \tau(t)\rho)u_{tt}(\pi,\, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho.
\end{aligned}$$

Recalling that $z(\rho,t) = u_t(\pi,\, t - \tau(t)\rho)$, we see that

$$\begin{aligned}\int_0^1 u_t(\pi,\, t-\tau(t)\rho)u_{tt}(\pi,\, t-\tau(t)\rho)(1-\dot{\tau}(t)\rho)d\rho &= -\frac{\dot{\tau}(t)}{2\tau(t)}\int_0^1 u_t^2(\pi,\, t-\tau(t)\rho)d\rho \\ &\quad -\frac{1-\dot{\tau}(t)}{2\tau(t)}u_t^2(\pi,\, t-\tau(t)) \\ &\quad +\frac{1}{2\tau(t)}u_t^2(\pi,\, t).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}E'(t) &= au_x(\pi,\, t)u_t(\pi,\, t) + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi,\, t-\tau(t)\rho)d\rho \\ &\quad -\frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi,\, t-\tau(t)\rho)d\rho - q\frac{1-\dot{\tau}(t)}{2}u_t^2(\pi,\, t-\tau(t)) + \frac{q}{2}u_t^2(\pi,\, t),\end{aligned}$$

which implies

$$E'(t) = -a\mu_0 u_t^2(\pi,\, t) - a\mu_1 u_t(\pi,\, t-\tau(t))u_t(\pi,\, t) - \frac{q}{2}(1-\dot{\tau}(t))u_t^2(\pi,\, t-\tau(t)) + \frac{q}{2}u_t^2(\pi,\, t).$$

By the condition (4.9) we can see that this identity yields

$$E'(t) \le (u_t(\pi,\, t),\, u_t(\pi,\, t-\tau(t)))\, \Psi_q \,(u_t(\pi,\, t),\, u_t(\pi,\, t-\tau(t)))^\top .$$

This concludes the proof as Ψ_q is negative definite. ∎

4.3.3 Exponential stability

In this section, under the assumptions (4.9) and (4.21), we prove the exponential stability of the wave equation (4.32) by using the following Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma\left(2\int_0^\pi xu_tu_x dx + \mathcal{E}_2(t)\right), \tag{4.46}$$

where $\gamma > 0$ is a parameter that will be fixed small enough later on, E is the standard energy defined by (4.44) with q a positive constant fixed such that Ψ_q is negative definite and $\mathcal{E}_2$ is defined by

$$\mathcal{E}_2(t) = q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u_t^2(\pi,\, s)ds = q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u_t^2(\pi,\, t-\tau(t)\rho)d\rho, \tag{4.47}$$

where $\delta > 0$ is a fixed positive real number.

The Lyapunov functional $E(t) + 2\gamma\int_0^\pi xu_tu_x dx$ is standard in problems with boundary conditions with memory (see for instance [86]). We have added the two terms to the standard energy $E(t)$ in order to take into account the dependence of τ with respect to t.

First we notice that the energies E and $\mathcal{E}$ are equivalent.

Lemma 4.3.5. *For γ small enough, there exist two positive constants $C_1(\gamma)$ and $C_2(\gamma)$ such that*

$$C_1(\gamma)E(t) \leq \mathcal{E}(t) \leq C_2(\gamma)E(t).$$

Proof: We have

$$2\gamma \int_0^\pi xu_x u_t dx \leq \gamma\pi \int_0^\pi (u_x^2 + u_t^2)dx \leq \gamma\pi c \int_0^\pi (au_x^2 + u_t^2)dx,$$

where $c = \max(1, \frac{1}{a})$ and

$$\gamma q\tau(t) \int_0^1 e^{-2\delta\tau(t)\rho} u_t^2(\pi, t - \tau(t)\rho)d\rho \leq \gamma q\tau(t) \int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho.$$

Since $c \geq 1$, these estimates yield

$$\mathcal{E}(t) \leq (1 + 2\gamma c\pi)E(t).$$

Moreover, by definition we have

$$\gamma\mathcal{E}_2(t) \geq 0$$

and by Cauchy-Schwarz's inequality

$$2\gamma \int_0^\pi xu_x u_t dx \geq -\gamma\pi \int_0^\pi (u_x^2 + u_t^2)dx.$$

Then

$$\mathcal{E}(t) \geq E(t) - c\gamma\pi \int_0^\pi (au_x^2 + u_t^2)dx,$$

and therefore, for γ small enough ($\gamma < \frac{1}{2c\pi}$), we obtain

$$\mathcal{E}(t) \geq (1 - 2c\pi\gamma)E(t).$$

■

We are ready to state our result about the decay of the energy E :

Theorem 4.3.6. *Let (4.3), (4.9) and (4.21) be satisfied. Then the energy E decays exponentially, more precisely there exist two positive constants α and C such that*

$$E(t) \leq Ce^{-\alpha t}E(0), \forall t > 0.$$

Proof: First we remark that

$$\begin{aligned}
\frac{d}{dt}\left(2\int_0^\pi xu_t u_x dx\right) &= 2\int_0^\pi xu_{tt}u_x dx + 2\int_0^\pi xu_t u_{xt} dx \\
&= 2a\int_0^\pi xu_{xx}u_x dx + 2\int_0^\pi xu_t u_{xt} dx \\
&= a\int_0^\pi x\partial_x(u_x^2)dx + 2\int_0^\pi xu_t u_{xt} dx \\
&= -a\int_0^\pi u_x^2 dx + a\pi u_x^2(\pi, t) + 2\int_0^\pi xu_t u_{xt} dx.
\end{aligned}$$

But by integrating by parts in space and by (4.32), we have

$$\int_0^\pi x u_t u_{xt} dx = -\int_0^\pi x u_{xt} u_t dx - \int_0^\pi u_t^2 dx + \pi u_t^2(\pi, t),$$

that is to say

$$2\int_0^\pi x u_t u_{xt} dx = -\int_0^\pi u_t^2 dx + \pi u_t^2(\pi, t).$$

Thus

$$\frac{d}{dt}\left(2\int_0^\pi x u_t u_x dx\right) = -\int_0^\pi (u_t^2 + a u_x^2) dx + \pi u_t^2(\pi, t) + a\pi u_x^2(\pi, t).$$

By the boundary conditions in (4.32) and Cauchy-Schwarz's inequality, we finally find

$$\frac{d}{dt}\left(2\int_0^\pi x u_t u_x dx\right) \leq -\int_0^\pi (u_t^2 + a u_x^2) dx + \pi(1 + 2a\mu_0^2) u_t^2(\pi, t) + 2a\pi\mu_1^2 u_t^2(\pi, t - \tau(t)). \tag{4.48}$$

Then, we differentiate $\mathcal{E}_2$ to have

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t)\int_0^1 (-2\delta\dot{\tau}(t)\rho) e^{-2\delta\tau(t)\rho} u_t^2(\pi, t - \tau(t)\rho) d\rho + J_w,$$

where

$$J_w = 2q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho} u_t(\pi, t - \tau(t)\rho) u_{tt}(\pi, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho) d\rho.$$

As in the proof of Theorem 4.2.6, by integration by parts, we have

$$\begin{aligned} J_w &= q\int_0^1 (-2\delta\tau(t)(1 - \dot{\tau}(t)\rho) - \dot{\tau}(t)) u_t(\pi, t - \tau(t)\rho) e^{-2\delta\tau(t)\rho} d\rho \\ &\quad - q e^{-2\delta\tau(t)} u_t^2(\pi, t - \tau(t))(1 - \dot{\tau}(t)) + q u_t^2(\pi, t). \end{aligned}$$

These identities yield

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &= \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\int_0^1 (-2\delta\tau(t) - \dot{\tau}(t)) u_t^2(\pi, t - \tau(t)\rho) e^{-2\delta\tau(t)\rho} d\rho \\ &\quad - q e^{-2\delta\tau(t)} u_t^2(\pi, t - \tau(t))(1 - \dot{\tau}(t)) + q u_t^2(\pi, t) \\ &= \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + \frac{-2\delta\tau(t) - \dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) - q e^{-2\delta\tau(t)} u_t^2(\pi, t - \tau(t))(1 - \dot{\tau}(t)) + q u_t^2(\pi, t). \end{aligned}$$

Since $\dot{\tau}(t) < 1$ (see (4.2)), we obtain

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -2\delta\mathcal{E}_2(t) + q u_t^2(\pi, t). \tag{4.49}$$

Consequently, gathering (4.46), (4.48) and (4.49), we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(t) \leq -\gamma\int_0^\pi (u_t^2 + a u_x^2) dx - 2\gamma\delta\mathcal{E}_2(t) \\ &+ (u_t(\pi, t), u_t(\pi, t - \tau(t)))\,\tilde{\Phi}_q\,(u_t(\pi, t), u_t(\pi, t - \tau(t)))^\top, \end{aligned}$$

where $\tilde{\Phi}_q$ is the matrix defined by

$$\begin{aligned}\tilde{\Phi}_q &= \frac{1}{2}\begin{pmatrix} q(1+2\gamma)-2a\mu_0+2\gamma\pi(1+2a\mu_0^2) & -a\mu_1 \\ -a\mu_1 & 4a\gamma\pi\mu_1^2-q(1-d)\end{pmatrix} \\ &= \Psi_q+\gamma\begin{pmatrix} q+\pi(1+2a\mu_0^2) & 0 \\ 0 & 2a\pi\mu_1^2\end{pmatrix}.\end{aligned}$$

Noticing that $\max\left(q+\pi(1+2a\mu_0^2),\, 2a\pi\mu_1^2\right)=q+\pi(1+2a\mu_0^2)$ by (4.21), for γ sufficiently small, i.e. $\gamma\leq\frac{-\lambda}{q+\pi(1+2a\mu_0^2)}$, where λ is the greatest negative eigenvalue of Ψ_q given by (4.30), $\tilde{\Phi}_q$ is negative and therefore

$$\frac{d}{dt}\mathcal{E}(t)\leq-\gamma\int_0^\pi(u_t^2+au_x^2)dx-2\delta\gamma\mathcal{E}_2(t).$$

By the definition (4.47) of $\mathcal{E}_2$, this estimate becomes

$$\frac{d}{dt}\mathcal{E}(t)\leq-\gamma\int_0^\pi(u_t^2+au_x^2)dx-2\delta\gamma q\tau(t)e^{-2\delta\tau(t)}\int_0^1u_t^2(\pi,\, t-\tau(t)\rho)d\rho.$$

Since $\tau(t)\leq M$ (see (4.3)), in view of the definition of E, there exists a constant $\gamma'>0$ (depending on γ and $\delta:\gamma'\leq2\gamma\min\left(1,\,2\delta e^{-2\delta M}\right)$) such that

$$\frac{d}{dt}\mathcal{E}(t)\leq-\gamma' E(t).$$

By applying Lemma 4.3.5, we arrive at

$$\frac{d}{dt}\mathcal{E}(t)\leq-\alpha\mathcal{E}(t),$$

where α is explicitely given by $\alpha=\frac{\gamma'}{1+2\gamma c\pi}$. Therefore

$$\mathcal{E}(t)\leq\mathcal{E}(0)e^{-\alpha t},$$

and Lemma 4.3.5 allows to conclude the proof :

$$E(t)\leq\frac{1}{1-2c\pi\gamma}\mathcal{E}(t)\leq\frac{1}{1-2c\pi\gamma}\mathcal{E}(0)e^{-\alpha t}\leq\frac{1+2\gamma c\pi}{1-2c\pi\gamma}E(0)e^{-\alpha t}.$$

■

Remark 4.3.7. *In the case where the delay is constant in time and $a=1$, we recover some results from [85, 89] and chapter 1. Moreover, in [85, 89], the energy is defined by*

$$E(t)=\frac{1}{2}\int_0^\pi(u_t^2(x,\, t)+u_x^2(x,\, t))dx+\frac{\xi}{2}\int_0^1u_t^2(\pi,\, t-\tau\rho)d\rho,$$

where ξ is a positive constant satisfying

$$\tau\mu_1\leq\xi\leq\tau(2\mu_0-\mu_1),$$

under the condition (4.21), which corresponds to the definition (4.44) of E with $q=\frac{\xi}{\tau}$.

Remark 4.3.8. *In the proof of Theorem 4.3.6, we notice that we can explicitely calculate the decay rate α of the energy, given by*

$$\alpha = \frac{2\gamma}{1+2\gamma c\pi} \min\left(1,\, 2\delta e^{-2\delta M}\right),$$

with

$$\gamma < \frac{1}{2c\pi} \text{ and } \gamma \le \frac{-\lambda}{q+\pi(1+2a\mu_0^2)}$$

(by Lemma 4.3.5 and Theorem 4.3.6) and $c = \max\left(1,\, \frac{1}{a}\right)$ *where λ is given by (4.30), q by (4.25) and δ is a positive real number. Therefore, we can choose δ such that the decay of the energy is as quick as possible. By Remark 4.2.7, we get that the larger decay rate of the energy is given by*

$$\alpha_{max} = \frac{2\gamma}{1+2\gamma c\pi} \min\left(1,\, \frac{1}{Me}\right).$$

Chapitre 5

Quasi exponential decay of a finite difference space discretization of the 1-d wave equation by pointwise interior stabilization

5.1 Introduction

Recently boundary (or internal) controllability and stability results for the wave equation have been obtained using different methods, like the multipliers method, the frequency domain method, the microlocal analysis, the differential geometry or a combination of them [61, 70, 114, 115]. Usually these results are based on an observability estimate. On the other hand, similar results at a discrete level, namely when the continuous operator is replaced by a discrete one (obtained by a finite difference, a finite element or a finite volume scheme, for instance) are less developed. In their pioneer works, R. Glowinski et al. in [47–50] considered the boundary controllability of the numerical approximation of the wave equation using the so-called HUM method. They observed a bad behavior of the control of the numerical scheme, due to the spurious modes that the numerical scheme introduces at high frequencies. The main idea is that, in general, any discrete scheme generates spurious high frequency oscillations that do not exist at the continuous level. Therefore the discretization and the control processes do not commute; in other words, the control of the discrete model do not converge to the control of the continuous model. Some remedies are then necessary to restore the convergence of the discrete control to the continuous one. In these papers different remedies were proposed, like a Tychonoff regularization [49, 50], a bi-grid algorithm [47] or mixed finite element method [48], providing good numerical results. For the wave equation, rigorous convergence analyses of these cures for some schemes, like finite differences (in space and in time), finite elements and mixed finite elements, are made in [30,31,56,78,80,102,112,114,115]. For internal stability results this phenomenon was underlined numerically in [14], where numerical simulations suggest that the exponential decay of the discrete energy might be not uniformly exponential with respect to the mesh and step sizes. As remedy the authors propose without any proof to use a mixed method. More recently the stabilization of a locally damped wave equation in [102] and for a boundary damping in [103] were rigourously analyzed by adding a numerical viscosity term. But to our knowledge such results are not yet proved for pointwise internal damping.

Consequently the main goal of our chapter is threefold :
i) As for the continuous model, we furnish a necessary and sufficient condition that guarantees the decay to zero of the discrete energy.
ii) We show that the exponential decay of the discrete energy of the discretization of the wave equation with a pointwise internal damping by finite difference is not uniform.
iii) We show that a filtering technique allows to restore a quasi-optimal uniform exponential decay.
iv) We prove that without filtering we cannot expect quasi-exponential decay.

The schedule of the chapter is as follows : Section 5.2 is devoted to the introduction of the one-dimensional wave equation with an internal pointwise damping and its discretization. Then in section 5.3 we give a necessary and sufficient condition that guarantees the decay to zero of the discrete energy. The non uniform exponential decay of this energy is proved in section 5.4. Finally the filtering method is described in section 5.5 and the quasi-optimal uniform exponential decay is established. Some numerical results are presented that show the advantage of the filtering technique.

5.2 The continuous and discrete problems

We consider the wave equation on an interval of length 1 with an interior damping at ξ, in other words we study the solution y of the following problem

$$\left\{\begin{array}{ll} y_{tt}(x,t) - y_{xx}(x,t) = 0 & 0 < x < 1,\ t > 0, \\ y(0,t) = 0,\ y_x(1,t) = 0 & t > 0, \\ y(\xi_-,t) = y(\xi_+,t) & t > 0, \\ y_x(\xi_-,t) - y_x(\xi_+,t) = -\alpha y_t(\xi,t) & t > 0, \\ y(t=0) = y^{(0)},\ y_t(t=0) = y^{(1)} & 0 < x < 1, \end{array}\right. \tag{5.1}$$

where $(y^{(0)}, y^{(1)}) \in V \times L^2(0,1)$, $V = \{y \in H^1(0,1)\,;\, y(0) = 0\}$ and α is a fixed positive constant. It is easy to check [6] that this system (5.1) is well-posed in the energy space $V \times L^2(0,1)$. More precisely, for any $(y^{(0)}, y^{(1)}) \in V \times L^2(0,1)$ there exists a unique solution $y \in C((0,T), V) \cap C^1((0,T), L^2(0,1))$ of (5.1).

The energy of the solution of system (5.1) is given by

$$E(t) = \frac{1}{2}\int_0^1 (|y_t(x,t)|^2 + |y_x(x,t)|^2)dx$$

and obeys the following dissipation law

$$\frac{dE(t)}{dt} = -\alpha\,|y_t(\xi,t)|^2\,. \tag{5.2}$$

This implies that the energy is decreasing. Moreover, we know that (see [6] and chapter 1) $\lim_{t\to\infty} E(t) = 0$ for any initial data in $V \times L^2(0, 1)$ if and only if

$$\xi \neq \frac{2p}{2q+1}, \forall p, q \in \mathbb{N}.$$

Moreover, as it was proved in [6] and in chapter 1, the exponential decay property of the solution of (5.1) is equivalent to an observability estimate for the corresponding conservative system

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) = 0 & 0 < x < 1, t > 0, \\ u(0, t) = 0, u_x(1, t) = 0 & t > 0, \\ u(t=0) = y^{(0)}, u_t(t=0) = y^{(1)} & 0 < x < 1, \end{cases} \tag{5.3}$$

where $(y^{(0)}, y^{(1)}) \in V \times L^2(0, 1)$. In this case, the observability estimate holds if and only if ξ is a rational number with an irreductible fraction

$$\xi = \frac{p}{q}, \text{ where } p \text{ is odd},$$

and therefore under this condition, the system (5.1) is exponentially stable in the energy space. In the remainder of this chapter, ξ is fixed and is supposed to satisfy these two conditions.

We now introduce the finite difference scheme used later on. For this purpose, let N be a nonnegative integer. Set $h = \frac{1}{N+1}$ and consider the subdivision of $(0, 1)$ given by

$$0 = x_0 < x_1 = h < ... < x_{j-1} < x_j = jh < x_{j+1} < ... < x_N < x_{N+1} = 1,$$

i.e. $x_j = jh$ for all $j = 0, ..., N+1$. As ξ is not necessarily equal to jh for some j, we fix $j_N \in \mathbb{N} \cap (0, N+1)$ such that $x_{j_N} \to \xi$ when $N \to \infty$.

The finite-difference space semi-discretization of system (5.1) that we consider is the following

$$\begin{cases} y_j'' - \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = 0 & t > 0, j = 1, ..., N, j \neq j_N, \\ y_0 = 0, y_{N+1} - y_N = 0 & t > 0, \\ \frac{y_{j_N+1} - 2y_{j_N} + y_{j_N-1}}{h} = \alpha y_{j_N}' & t > 0, \\ y_j(t=0) = y_j^{(0)}, y_j'(t=0) = y_j^{(1)} & j = 1, ..., N. \end{cases} \tag{5.4}$$

Obviously, $y_j(t)$ is an approximation of $y(x_j, t)$, y being the solution of (5.1), provided the initial data $(y_j^{(0)}, y_j^{(1)})$, $j = 0, ..., N+1$ are approximations of the initial data in (5.1). Note further that the condition $\frac{y_{j_N+1} - 2y_{j_N} + y_{j_N-1}}{h} = \alpha y_{j_N}'$ is a natural approximation of the condition $y_x(\xi_-, t) - y_x(\xi_+, t) = -\alpha y_t(\xi, t)$. Alternatively (see [24]) we could chose the approximated condition

$$hy_{j_N}'' - \frac{y_{j_N+1} - 2y_{j_N} + y_{j_N-1}}{h} = -\alpha y_{j_N}',$$

but this choice did not allow us to prove any positive results.

We now set $y_h = (y_j)_j$, $y_h^{(0)} = (y_j^{(0)})_j$ and $y_h^{(1)} = (y_j^{(1)})_j$.

The system (5.4) may be equivalently written

$$\begin{cases} v_h' = \mathcal{A}_h v_h \\ v_h(0) = v_{h0} \end{cases}, \tag{5.5}$$

where $v_h = \begin{pmatrix} y_h \\ y_h' \end{pmatrix}$, $v_{h0} = \begin{pmatrix} y_h^{(0)} \\ y_h^{(1)} \end{pmatrix}$ and $\mathcal{A}_h$ is a $2N \times 2N$ matrix. This system is therefore uniquely solvable (as a differential system). Let us now show that it is dissipative : introduce its energy given by

$$E_h(t) = \frac{h}{2} \sum_{j=0,\, j\neq j_N}^{N} \left|y_j'(t)\right|^2 + \frac{h}{2}\sum_{j=0}^{N}\left|\frac{y_{j+1}(t) - y_j(t)}{h}\right|^2, \tag{5.6}$$

which is a discretization of the continuous energy E.

Lemma 5.2.1. *The energy of system (5.4) is a nonincreasing function of the time t and its derivative is given by*

$$E_h'(t) = -\alpha(y_{j_N}'(t))^2. \tag{5.7}$$

Proof: Deriving (5.6) and using (5.4), we obtain

$$\begin{aligned}
E_h'(t) &= h \sum_{j=1,\, j\neq j_N}^{N} y_j'(t)\left(\frac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2}\right) \\
&\quad + h\sum_{j=0}^{N}\left(\frac{y_{j+1}(t) - y_j(t)}{h}\right)\left(\frac{y_{j+1}'(t) - y_j'(t)}{h}\right) \\
&= \frac{1}{h}\sum_{j=1,\, j\neq j_N}^{N} y_j'(y_{j+1} - y_j) - \frac{1}{h}\sum_{j=1,\, j\neq j_N}^{N}(y_j - y_{j-1})y_j' + \frac{1}{h}\sum_{j=0}^{N}(y_{j+1} - y_j)(y_{j+1}' - y_j') \\
&= \frac{1}{h}\sum_{j=0,\, j\neq j_N}^{N} y_j'(y_{j+1} - y_j) - \frac{1}{h}\sum_{j=0,\, j\neq j_N-1}^{N}(y_{j+1} - y_j)y_{j+1}' + \frac{1}{h}\sum_{j=0}^{N}(y_{j+1} - y_j)(y_{j+1}' - y_j') \\
&= \frac{1}{h}y_{j_N}'(y_{j_N} - y_{j_N-1}) - \frac{1}{h}y_{j_N}'(y_{j_N+1} - y_{j_N}) \\
&= -y_{j_N}'(t)\left(\frac{y_{j_N+1}(t) - 2y_{j_N}(t) + y_{j_N-1}(t)}{h}\right) = -\alpha(y_{j_N}'(t))^2 \leq 0 \text{ by (5.4)}.
\end{aligned}$$

This proves (5.7) and, therefore, the energy is nonincreasing. ∎

Observe that (5.7) is the semi-discrete analogue of the dissipation law (5.2).

We also need to consider the finite-difference space semi-discretization of problem without damping (5.3)

$$\left\{\begin{array}{ll} u_j'' - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 0 & t>0,\, j=1,\dots,N \\ u_0 = 0,\, u_{N+1}-u_N = 0 & t>0 \\ u_j(t=0) = y_j^{(0)},\, u_j'(t=0) = y_j^{(1)} & j=1,\dots,N. \end{array}\right. \tag{5.8}$$

We observe that the system (5.8) can be rewritten in the following simplified form

$$U_h'' + A_h U_h = 0,\, t>0$$

where U_h stands for the column vector $(u_1,\dots,u_N)^T$, A_h denotes the matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & . & . \\ \vdots & . & . & . & . & . \\ 0 & . & . & . & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & -1 & 1 \end{pmatrix}$$

entering in the finite difference discretization of the Laplacian with Dirichlet boundary condition at the left endpoint, and Neumann one at the right endpoint. The eigenvectors of the matrix A_h satisfy the eigenvalue system

$$\left\{\begin{array}{ll} -\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} = \lambda\varphi_j & j=1,\dots,N, \\ \varphi_0 = 0,\, \varphi_{N+1}-\varphi_N = 0. & \end{array}\right. \tag{5.9}$$

It was shown in [58] (see also [103]) that

$$\begin{gathered} \varphi_j^{k,h} = \sin\left(\frac{(2k+1)\pi jh}{2-h}\right),\, j=0,1,\dots,N, \\ \lambda_{k,h} = \frac{4}{h^2}\sin^2\left(\frac{(2k+1)\pi h}{2(2-h)}\right),\, k=0,1,\dots,N-1. \end{gathered} \tag{5.10}$$

We have the following property of the eigenvectors of (5.9) :

Lemma 5.2.2. *For any eigenvector φ with eigenvalue λ of (5.9), the following identity holds*

$$\sum_{j=0}^{N}\left|\frac{\varphi_{j+1}-\varphi_j}{h}\right|^2 = \lambda\sum_{j=1}^{N}|\varphi_j|^2.$$

Proof: This result is well known (see [56]), we give its proof for the sake of completeness. Multiplying the first identity of (5.9) by φ_j and by adding on $j = 1, ..., N$, we obtain

$$\begin{aligned} -\lambda\sum_{j=1}^{N}|\varphi_j|^2 &= \sum_{j=1}^{N}(\tfrac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2})\varphi_j \\ &= \sum_{j=1}^{N}(\tfrac{\varphi_{j+1}-\varphi_j}{h^2})\varphi_j - \sum_{j=1}^{N}(\tfrac{\varphi_j-\varphi_{j-1}}{h^2})\varphi_j \\ &= \sum_{j=1}^{N}(\tfrac{\varphi_{j+1}-\varphi_j}{h^2})\varphi_j - \sum_{j=0}^{N-1}(\tfrac{\varphi_{j+1}-\varphi_j}{h^2})\varphi_{j+1}. \end{aligned}$$

and we conclude by the boundary condition $\varphi_{N+1} - \varphi_N = 0$. ■

The solution of (5.8) admits a Fourier development in the basis of eigenvectors of system (5.9). More precisely, every solution $u_h = (u_j)_j$ of (5.8) can be written as

$$u_h(t) = \sum_{k=0}^{N-1}\left(a_k\cos(\sqrt{\lambda_{k,h}}t) + \frac{b_k}{\sqrt{\lambda_{k,h}}}\sin(\sqrt{\lambda_{k,h}}t)\right)\varphi^{k,h}$$

for suitable coefficients $a_k, b_k \in \mathbb{R}$, $k = 0, ..., N-1$, that can be computed explicitely in terms of the initial data.

The energy of the problem without damping (5.8) is given by

$$E_{j_N}(u_h, t) = \frac{h}{2}\sum_{j=0}^{N}\left(|u_j'(t)|^2 + \left|\frac{u_{j+1}(t) - u_j(t)}{h}\right|^2\right). \tag{5.11}$$

Obviously, this energy $E_{j_N}(u_h, .)$ is constant (proved like Lemma 5.2.1).

5.3 Decay of the energy to 0

Proposition 5.3.1. *We have*

$$\lim_{t\to\infty} E_h(t) = 0$$

if and only if

$$j_N h \neq \frac{(2-h)l}{2k+1},\ \forall k = 0, ..., N-1,\ l \in \mathbb{N}.$$

Proof: We use the same kind of proof that in the continuous case (see [6] or chapter 1).

$\boxed{\Leftarrow}$ Let $S(t) = e^{\mathcal{A}_h t}$ be the exponential of the matrix $\mathcal{A}_h$. It suffices to show that

$$\lim_{t\to\infty} S(t)\begin{pmatrix} y_h^{(0)} \\ y_h^{(1)} \end{pmatrix} = 0.$$

For $U = \begin{pmatrix} y_j \\ y'_j \end{pmatrix}_j$, we set

$$\|U\|^2 = \frac{h}{2} \sum_{j=0, j\neq j_N}^{N} |y'_j|^2 + \frac{h}{2} \sum_{j=0}^{N} \left| \frac{y_{j+1} - y_j}{h} \right|^2 .$$

Let us fix $U_0 = \begin{pmatrix} y_h^{(0)} \\ y_h^{(1)} \end{pmatrix} \in \mathbb{R}^{2N}$. The set $orb(U_0) = \bigcup_{t\geq 0} S(t)U_0$ is precompact in $\mathbb{R}^{2N}$ because, for any sequence $(t_n)_n$

$$\begin{aligned} \|S(t_n)U_0\| + \|\mathcal{A}_h S(t_n)U_0\| &= \|S(t_n)U_0\| + \|S(t_n)\mathcal{A}_h U_0\| \\ &\leq \|U_0\| + \|\mathcal{A}_h U_0\| = cst. \end{aligned}$$

Therefore, there exists a subsequence, still denote by $S(t_n)U_0$, which converges in $\mathbb{R}^{2N}$. In this case, the ω-limit of U_0 defined by

$$\omega(U_0) = \{U \in \mathbb{R}^{2N}\,;\, \exists t_n,\, t_n \to \infty,\, S(t_n)U_0 \to U,\, t \to \infty\}$$

is non empty. On the other hand, if $\Phi = \begin{pmatrix} \phi_j^0 \\ \phi_j^1 \end{pmatrix}_j \in \omega(U_0)$, then $S(t)\Phi = \begin{pmatrix} \phi_j(t) \\ \phi'_j(t) \end{pmatrix} \in \omega(U_0)$. We can now apply LaSalle's invariance principle [32] with the relatively compact set $\bigcup_{t\geq 0} S(t)U_0$ and the Lyapounov functional $\|.\|$. As $\begin{pmatrix} \phi_j^0 \\ \phi_j^1 \end{pmatrix}$ and $S(t)\begin{pmatrix} \phi_j^0 \\ \phi_j^1 \end{pmatrix} = \begin{pmatrix} \phi_j(t) \\ \phi'_j(t) \end{pmatrix}$ belong to $\omega(U_0)$, we find that

$$\left\| \begin{pmatrix} \phi_j^0 \\ \phi_j^1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \phi_j(t) \\ \phi'_j(t) \end{pmatrix} \right\| = L,\ \forall t \geq 0.$$

Therefore $\phi(t)$ satisfies the problem (5.4) with initial condition ϕ_j^0 and ϕ_j^1. Moreover, it holds

$$0 = L - L = \left\| \begin{pmatrix} \phi_j(T) \\ \phi'_j(T) \end{pmatrix} \right\| - \left\| \begin{pmatrix} \phi_j^0 \\ \phi_j^1 \end{pmatrix} \right\| = -\alpha \int_0^T \left|\phi'_{j_N}\right|^2 dt,$$

which implies that

$$\phi'_{j_N}(t) = 0,\ \forall t > 0. \tag{5.12}$$

In particular, this implies that ϕ is solution of problem (5.8) with initial data ϕ_j^0 and ϕ_j^1 because the damping term disappears and $\phi''_{j_N} = 0$.

Let us now write

$$\phi^0 = \sum_{k=0}^{N-1} a_k \varphi^{k,h}, \quad \phi^1 = \sum_{k=0}^{N-1} b_k \varphi^{k,h}.$$

Then ϕ is given by

$$\phi(t) = \sum_{k=0}^{N-1} \left(a_k \cos(\sqrt{\lambda_{k,h}}t) + \frac{b_k}{\sqrt{\lambda_{k,h}}} \sin(\sqrt{\lambda_{k,h}}t) \right) \varphi^{k,h}.$$

Consequently by (5.12)

$$0 = \phi'_{j_N}(t) = \sum_{k=0}^{N-1}(-a_k\sqrt{\lambda_{k,h}}\varphi_{j_N}^{k,h})\sin(\sqrt{\lambda_{k,h}}t) + \sum_{k=0}^{N-1}(b_k\varphi_{j_N}^{k,h})\cos(\sqrt{\lambda_{k,h}}t).$$

As the size h of the discretization is fixed, we can use Ingham's inequality (see [57]) because the gap condition is verified. We find that

$$0 = \int_0^T \left|\phi'_{j_N}\right|^2 dt \geq C(h)\sum_{k=0}^{N-1}[(a_k\sqrt{\lambda_{k,h}}\varphi_{j_N}^{k,h})^2 + (b_k\varphi_{j_N}^{k,h})^2] \geq 0,$$

for T large enough and for a constant $C(h) > 0$ which depends on the size h of the discretization. Consequently

$$a_k\varphi_{j_N}^{k,h} = 0 \text{ and } b_k\varphi_{j_N}^{k,h} = 0,\ \forall k = 0, ..., N-1.$$

But for all $k = 0, ..., N-1$,

$$\varphi_{j_N}^{k,h} \neq 0 \Leftrightarrow \sin\left(\frac{(2k+1)\pi j_N h}{2-h}\right) \neq 0 \Leftrightarrow j_N h \neq \frac{(2-h)l}{2k+1},\ \forall l \in \mathbb{N}.$$

Therefore, if $j_N h \neq \frac{(2-h)l}{2k+1}$, for all $l \in \mathbb{N}$, $k = 0, ..., N-1$, we will have $a_k = b_k = 0$, for all $k = 0, ..., N-1$ and therefore $\phi^0 = \phi^1 = 0$. We have shown that all $(\phi^0, \phi^1)^T \in \omega(U_0)$ is equal to zero and consequently $\lim_{t\to\infty} E(t) = 0$.

$\boxed{\Rightarrow}$ We use a contradiction argument. Suppose that there exists $k \in \{0, ..., N-1\}$ such that $j_N h = \frac{(2-h)l}{2k+1}$, so that $\varphi_{j_N}^{k,h} = 0$. Then

$$y_h(t) = \varphi^{k,h}\cos(\lambda_{k,h}t)$$

is solution of (5.4) with a constant energy. ■

Remark 5.3.2. Our previous result is a discrete version of the continuous one [6] that says that the energy of the solution of system (5.1) tends to zero if and only if

$$\xi \neq \frac{2p}{2q+1},\ \forall p, q \in \mathbb{N}.$$

5.4 Non exponential decay of the discrete energy

In this section, our goal is to prove the non exponential decay of solutions of (5.4). First, we need the two following lemmas.

Lemma 5.4.1. *If there exist positive constants M and ω such that for all $y_h^{(0)}$ and $y_h^{(1)}$ in $\mathbb{R}^N$,*

$$E_h(t) \le Me^{-\omega t}E_h(0), \forall t \ge 0 \tag{5.13}$$

holds, then there exist positive constants T and C such that for all $y_h^{(0)}$ and $y_h^{(1)}$ in $\mathbb{R}^N$, one has

$$E_{j_N}(u_h, 0) \le C\int_0^T \left|u'_{j_N}\right|^2 dt + Ch^2\int_0^T |(A_h u_h)_{j_N}|^2 dt, \tag{5.14}$$

where u_h is solution of (5.8) and $E_{j_N}(u_h, .)$ is defined by (5.11).

Proof: Let us suppose that (5.13) holds. It follows from the dissipation law (5.7) that for all $T > 0$,

$$E_h(0) - E_h(T) = \alpha\int_0^T \left|y'_{j_N}\right|^2 dt. \tag{5.15}$$

By (5.13), for T large enough (any $T \ge \frac{\ln(\frac{4M}{3})}{\omega}$ suffices), one has $E_h(T) \le \frac{3}{4}E_h(0)$, and so

$$\alpha\int_0^T \left|y'_{j_N}\right|^2 dt \ge E_h(0) - \frac{3}{4}E_h(0) = \frac{1}{4}E_h(0) = \frac{1}{4}E_h(u_h, 0). \tag{5.16}$$

Now, we split up y_h solution of (5.4) as follows

$$y_h = u_h + w_h, \tag{5.17}$$

where $u_h = (u_j)_j$ is solution of problem without damping (5.8) and the difference $w_h = (w_j)_j$ solves

$$\begin{cases} w_j'' - \frac{w_{j+1}-2w_j+w_{j-1}}{h^2} = 0 & t > 0,\ j = 1, ..., N,\ j \ne j_N \\ w_0 = 0,\ w_{N+1} - w_N = 0 & t > 0 \\ \frac{w_{j_N+1}-2w_{j_N}+w_{j_N-1}}{h} = \alpha y'_{j_N} - h(A_h u_h)_{j_N} & t > 0 \\ w_j(t=0) = 0,\ w_j'(t=0) = 0 & j = 1, ..., N. \end{cases} \tag{5.18}$$

Denote by

$$E_h(w_h, t) = \frac{h}{2}\sum_{j=0, j\ne j_N}^{N} \left|w_j'(t)\right|^2 + \frac{h}{2}\sum_{j=0}^{N}\left|\frac{w_{j+1}(t) - w_j(t)}{h}\right|^2. \tag{5.19}$$

We easily check, like in the proof of Lemma 5.2.1, that

$$E_h'(w_h, t) = -w'_{j_N}(\alpha y'_{j_N} - h(A_h u_h)_{j_N}). \tag{5.20}$$

Therefore

$$\begin{aligned} E_h(w_h, t) = \int_0^t E_h'(w_h, t)dt &= -\alpha\int_0^t w'_{j_N}y'_{j_N}dt + h\int_0^t w'_{j_N}(A_h u_h)_{j_N}dt \\ &= -\alpha\int_0^t w'_{j_N}u'_{j_N}dt - \alpha\int_0^t (w'_{j_N})^2dt + h\int_0^t w'_{j_N}(A_h u_h)_{j_N}dt, \end{aligned}$$

by (5.17), which implies that

$$E_h(w_h,\, t) + \alpha \int_0^t (w'_{j_N})^2 dt = -\alpha \int_0^t w'_{j_N} u'_{j_N} dt + h \int_0^t w'_{j_N} (A_h u_h)_{j_N} dt.$$

By Cauchy-Schwarz's inequality, we deduce that

$$\begin{aligned} \alpha \int_0^t (w'_{j_N})^2 dt &\le -\alpha \int_0^t w'_{j_N} u'_{j_N} dt + h \int_0^t w'_{j_N} (A_h u_h)_{j_N} dt \\ &\le \left(\int_0^t (w'_{j_N})^2 dt \right)^{\frac{1}{2}} \left(\alpha \left(\int_0^t (u'_{j_N})^2 dt \right)^{\frac{1}{2}} + h \left(\int_0^t \left|(A_h u_h)_{j_N}\right|^2 dt \right)^{\frac{1}{2}} \right), \end{aligned}$$

and therefore

$$\alpha \left(\int_0^t (w'_{j_N})^2 dt \right)^{\frac{1}{2}} \le \alpha \left(\int_0^t (u'_{j_N})^2 dt \right)^{\frac{1}{2}} + h \left(\int_0^t \left|(A_h u_h)_{j_N}\right|^2 dt \right)^{\frac{1}{2}}.$$

By a convex inequality, we obtain that

$$\alpha^2 \int_0^t (w'_{j_N})^2 dt \le 2\alpha^2 \int_0^t (u'_{j_N})^2 dt + 2h^2 \int_0^t \left|(A_h u_h)_{j_N}\right|^2 dt. \tag{5.21}$$

By the estimate (5.16) and by (5.17), we find that

$$E_h(u_h,\, 0) \le 4\alpha \int_0^T \left|y'_{j_N}\right|^2 dt \le 8\alpha \left(\int_0^T \left|u'_{j_N}\right|^2 dt + \int_0^T \left|w'_{j_N}\right|^2 dt \right)$$

and so, by the estimate (5.21), we obtain

$$\begin{aligned} E_h(u_h,\, 0) &\le 8\alpha \int_0^T \left|u'_{j_N}\right|^2 dt + 16\alpha \int_0^T \left|u'_{j_N}\right|^2 dt + \frac{16}{\alpha} h^2 \int_0^T \left|(A_h u_h)_{j_N}\right|^2 dt \\ &\le C \int_0^T \left|u'_{j_N}\right|^2 dt + C h^2 \int_0^T \left|(A_h u_h)_{j_N}\right|^2 dt. \end{aligned} \tag{5.22}$$

The difference between (5.22) and (5.14) relies on the left-hand side, i.e. the energy $E_h(u_h,\, 0)$ instead of $E_{j_N}(u_h,\, 0)$. To solve this problem, we notice that

$$E_{j_N}(u_h,\, t) = E_h(u_h,\, t) + \frac{h}{2} (u'_{j_N}(t))^2,\ \forall t > 0. \tag{5.23}$$

Deriving the previous identity (5.23), we have

$$0 = E'_{j_N}(u_h,\, t) = E'_h(u_h,\, t) + h u'_{j_N}(t) u''_{j_N}(t),\ \forall t > 0,$$

and therefore

$$E_h(u_h,\, t) = E_h(u_h,\, 0) - h \int_0^t u'_{j_N}(s) u''_{j_N}(s) ds,\ \forall t > 0. \tag{5.24}$$

Moreover, since $E_{j_N}(u_h, .)$ is constant, for all $\epsilon > 0$, we have

$$\begin{aligned}
E_{j_N}(u_h, 0) &= \frac{1}{\epsilon}\int_0^\epsilon E_{j_N}(u_h, t)dt \\
&= \frac{1}{\epsilon}\int_0^\epsilon \left(E_h(u_h, t) + \frac{h}{2}(u'_{j_N}(t))^2\right) dt \text{ by (5.23)} \\
&= \frac{1}{\epsilon}\int_0^\epsilon (E_h(u_h, 0) + \frac{h}{2}(u'_{j_N}(t))^2)dt - \frac{1}{\epsilon}\int_0^\epsilon h\left(\int_0^t u'_{j_N}(s)u''_{j_N}(s)ds\right) dt \text{ by (5.24)} \\
&\leq E_h(u_h, 0) + \frac{h}{2\epsilon}\int_0^\epsilon (u'_{j_N}(t))^2 dt + \frac{1}{2}\int_0^\epsilon (u'_{j_N}(s))^2 ds + \frac{h^2}{2}\int_0^\epsilon (u''_{j_N}(s))^2 ds \\
&\leq E_h(u_h, 0) + \frac{h}{2\epsilon}\int_0^\epsilon (u'_{j_N}(t))^2 dt + \frac{1}{2}\int_0^\epsilon (u'_{j_N}(t))^2 dt + \frac{h^2}{2}\int_0^\epsilon |(A_h u_h)_{j_N}|^2 dt
\end{aligned}$$

by (5.8). Taking $\epsilon = h$, we have

$$E_{j_N}(u_h, 0) \leq E_h(u_h, 0) + \int_0^h (u'_{j_N}(t))^2 dt + \frac{h^2}{2}\int_0^h |(A_h u_h)_{j_N}|^2 dt. \tag{5.25}$$

Therefore with (5.22) and for $T > h$, we obtain (5.14). ∎

Lemma 5.4.2. *Assume that N is a multiple of q and $j_N = N\frac{p}{q}$, where p is odd (so that $x_{j_N} \to \frac{p}{q}$). Then for all $T > 0$ there exist a positive constant $C(T)$ and initial data such that the solution u_h of (5.8) with these initial data satisfies*

$$E_{j_N}(u_h, 0) \geq \frac{C(T)}{h^2}\left(\int_0^T \left|u'_{j_N}(t)\right|^2 dt + h^2\int_0^T |(A_h u_h)_{j_N}|^2 dt\right).$$

Proof: Choose

$$y_j^{(0)} = \varphi_j^{k,h}, \quad y_j^{(1)} = 0,$$

where $k \in \{0, ..., N-1\}$ will be chosen later. It is easy to check that

$$u_h = \cos(\sqrt{\lambda_{k,h}}t)\varphi^{k,h}$$

solves (5.8). On the other hand, we can verify (see [103] and Lemma 5.2.2) that

$$E_{j_N}(u_h, 0) = \frac{2-h}{8}\lambda_{k,h}.$$

Moreover

$$\int_0^T \left|u'_{j_N}(t)\right|^2 dt \leq \int_0^T \lambda_{k,h}\left|\varphi_{j_N}^{k,h}\right|^2 dt = \lambda_{k,h}T\left|\varphi_{j_N}^{k,h}\right|^2. \tag{5.26}$$

Since we assume that N is a multiple of q, that is to say $N = mq$, where $m \in \mathbb{N}$, then $j_N = N\frac{p}{q} = mp$. First, we notice that

$$x_{j_N} = j_N h = \frac{Np}{q(N+1)} = \frac{mp}{N+1} \to \frac{p}{q}.$$

Secondly, we have

$$\varphi_{j_N}^{k,h} = \sin\left(\frac{(2k+1)\pi j_N h}{2-h}\right) = \sin\left(\frac{(2k+1)\pi mp}{2N+1}\right)$$

because $h = \frac{1}{N+1}$ and so $2-h = \frac{2N+1}{N+1}$. Now, we fix k in the form

$$k = N + \beta,$$

with $\beta < 0$ to determine. Therefore, by simple calculations, we get

$$\frac{(2k+1)j_N h}{2-h} = \frac{Np}{q} + \frac{2\beta Np}{(2N+1)q}.$$

Thus, by choosing $2\beta = -2q$, that is to say $\beta = -q$, then $k = N - q \in \{0, ..., N-1\}$ for N large enough. Again simple calculations yield

$$\frac{(2k+1)j_N h}{2-h} = mp - p + \frac{p}{2N+1}.$$

As a consequence

$$\varphi_{j_N}^{N-q,h} = \sin\left((mp-p)\pi + \frac{p\pi}{2N+1}\right) = \pm\sin\left(\frac{p\pi}{2N+1}\right).$$

Because $2N+1 = \frac{2-h}{h}$, we obtain

$$\varphi_{j_N}^{N-q,h} \sim \pm(\pi p)\frac{h}{2}, \tag{5.27}$$

when $h \to 0$. Thus, for N large enough, by (5.26) and (5.10), there exists $C > 0$ such that

$$\int_0^T \left|u'_{j_N}(t)\right|^2 dt \leq C\left(\lambda_{N-q,h} T (\pi p)^2 \frac{h^2}{4}\right).$$

Similarly we have

$$h^2 \int_0^T |(A_h u_h)_{j_N}|^2 \, dt \leq h^2 \lambda_{N-q,h}^2 \left|\varphi_{j_N}^{N-q,h}\right|^2 T \leq C(h^2 \lambda_{N-q,h} T) \text{ for a } C > 0,$$

due to (5.27) and because $h^2\lambda_{N-q,h} \sim 1$. Indeed, $\frac{(2(N-q)+1)\pi h}{2(2-h)} = \frac{\pi}{2} - \frac{q\pi}{2N+1}$ and thus, by (5.10),

$$\lambda_{N-q,h} = \frac{4}{h^2}\cos^2\left(\frac{q\pi}{2N+1}\right) \sim \frac{4}{h^2}. \tag{5.28}$$

Therefore, there exists $C(T) > 0$ such that for h small enough,

$$\begin{aligned} \int_0^T \left|u'_{j_N}(t)\right|^2 dt + h^2 \int_0^T |(A_h u_h)_{j_N}|^2 \, dt &\leq C(T)\lambda_{N-q,h} h^2 \\ &\leq C(T) h^2 E_{j_N}(u_h, 0). \end{aligned}$$

■

The main result of this section is the following theorem

Theorem 5.4.3. *Assume that N is a multiple of q and $j_N = N\frac{p}{q}$ (hence $x_{j_N} \to \frac{p}{q}$). Then the decay of E_h to zero is not uniformly exponential with respect to h. More precisely there do not exist positive constants M and ω which are independent of h such that for all $h > 0$ and $y_h^{(0)}$ and $y_h^{(1)}$ in $\mathbb{R}^N$, (5.13) holds.*

Proof: Lemma 5.4.2 shows that the estimate (5.14) is not uniform with respect to h. By Lemma 5.4.1, (5.13) cannot hold with M and ω independent of h. ∎

In conclusion, the decay of E_h to zero is not uniformly exponential with respect to h due to the existence of high frequency spurious solutions of the semi-discrete model. To overcome this obstacle, we rule out the high frequency spurious modes.

5.5 Filtering technique

5.5.1 Interior observability of the discrete wave equation

We recall that every solution of (5.8) can be developped in Fourier series as follow

$$u_h(t) = \sum_{k=0}^{N-1} \left(a_k \cos(\sqrt{\lambda_{k,h}}t) + \frac{b_k}{\sqrt{\lambda_{k,h}}} \sin(\sqrt{\lambda_{k,h}}t) \right) \varphi^{k,h}$$

where a_k, $b_k \in \mathbb{R}$, $k = 0, \dots, N-1$, are the Fourier coefficients of the initial data, i.e.

$$y_h^{(0)} = \sum_{k=0}^{N-1} a_k \varphi^{k,h}, \; y_h^{(1)} = \sum_{k=0}^{N-1} b_k \varphi^{k,h}.$$

In order to obtain a positive counterpart to Theorem 5.4.3, we have to introduce suitable subclasses of solutions of (5.8) generated by eigenvectors of (5.9) associated with eigenvalues such that $\lambda h^2 \leq \gamma$. More precisely, introduce

$$C_h(\gamma) := \left\{ \sum_{\lambda_{k,h} \leq \frac{\gamma}{h^2}} a_k \varphi^{k,h} \text{ with } a_k \in \mathbb{R} \right\},$$

the space spanned by the eigenvectors $\varphi^{k,h}$ for which $\lambda_{k,h} \leq \frac{\gamma}{h^2}$.

Moreover, in order to use Ingham's inequality, we need an estimate between the squareroots of consecutive eigenvalues entering in the Fourier development of the solutions of (5.8) in the class $C_h(\gamma)$. This is stated in the next lemma proved in [56].

Lemma 5.5.1. *Assume that*

$$\gamma = 4 \sin^2\left(\frac{\pi\epsilon}{2}\right) \tag{5.29}$$

for some $0 \leq \epsilon < 1$. Then

$$\sqrt{\lambda_{k,h}} - \sqrt{\lambda_{k-1,h}} \geq \pi \cos\left(\frac{\pi\epsilon}{2}\right)$$

for all eigenvalues in the range $\lambda_{k,h} h^2 \leq \gamma$.

Remark 5.5.2. Note that every $0 \leq \gamma < 4$ can be written in the form (5.29) for some $0 \leq \epsilon < 1$.

Proof: This result is well known (see [56]), we give its proof for the sake of completeness. Let us compute the gap

$$\begin{aligned}
\sqrt{\lambda_{k,h}} - \sqrt{\lambda_{k-1,h}} &= \frac{2}{h}\left(\sin\left(\frac{(2k+1)\pi h}{2(2-h)}\right) - \sin\left(\frac{(2k-1)\pi h}{2(2-h)}\right)\right) \\
&= \frac{2}{h}\left(\frac{(2k+1)\pi h}{2(2-h)} - \frac{(2k-1)\pi h}{2(2-h)}\right)\cos(\xi) \\
&= \frac{2\pi}{2-h}\cos\xi,
\end{aligned}$$

for $\xi \in [\frac{(2k-1)\pi h}{2(2-h)}, \frac{(2k+1)\pi h}{2(2-h)}]$. Moreover, the eigenvalue $\lambda_{k,h} = \frac{4}{h^2}\sin^2(\frac{(2k+1)\pi h}{2(2-h)})$ satisfies $\lambda h^2 \leq \gamma = 4\sin^2(\frac{\pi\epsilon}{2})$ if and only if $\sin^2(\frac{(2k+1)\pi h}{2(2-h)}) \leq \sin^2(\frac{\pi\epsilon}{2})$, that is to say if and only if

$$\frac{(2k+1)\pi h}{2(2-h)} \leq \frac{\pi\epsilon}{2}$$

because $0 < \frac{\pi\epsilon}{2} < \frac{\pi}{2}$ and $0 < \frac{(2k+1)\pi h}{2(2-h)} = \frac{(2k+1)\pi}{2(2N+1)} \leq \frac{\pi}{2}$ $(k \leq N)$ and because the function sinus is increasing on $[0, \frac{\pi}{2}]$. Since $0 \leq \frac{(2k-1)\pi h}{2(2-h)} \leq \xi \leq \frac{(2k+1)\pi h}{2(2-h)} \leq \frac{\pi}{2}\epsilon < \frac{\pi}{2}$ and because the function cosinus is decreasing on $[0, \frac{\pi}{2}]$, we obtain

$$\sqrt{\lambda_{k,h}} - \sqrt{\lambda_{k-1,h}} = \frac{2\pi}{2-h}\cos(\xi) \geq \frac{2\pi}{2-h}\cos\left(\frac{\pi}{2}\epsilon\right) \geq \pi\cos\left(\frac{\pi}{2}\epsilon\right).$$

■

Now we take the sequence $j_N \in \mathbb{N}$ such that

$$x_{j_N} = j_N h \longrightarrow \xi = \frac{p}{q}, \text{ when } h \to 0,$$

defined by

$$j_N = \left[\frac{p(2N+1)}{2q}\right] \in \mathbb{N}, \tag{5.30}$$

where $[x]$ means the integral part of x.

In the sequel, we need the following lemma which is proved in Lemma 3.1 of [103] by the multiplier technique.

Lemma 5.5.3. *For every $T > 2$, $h > 0$ and every solution u_h of (5.8), we have*

$$(T-2)E_{j_N}(u_h, 0) \leq \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left(\frac{u'_{j+1} - u'_j}{h}\right)^2 dt + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 dt.$$

We also use the following lemma proved in Lemma 2.9 of [97].

Lemma 5.5.4. *s is a rational number with an irreductible fraction*

$$s = \frac{p}{q}, \text{ where } p \text{ is odd}$$

if and only if there exists $\alpha > 0$ such that

$$\left|\sin\left((k+\frac{1}{2})\pi s\right)\right| > \alpha, \ \forall k \in \mathbb{N}.$$

We are ready to prove the following uniform interior observability of the discrete wave equation :

Proposition 5.5.5. *Assume that $\epsilon \in (0, 1)$ is small enough such that*

$$\alpha > \tan\left(\frac{\pi\epsilon}{2}\right),$$

where α verifies

$$\left|\sin\left((k+\frac{1}{2})\pi\xi\right)\right| > \alpha, \ \forall k \in \mathbb{N}$$

(which is possible, see Lemma 5.5.4). Then there exist $T = T(\gamma) > 2$ and $C = C(\gamma, T) > 0$ such that for every solution u_h of (5.8) in the class $C_h(\gamma)$, we have

$$(T-2)E_{j_N}(u_h, 0) \leq C\int_0^T \left|u'_{j_N}(t)\right|^2 dt.$$

Proof: We rewrite the solution $u_h \in C_h(\gamma)$ of (5.8) as

$$u_h = \sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} c_k e^{i\mu_{k,h}t}\varphi^{k,h},$$

where

$$\mu_{-k,h} = -\mu_{k,h}, \ \mu_{k,h} = \sqrt{\lambda_{k,h}}, \ \varphi^{-k,h} = \varphi^{k,h}, \ c_k = \frac{a_k - ib_k}{2}, \ c_{-k} = \bar{c}_k.$$

Then, applying Ingham's Theorem (see Lemma 5.5.1), there exist $C > 0$ and $T > \frac{2}{\cos(\frac{\pi\epsilon}{2})}$ such that

$$\begin{aligned}\int_0^T \left|u'_{j_N}(t)\right|^2 dt &\geq C\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2 |\mu_{k,h}|^2 \left|\varphi^{k,h}_{j_N}\right|^2 \\ &= C\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2 \lambda_{k,h} \left|\varphi^{k,h}_{j_N}\right|^2.\end{aligned} \tag{5.31}$$

Now, we look for a condition that guarantees

$$\left|\varphi^{k,h}_{j_N}\right|^2 \geq \alpha, \ \forall k \in \{0, ..., N-1\} : \ \lambda_{k,h}h^2 \leq \gamma.$$

Recalling that $j_N = \left[\frac{p(2N+1)}{2q}\right] \in \mathbb{N}$, we may write

$$x_{j_N} = j_N h = \frac{p}{q}\frac{2-h}{2} \pm \eta, \text{ with } \eta \to 0 \text{ as } h \to 0.$$

Then

$$\begin{aligned}
\varphi_{j_N}^{k,h} &= \sin\left((k+\frac{1}{2})\pi\frac{2j_N h}{2-h}\right) = \sin\left((k+\frac{1}{2})\pi\left(\frac{p}{q} \pm \eta\frac{2}{2-h}\right)\right)\\
&= \sin\left((k+\frac{1}{2})\pi\frac{p}{q}\right)\cos\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right) \pm \cos\left((k+\frac{1}{2})\pi\frac{p}{q}\right)\sin\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right).
\end{aligned}$$

Thus, by Lemma 5.5.4, we obtain

$$\begin{aligned}
\left|\varphi_{j_N}^{k,h}\right| &\geq \left|\sin\left((k+\frac{1}{2})\pi\frac{p}{q}\right)\right|\left|\cos\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right)\right|\\
&\quad -\left|\cos\left((k+\frac{1}{2})\pi\frac{p}{q}\right)\right|\left|\sin\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right)\right|\\
&\geq \alpha\left|\cos\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right)\right| - \left|\sin\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right)\right|.
\end{aligned}$$

Now, by the choice of γ, we have

$$\begin{aligned}
\lambda_{k,h}h^2 \leq \gamma \quad &\Leftrightarrow \quad \frac{(2k+1)\pi h}{2(2-h)} \leq \frac{\pi\epsilon}{2}\\
&\Leftrightarrow \quad 0 \leq \frac{(k+\frac{1}{2})2\pi\eta}{2-h} \leq \pi\epsilon\eta \leq \frac{\pi}{2}, \text{ because } \eta \to 0, \text{ so } \eta < \frac{1}{2}.
\end{aligned}$$

Since the function sinus is increasing on $[0, \frac{\pi}{2}]$, we may write

$$\sin\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right) \leq \sin(\pi\epsilon\eta) \leq \sin\left(\frac{\pi\epsilon}{2}\right),$$

and then

$$\left|\cos\left((k+\frac{1}{2})\pi\frac{2\eta}{2-h}\right)\right| \geq \cos\left(\frac{\pi\epsilon}{2}\right).$$

Thus, by the assumption $\alpha > \tan(\frac{\pi\epsilon}{2})$, we get

$$\left|\varphi_{j_N}^{k,h}\right| \geq \alpha\cos\left(\frac{\pi\epsilon}{2}\right) - \sin\left(\frac{\pi\epsilon}{2}\right) > C > 0.$$

Now, we can return to the inequality (5.31) and deduce that

$$\int_0^T \left|u'_{j_N}(t)\right|^2 dt \geq C \sum_{\left|\mu_{k,h}\right|h \leq \sqrt{\gamma}} |c_k|^2 \lambda_{k,h}. \tag{5.32}$$

Then, we recall that Lemma 5.5.3 states that, for $T > 2$, we have

$$(T-2)E_{j_N}(u_h, 0) \leq \frac{h}{4}\sum_{j=0}^{N}\int_0^T (u'_{j+1}-u'_j)^2 dt + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 dt.$$

Moreover, applying Ingham's inequality to the two terms of this right hand side, we get

$$\begin{aligned}
\frac{h}{4}\sum_{j=0}^{N}\int_0^T (u'_{j+1}-u'_j)^2 dt &\leq Ch\sum_{j=0}^{N}\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}(\varphi_{j+1}^{k,h}-\varphi_j^{k,h})^2 \\
&= Ch^3\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}\sum_{j=0}^{N}\left(\frac{\varphi_{j+1}^{k,h}-\varphi_j^{k,h}}{h}\right)^2 \\
&= Ch^3\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}^2\sum_{j=0}^{N}(\varphi_j^{k,h})^2 \text{ by Lemma 5.2.2} \\
&\leq C\gamma\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}h\sum_{j=0}^{N}(\varphi_j^{k,h})^2 \text{ because } u\in C_h(\gamma),
\end{aligned}$$

and

$$\begin{aligned}
\frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 dt &\leq C\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}(\varphi_{N+1}^{k,h})^2 \\
&\leq C\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h} \text{ because } \varphi_{N+1}^{k,h} \text{ is bounded by } 1.
\end{aligned}$$

Therefore, since $h\sum_{j=0}^{N}(\varphi_j^{k,h})^2 \sim 1$, for $T > \frac{2}{\cos(\frac{\pi\epsilon}{2})}$, there exists $C > 0$ such that

$$(T-2)E_{j_N}(u_h, 0) \leq C\sum_{|\mu_{k,h}|h\leq\sqrt{\gamma}} |c_k|^2\lambda_{k,h}.$$

This estimate and (5.32) lead to the conclusion. ■

5.5.2 Some estimates

Lemma 5.5.6. *There exist $T > 0$ and $C(T) > 0$ such that the solution w_h of (5.18) verifies*

$$\int_0^{2T} w'_{j_N}w'_{j_N-1}(2j_Nh-h)(2T-t)dt \leq C(T)\int_0^{2T} E_h(w_h, t)dt + \frac{h}{4}\int_0^{2T}(w'_{j_N})^2(2T-t)dt.$$

Proof: We multiply the first identity of (5.18) by $(jh)(2T-t)\frac{w_{j+1}-w_{j-1}}{h}$ (which is the discrete version of the multiplier $x(2T-t)w_x(x,t)$), we add on $j=1,\ldots,j_N-1$ and integrate between 0 and $2T$. This yields

$$I_1 - I_2 = 0, \tag{5.33}$$

where

$$I_1 = h\sum_{j=1}^{j_N-1}\int_0^{2T} w_j''(jh)(2T-t)\frac{w_{j+1}-w_{j-1}}{h}dt,$$

$$I_2 = h\sum_{j=1}^{j_N-1}\int_0^{2T} \frac{w_{j+1}-2w_j+w_{j-1}}{h^2}(jh)(2T-t)\frac{w_{j+1}-w_{j-1}}{h}dt.$$

For the first term I_1, an integration by parts yields (recalling that $w_j'(t=0)=0$)

$$\begin{aligned} I_1 &= -h\sum_{j=1}^{j_N-1}\int_0^{2T} w_j'(jh)\frac{d}{dt}((2T-t)\frac{w_{j+1}-w_{j-1}}{h})dt \\ &= h\sum_{j=1}^{j_N-1}\int_0^{2T} w_j'(jh)\frac{w_{j+1}-w_{j-1}}{h}dt - I_3, \end{aligned}$$

where

$$I_3 = h\sum_{j=1}^{j_N-1}\int_0^{2T} w_j'(jh)(2T-t)\frac{w_{j+1}'-w_{j-1}'}{h}dt.$$

Successive changes of variables lead to

$$\begin{aligned} I_3 &= \sum_{j=2}^{j_N}\int_0^{2T} w_{j-1}'w_j'(j-1)h(2T-t)dt - \sum_{j=0}^{j_N-2}\int_0^{2T} w_{j+1}'w_j'(j+1)h(2T-t)dt \\ &= \sum_{j=1}^{j_N}\int_0^{2T} w_{j-1}'w_j'(j-1)h(2T-t)dt - \sum_{j=1}^{j_N-2}\int_0^{2T} w_{j+1}'w_j'(j+1)h(2T-t)dt \text{ because } w_0=0 \\ &= -\sum_{j=1}^{j_N-1}\int_0^{2T} w_j'(w_{j+1}'-w_{j-1}')jh(2T-t)dt - h\sum_{j=1}^{j_N}\int_0^{2T} w_{j-1}'w_j'(2T-t)dt \\ &\quad + \int_0^{2T} w_{j_N-1}'w_{j_N}'(j_N-1)h(2T-t)dt - h\sum_{j=1}^{j_N-2}\int_0^{2T} w_{j+1}'w_j'(2T-t)dt \\ &\quad + \int_0^{2T} w_{j_N}'w_{j_N-1}'j_Nh(2T-t)dt \\ &= -I_3 - h\sum_{j=1}^{j_N}\int_0^{2T} w_{j-1}'w_j'(2T-t)dt - h\sum_{j=1}^{j_N-2}\int_0^{2T} w_{j+1}'w_j'(2T-t)dt \\ &\quad + \int_0^{2T} w_{j_N}'w_{j_N-1}'(2j_Nh-h)(2T-t)dt. \end{aligned}$$

This implies that

$$
\begin{aligned}
I_3 &= -\frac{h}{2}\sum_{j=1}^{j_N}\int_0^{2T} w'_{j-1}w'_j(2T-t)dt - \frac{h}{2}\sum_{j=1}^{j_N-2}\int_0^{2T} w'_{j+1}w'_j(2T-t)dt \\
&\quad +\frac{1}{2}\int_0^{2T} w'_{j_N}w'_{j_N-1}(2j_Nh-h)(2T-t)dt,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
I_1 &= h\sum_{j=1}^{j_N-1}\int_0^{2T} w'_j(jh)\frac{w_{j+1}-w_{j-1}}{h}dt + \frac{h}{2}\sum_{j=1}^{j_N}\int_0^{2T} w'_{j-1}w'_j(2T-t)dt \\
&\quad +\frac{h}{2}\sum_{j=1}^{j_N-2}\int_0^{2T} w'_{j+1}w'_j(2T-t)dt - \frac{1}{2}\int_0^{2T} w'_{j_N}w'_{j_N-1}(2j_Nh-h)(2T-t)dt.
\end{aligned} \tag{5.34}
$$

Again performing some elementary changes of variables, we get

$$
I_2 = -h\sum_{j=0}^{j_N-1}\int_0^{2T}\left(\frac{w_{j+1}-w_j}{h}\right)^2(2T-t)dt + \int_0^{2T}\left(\frac{w_{j_N}-w_{j_N-1}}{h}\right)^2 j_Nh(2T-t)dt. \tag{5.35}
$$

Inserting (5.34) and (5.35) into the identity (5.33), we have obtained that

$$
\begin{aligned}
\frac{1}{2}\int_0^{2T} w'_{j_N}w'_{j_N-1}(2j_Nh-h)(2T-t)dt &= h\sum_{j=1}^{j_N-1}\int_0^{2T} w'_j(jh)\frac{w_{j+1}-w_{j-1}}{h}dt \\
&\quad +\frac{h}{2}\sum_{j=1}^{j_N}\int_0^{2T} w'_{j-1}w'_j(2T-t)dt + \frac{h}{2}\sum_{j=1}^{j_N-2}\int_0^{2T} w'_{j+1}w'_j(2T-t)dt \\
&\quad +h\sum_{j=0}^{j_N-1}\int_0^{2T}(\frac{w_{j+1}-w_j}{h})^2(2T-t)dt - \int_0^{2T}(\frac{w_{j_N}-w_{j_N-1}}{h})^2 j_Nh(2T-t)dt.
\end{aligned}
$$

We conclude by the inequality $ab \leq \frac{a^2}{2}+\frac{b^2}{2}$, by (5.19) and by the fact that $jh \leq 1$. ∎

In the same way, we have the following lemma

Lemma 5.5.7. *There exist $T>0$ and $C(T)>0$ such that the solution w_h of (5.18) verify*

$$
\int_0^{2T} w'_{j_N}w'_{j_N+1}(2T-t)(2-2j_Nh-h)dt \leq C(T)\int_0^{2T} E_h(w_h,t)dt + \frac{h}{4}\int_0^{2T}(w'_{j_N})^2(2T-t)dt.
$$

Proof: The proof is similar to the one of the previous lemma. Here, we multiply (5.18) by $(1-jh)(2T-t)\,\frac{w_{j+1}-w_{j-1}}{h}$, we add this time on $j=j_N+1,\ldots,N$ and integrate between 0 and $2T$. ∎

Remark 5.5.8. *Lemmas 5.5.6 and 5.5.7 imply that there exist $T > 0$, a constant $C(T) > 0$ and another constant $C > 0$ (independent of T) such that*

$$\int_0^{2T} (w'_{j_N+1} + w'_{j_N-1})w'_{j_N}(2T-t)dt \le C(T)\int_0^{2T} E_h(w_h, t)dt + Ch\int_0^{2T} (w'_{j_N})^2(2T-t)dt,$$

because $h \to 0$ and $x_{j_N} = j_N h \to \frac{p}{q} \in (0, 1)$.

Now, we are ready to prove the main estimate of this subsection.

Proposition 5.5.9. *There exist $T > 0$ and $C(T) > 0$ such that the solution w_h of (5.18) verifies*

$$\int_0^{T} (w'_{j_N})^2 dt \le C(T)\left(\int_0^{2T} (y'_{j_N})^2 dt + h^2\int_0^{2T} |(A_h u_h)_{j_N}(t)|^2\, dt + h^4\int_0^{2T} |(A_h u'_h)_{j_N}(t)|^2\, dt\right),$$

where y_h (respectively u_h) is solution of (5.4) (respectively (5.8)).

Proof: By (5.18), we have

$$\frac{w_{j_N+1} - 2w_{j_N} + w_{j_N-1}}{h} = \alpha y'_{j_N} - h(A_h u_h)_{j_N},$$

which implies that

$$2(w'_{j_N})^2 = (w'_{j_N+1} + w'_{j_N-1})w'_{j_N} - \alpha h y''_{j_N} w'_{j_N} + h^2 (A_h u'_h)_{j_N} w'_{j_N}.$$

We multiply this identity by $2T - t$ and integrate between 0 and $2T$ to find

$$\begin{aligned} 2\int_0^{2T} (2T-t)(w'_{j_N})^2 dt &= \int_0^{2T} (w'_{j_N+1} + w'_{j_N-1})w'_{j_N}(2T-t)dt \\ &\quad -\alpha h\int_0^{2T} y''_{j_N} w'_{j_N}(2T-t)dt + h^2\int_0^{2T} (A_h u'_h)_{j_N} w'_{j_N}(2T-t)dt. \end{aligned}$$

By Remark 5.5.8, we obtain

$$\begin{aligned} 2\int_0^{2T} (2T-t)(w'_{j_N})^2 dt &\le C(T)\int_0^{2T} E_h(w_h, t)dt + Ch\int_0^{2T} (w'_{j_N})^2(2T-t)dt \\ &\quad -\alpha h\int_0^{2T} y''_{j_N} w'_{j_N}(2T-t)dt + h^2\int_0^{2T} (A_h u'_h)_{j_N} w'_{j_N}(2T-t)dt. \end{aligned} \tag{5.36}$$

Moreover, by (5.17)

$$-\alpha h\int_0^{2T} y''_{j_N} w'_{j_N}(2T-t)dt = -\alpha h\int_0^{2T} u''_{j_N} w'_{j_N}(2T-t)dt - \alpha h\int_0^{2T} w''_{j_N} w'_{j_N}(2T-t)dt.$$

Now, observe that

$$\int_0^{2T} w''_{j_N} w'_{j_N}(2T-t)dt = \frac{1}{2}\int_0^{2T} \frac{d}{dt}\left((w'_{j_N})^2\right)(2T-t)dt = \frac{1}{2}\int_0^{2T} (w'_{j_N})^2 dt,$$

by integration by parts and because $w'_j(t=0)=0$, $\forall j$. This identity in the previous one yields

$$-\alpha h \int_0^{2T} y''_{j_N} w'_{j_N}(2T-t)dt = -\alpha h \int_0^{2T} u''_{j_N} w'_{j_N}(2T-t)dt - \frac{\alpha h}{2}\int_0^{2T} (w'_{j_N})^2 dt. \quad (5.37)$$

On the other hand, by Young's inequality, we have for all $\epsilon > 0$:

$$-\alpha h \int_0^{2T} u''_{j_N} w'_{j_N}(2T-t)dt \leq \epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt + \frac{\alpha^2 h^2 T}{2\epsilon}\int_0^{2T} (u''_{j_N})^2 dt.$$

Then (5.37) becomes

$$-\alpha h \int_0^{2T} y''_{j_N} w'_{j_N}(2T-t)dt \leq \epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt + \frac{\alpha^2 h^2 T}{2\epsilon}\int_0^{2T} (u''_{j_N})^2 dt - \frac{\alpha h}{2}\int_0^{2T} (w'_{j_N})^2 dt. \quad (5.38)$$

For the last term of the right hand side of (5.36), we still use Young's inequality and for any $\epsilon > 0$, we get

$$h^2 \int_0^{2T} (A_h u'_h)_{j_N} w'_{j_N}(2T-t)dt \leq \frac{h^4}{4\epsilon}\int_0^{2T} ((A_h u'_h)_{j_N})^2 (2T-t)dt + \epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt$$
$$\leq \frac{h^4 T}{2\epsilon}\int_0^{2T} ((A_h u'_h)_{j_N})^2 dt + \epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt. \quad (5.39)$$

Inserting (5.38) and (5.39) into the estimate (5.36), we have obtained that

$$2\int_0^{2T} (2T-t)(w'_{j_N})^2 dt \leq C(T)\int_0^{2T} E_h(w_h, t)dt + Ch\int_0^{2T} (w'_{j_N})^2 (2T-t)dt$$
$$+\epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt + \frac{\alpha^2 h^2 T}{2\epsilon}\int_0^{2T} (u''_{j_N})^2 dt - \frac{\alpha h}{2}\int_0^{2T} (w'_{j_N})^2 dt$$
$$+\frac{h^4 T}{2\epsilon}\int_0^{2T} ((A_h u'_h)_{j_N})^2 dt + \epsilon \int_0^{2T} (w'_{j_N})^2 (2T-t)dt,$$

which implies that

$$\int_0^{2T} (2T-t)(2-Ch-2\epsilon)(w'_{j_N})^2 dt \leq C(T)\int_0^{2T} E_h(w_h, t)dt + \frac{\alpha^2 h^2 T}{2\epsilon}\int_0^{2T} (u''_{j_N})^2 dt$$
$$+\frac{h^4 T}{2\epsilon}\int_0^{2T} ((A_h u'_h)_{j_N})^2 dt.$$

Taking $\epsilon = \frac{1}{4}$ and h small enough such that $Ch \leq 1$, we arrive at

$$\int_0^{2T} (w'_{j_N})^2(2T-t)dt \leq C(T)\int_0^{2T} E_h(w_h, t)dt + 4\alpha^2h^2T\int_0^{2T}(u''_{j_N})^2dt + 4h^4T\int_0^{2T}((A_hu'_h)_{j_N})^2dt. \tag{5.40}$$

Now, by (5.20), we have

$$E_h(w_h, t) = E_h(w_h, t) - E_h(w_h, 0) = -\int_0^t w'_{j_N}(\alpha y'_{j_N} - h(A_hu_h)_{j_N})ds,$$

which implies by Young's inequality, for all $\epsilon > 0$ and $t \leq 2T$,

$$E_h(w_h, t) \leq (1+\alpha)\epsilon\int_0^t (w'_{j_N})^2ds + \frac{\alpha}{4\epsilon}\int_0^{2T}(y'_{j_N})^2ds + \frac{h^2}{4\epsilon}\int_0^{2T}(A_hu_h)^2_{j_N}ds.$$

Fubini's theorem yields

$$\int_0^{2T}\int_0^t (w'_{j_N}(s))^2dsdt = \int_0^{2T}\int_s^{2T}(w'_{j_N}(s))^2dtds = \int_0^{2T}(w'_{j_N}(s))^2(2T-s)ds.$$

Thus,

$$\begin{aligned}\int_0^{2T} E_h(w_h, t)dt \;&\leq\; (1+\alpha)\epsilon\int_0^{2T}(w'_{j_N})^2(2T-t)dt + \frac{\alpha T}{2\epsilon}\int_0^{2T}(y'_{j_N})^2dt \\ &\quad + \frac{h^2T}{2\epsilon}\int_0^{2T}(A_hu_h)^2_{j_N}dt.\end{aligned} \tag{5.41}$$

Inserting (5.41) into (5.40) and taking ϵ small enough, we obtain

$$\begin{aligned}\int_0^{2T}(w'_{j_N})^2(2T-t)dt \;&\leq\; C(T)[\int_0^{2T}(y'_{j_N})^2dt + h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt \\ &\quad + h^2\int_0^{2T}(u''_{j_N})^2dt + h^4\int_0^{2T}((A_hu'_h)_{j_N})^2dt] \\ &= C(T)\left(\int_0^{2T}(y'_{j_N})^2dt + 2h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt + h^4\int_0^{2T}((A_hu'_h)_{j_N})^2dt\right).\end{aligned}$$

As

$$\int_0^{2T}(w'_{j_N})^2(2T-t)dt \geq \int_0^T (w'_{j_N})^2(2T-t)dt \geq T\int_0^T(w'_{j_N})^2dt,$$

we have proved the requested estimate. ∎

5.5.3 "Quasi" exponential decay

We are ready to state the main result of this chapter

Theorem 5.5.10. *Under the assumptions of Proposition 5.5.5, there exist $T > 0$, $K > 0$, $\omega > 0$ and $\beta \in (0, 1)$ such that any solution of (5.4) with initial data in the class $C_h(\gamma)$ verifies, for all $t > 0$,*

$$E_h(y_h, t) \le Ke^{-\omega t}E_h(y_h, 0)+\sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}ds + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}ds\right), \tag{5.42}$$

where n is such that $2(n-1)T < t \le 2nT$ and u_h is solution of the system (5.8).

Proof: Lemma 5.2.1 implies that

$$E_h(0) - E_h(2T) = \alpha\int_0^{2T}(y_{j_N}')^2dt. \tag{5.43}$$

Writing, like in the previous subsection, $y_h = u_h + w_h$, yields

$$\left|u_{j_N}'\right|^2 \le 2\left|y_{j_N}'\right|^2 + 2\left|w_{j_N}'\right|^2.$$

By Proposition 5.5.5, for filtered solution of (5.8), we have

$$(T-2)E_{j_N}(u_h, 0) \le C\int_0^T\left|u_{j_N}'(t)\right|^2dt. \tag{5.44}$$

By the definition (5.6) of E_h and (5.11) of E_{j_N}, we have

$$E_h(y_h, 0) = E_h(u_h, 0) \le E_{j_N}(u_h, 0).$$

Therefore

$$(T-2)E_h(y_h, 0) \le C\int_0^T(\left|y_{j_N}'(t)\right|^2 + \left|w_{j_N}'(t)\right|^2)dt.$$

Then, using Proposition 5.5.9, we obtain

$$(T-2)E_h(y_h, 0) \le C(T)\left(\int_0^{2T}(y_{j_N}')^2dt + h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt + h^4\int_0^{2T}((A_hu_h')_{j_N})^2dt\right).$$

So, for $T > 2$, and since the energy E_h is decreasing, we have

$$E_h(y_h, 2T) \le E_h(y_h, 0) \le C\int_0^{2T}(y_{j_N}')^2dt+C\left(h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt + h^4\int_0^{2T}((A_hu_h')_{j_N})^2dt\right).$$

The identity (5.43) yields

$$(1+C)E_h(y_h, 2T) \le CE_h(y_h, 0) + C\left(h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt + h^4\int_0^{2T}((A_hu_h')_{j_N})^2dt\right),$$

and therefore

$$E_h(y_h,\,2T) \le \frac{C}{1+C} E_h(y_h,\,0) + \frac{C}{1+C}\left(h^2\int_0^{2T}(A_hu_h)^2_{j_N}dt + h^4\int_0^{2T}((A_hu_h')_{j_N})^2dt\right).$$

We set $\beta = \frac{C}{1+C} < 1$. Since the system (5.4) is invariant by translation, by iteration, we find for all $n \in \mathbb{N}$,

$$E_h(y_h,\,2nT) \le \beta^n E_h(y_h,\,0) + \sum_{i=0}^{n-1}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}dt\right).$$

Setting $\omega = \frac{1}{T}\ln(\frac{1}{\beta})$, we have obtained

$$E_h(y_h,\,2nT) \le e^{-\omega nT} E_h(y_h,\,0) + \sum_{i=0}^{n-1}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}dt\right).$$

For arbitary $t > 0$, there exists $n \in \mathbb{N}$ such that $2(n-1)T < t \le 2nT$. By the decreasing property of E_h, we conclude that

$$\begin{aligned}
E_h(y_h,\,t) &\le E_h(y_h,\,2(n-1)T)\\
&\le e^{-\omega(n-1)T}E_h(y_h,\,0) + \sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}ds + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}ds\right)\\
&\le \frac{1}{\beta}e^{-\omega nT}E_h(y_h,\,0) + \sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}ds + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}ds\right),
\end{aligned}$$

which finishes the proof. ∎

In fact, we have the equivalence between the "quasi" exponential decay and the observability inequality :

Proposition 5.5.11. *Let the assumptions of Proposition 5.5.5 be satisfied, then the following assertions are equivalent*

(i) There exist $T > 0$, $K > 0$, $\omega > 0$ and $\beta \in (0,\,1)$ such that every solution of (5.4) with initial data in the class $C_h(\gamma)$ verifies, for all $t > 0$

$$E_h(y_h,\,t) \le Ke^{-\omega t}E_h(y_h,\,0) + \sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)^2_{j_N}ds + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')^2_{j_N}ds\right),$$

where n is such that $2(n-1)T < t \le 2nT$ and u_h is solution of the system (5.8).

(ii) There exist positive constants T_0 and C_0 such that every solution u_h of (5.8) verifies

$$E_{j_N}(u_h,\,0) \le C_0\int_0^{T_0}\left|u_{j_N}'\right|^2dt + C_0\left(h^2\int_0^{T_0}(A_hu_h)^2_{j_N}dt + h^4\int_0^{T_0}((A_hu_h')_{j_N})^2dt\right).$$

Proof: The implication (ii)$\Rightarrow$(i) is due to the proof of Theorem 5.5.10 just by adding the term $C_0(h^2\int_0^{2T}(A_hu_h)_{j_N}^2dt + h^4\int_0^{2T}((A_hu_h')_{j_N})^2dt)$ in the right hand side of (5.44).

The proof of the other implication (i)$\Rightarrow$(ii) follows the proof of Lemma 5.4.1. For T_0 large enough ($T_0 \geq \ln(\frac{4M}{3})/\omega$), by (i), we have

$$E_h(T_0) \leq \frac{3}{4}E_h(0) + \sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)_{j_N}^2dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')_{j_N}^2dt\right),$$

where $2(n-1)T < T_0 \leq 2nT$, and then, by (5.15), we have

$$E_h(u_h, 0) = E_h(0) \leq 4\alpha\int_0^{T_0}(y_{j_N}')^2dt + 4\sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)_{j_N}^2dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')_{j_N}^2dt\right).$$

Then, (5.17) and (5.21) imply

$$\begin{aligned} E_h(u_h, 0) \;\leq\;& C\int_0^{T_0}(u_{j_N}')^2dt + Ch^2\int_0^{T_0}(A_hu_h)_{j_N}^2dt \\ &+4\sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)_{j_N}^2dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')_{j_N}^2dt\right). \end{aligned}$$

Consequently, by (5.25), we obtain

$$\begin{aligned} E_{j_N}(u_h, 0) \;\leq\;& C\int_0^{T_0}(u_{j_N}')^2dt + Ch^2\int_0^{T_0}(A_hu_h)_{j_N}^2dt \\ &+4\sum_{i=0}^{n-2}\beta^{n-i}\left(h^2\int_{2iT}^{2(i+1)T}(A_hu_h)_{j_N}^2dt + h^4\int_{2iT}^{2(i+1)T}(A_hu_h')_{j_N}^2dt\right) \\ \leq\;& C\int_0^{T_0}(u_{j_N}')^2dt + Ch^2\int_0^{T_0}(A_hu_h)_{j_N}^2dt \\ &+\frac{4}{1-\beta}\left(h^2\int_0^{T_0}(A_hu_h)_{j_N}^2dt + h^4\int_0^{T_0}((A_hu_h')_{j_N})^2dt\right), \end{aligned}$$

because $2(i+1)T \leq 2(n-1)T < T_0$, which finish the proof. ∎

We have seen in Theorem 5.4.3 that without filter, the energy does not decay uniformly exponentially. Therefore, a natural question is : without filter, do we have a uniform "quasi" exponential decay of the energy E_h? The following proposition gives a negative result, which shows the advantage of the filtering technique.

Proposition 5.5.12. *Assume that N is a multiple of q and then $j_N = N\frac{p}{q}$. Then for all $T > 0$ there exist a positive constant $C(T)$ and initial data such that the solution u_h of (5.8) with these initial data satisfies*

$$E_{j_N}(u_h, 0) \geq \frac{C(T)}{h^2}\left(\int_0^T\left|u_{j_N}'(t)\right|^2dt + h^2\int_0^T\left|(A_hu_h)_{j_N}\right|^2dt + h^4\int_0^{T_0}((A_hu_h')_{j_N})^2dt\right).$$

Therefore the decay of E_h is not uniformly quasi exponential with respect to h by Proposition 5.5.11.

Proof: We follow the proof of Lemma 5.4.2. We assume that N is a multiple of q and, thus, we can easily prove that j_N given in (5.30) verify $j_N = N\frac{p}{q}$. Choose

$$y_j^{(0)} = \varphi_j^{N-q,h}, \quad y_j^{(1)} = 0.$$

It is easy to check that

$$u_h = \cos(\sqrt{\lambda_{N-q,h}}t)\varphi^{N-q,h}$$

solves (5.8). We have seen in Lemma 5.4.2 that

$$E_{j_N}(u_h, 0) = \frac{2-h}{8}\lambda_{N-q,h},$$

and there exists $C > 0$ such that

$$\int_0^T \left|u'_{j_N}(t)\right|^2 dt \le C\lambda_{N-q,h}T(\pi p)^2\frac{h^2}{4},$$

and

$$h^2\int_0^T (|(A_h u_h)_{j_N}|^2\, dt \le Ch^2\lambda_{N-q,h}T.$$

Moreover

$$(A_h u'_h)_{j_N}(t) = \lambda_{N-q,h}^{3/2}\varphi_{j_N}^{k,h}(-\sin(\sqrt{\lambda_{k,h}}t)).$$

Thus

$$h^4\int_0^T ((A_h u'_h)_{j_N})^2 dt \le h^4\lambda_{N-q,h}^3\left|\varphi_{j_N}^{k,h}\right|^2 T \le C\lambda_{N-q,h}h^2T,$$

by (5.27) and (5.28).

Therefore, there exists $C(T) > 0$ such that for h small enough,

$$\begin{aligned}\int_0^T \left|u'_{j_N}(t)\right|^2 dt + h^2\int_0^T |(A_h u_h)_{j_N}|^2\, dt + h^4\int_0^T ((A_h u'_h)_{j_N})^2 dt &\le C(T)\lambda_{N-q,h}h^2\\ &\le C(T)h^2E_{j_N}(u_h, 0).\end{aligned}$$

■

Remark 5.5.13. *We can easily verify that (5.42) implies that there exist $T > 0$, $K > 0$, $\omega > 0$ and $C > 0$ such that every solution of (5.4) with initial data in the class $C_h(\gamma)$ verifies, for all $t > 0$*

$$E_h(y_h, t) \le Ke^{-\omega t}E_h(y_h, 0) + C(h\tilde{E}_h(u_h, 0) + h^3\tilde{F}_h(u_h, 0)), \tag{5.45}$$

where $\tilde{E}_h$ is a constant energy for the problem without damping (5.8) defined by

$$\tilde{E}_h(u_h, t) = \frac{h}{2}\sum_{j=0}^{N}\left|\frac{u'_{j+1}(t) - u'_j(t)}{h}\right|^2 + \frac{h}{2}\sum_{j=1}^{N}|(A_h u_h)_j(t)|^2, \tag{5.46}$$

and $\tilde{F}_h$ is given by

$$\tilde{F}_h(u_h, t) = \frac{h}{2}\sum_{j=0}^{N}\left|\frac{u''_{j+1}(t) - u''_j(t)}{h}\right|^2 + \frac{h}{2}\sum_{j=1}^{N}|(A_h u'_h)_j(t)|^2$$

which is also constant and which is obtained by substituting u_h by u'_h in the energy $\tilde{E}_h$ (we observe that u'_h is a solution of (5.8) because (5.8) is linear).

In the general situation, the right hand side of (5.45) behaves as a non increasing exponential function for t between 0 and $-\frac{\ln h}{\omega} + c$ and is constant (proportional to h) for t large, see Figure 5.1. But for smooth enough initial data $y^{(0)}$ and $y^{(1)}$, and choosing appropriately $y_h^{(0)}$ and $y_h^{(1)}$, we have

$$\tilde{E}_h(u_h, 0) \to \tilde{E}(u, 0) \text{ and } \tilde{F}_h(u_h, 0) \to \tilde{F}(u, 0), \text{ when } h \to 0.$$

Therefore, the two last terms of the right hand side of (5.45) tends to zero as $h \to 0$, which means that the estimate (5.45) tends to the exponential stability estimate

$$E(y, t) \leq Ke^{-\omega t}E(y, 0).$$

In that sense our estimate (5.45) is quasi optimal.

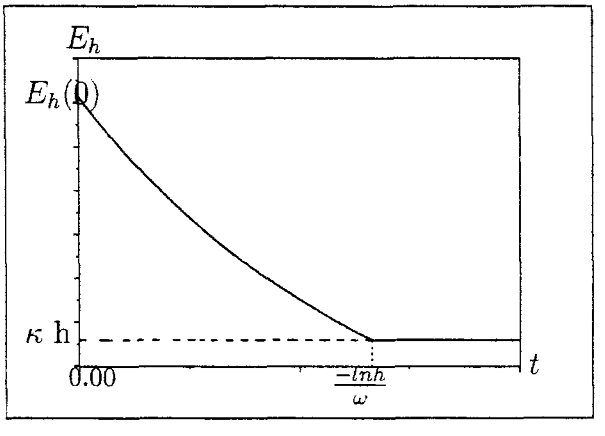

FIG. 5.1 – Behaviour of the right hand side of (5.45)

In order to illustrate our results from Theorem 5.5.10 and Proposition 5.5.12 (or Theorem 5.4.3), let us present the following numerical results. We consider the semi-discrete problem (5.4) with $N = 2^6$ and appropriate j_N such that $j_N h$ approaches $\xi = 1/2$. As initial data, we take either $y^{(0)} = \varphi^{N-2,h}$, $y^{(1)} = 0$ or $y^{(0)} = \varphi^{1,h}$, $y^{(1)} = 0$; corresponding to high or low frequency data respectively. The logarithms of the discrete energies are presented in Figure 5.2. The top curve of the figure is the logarithm of the energy of the high frequency datum, while the bottom curve of the figure is the logarithm of the energy of the low frequency datum. Hence we see that without filtering the decay of the energy is not exponential (actually the energy decays very slowly from $e^{8.3386}$ to $e^{8.3358}$). For the low frequency datum a quasi-exponential decay is detected (because the curve of the logarithm of the energy is mainly a line up to $t \simeq 15$ and mainly constant afterwards). These two tests confirm the advantage of the filtering technique.

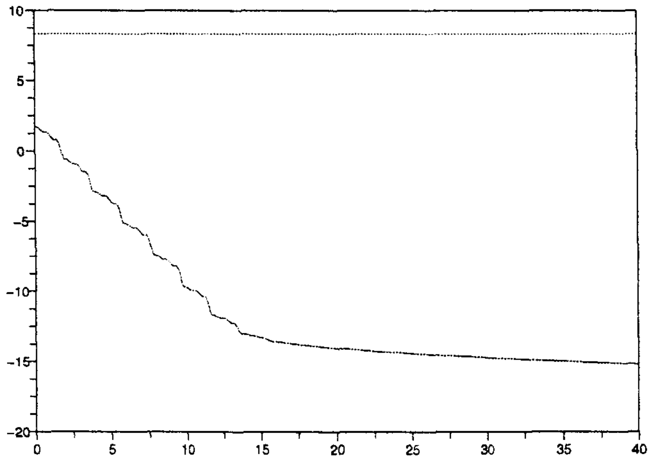

FIG. 5.2 – The energy for a low (bottom curve) and high (top curve) frequency data

Conclusion

Dans cette thèse nous avons étudié la stabilisation de quelques équations d'évolution par rétro-action (feedback) ; plus particulièrement notre attention s'est portée sur les équations des ondes, de la chaleur et des poutres. Nous nous sommes intéressés à trois axes principaux.

Tout d'abord, nous avons considéré la stabilisation de l'équation des ondes sur des réseaux 1-d par des feedbacks situés aux noeuds en mettant en oeuvre deux méthodes différentes. Dans le premier chapitre, nous nous sommes ramenés à l'étude d'une inégalité d'observabilité pour le problème conservatif, où l'on impose la condition de Neumann à la place des conditions de dissipation, et nous avons donné des conditions spectrales fournissant la décroissance exponentielle ou polynomiale.

Dans le second chapitre, nous avons transféré des inégalités d'observabilité à poids, déjà existantes pour le problème conservatif (où la condition de dissipation est remplacée par la condition de Dirichlet), en inégalités d'observabilité faibles pour le système dissipé. Grâce à une inégalité d'interpolation, nous avons obtenu des taux de décroissance explicites qui dépendent des propriétés géométriques et topologiques du réseau.

Le second axe d'étude est la stabilisation de systèmes avec un terme de retard dans les feedbacks.

En supposant que le poids du feedback avec retard est plus petit que celui sans retard, nous avons donné des conditions pour obtenir la stabilité forte, exponentielle ou polynomiale de l'équation des ondes sur des réseaux 1-d.

Nous avons ensuite développé une théorie abstraite pour les équations d'évolution du second ordre généralisant ces résultats.

Enfin nous avons étudié le cas où le retard dépend du temps pour les équations des ondes et de la chaleur. En émettant certaines hypothèses sur ce retard $\tau(t)$ (en particulier la dérivée de τ doit être majorée par 1 et τ borné) et en utilisant une fonctionnelle de Lyapounov appropriée, nous avons prouvé que l'énergie est exponentiellement décroissante et nous avons donné explicitement son taux de décroissance.

Le dernier axe de notre étude porte sur l'aspect numérique. Nous avons montré qu'une technique de filtrage permet d'obtenir une décroissance quasi-exponentielle de l'équation des ondes discrétisée en espace par différences finies avec un amortissement interne. Sans ce filtrage, le taux de décroissance ne serait pas uniforme par rapport au pas de discrétisation.

Une perspective de recherche intéressante porterait sur l'étude de la stabilité des équations d'évolution abstraites du second ordre avec des feedbacks avec retard dépendant du temps. Dans un tel cas, comme le système n'est pas invariant par translation, la méthode du chapitre 3 n'est plus utilisable et une fonctionnelle de Lyapounov appropriée, comme dans le chapitre 4, pourrait répondre au problème.

Nous pourrons étendre la stabilisation de l'équation des ondes dans un domaine de $\mathbb{R}^n$ avec un terme de retard dépendant du temps sur une partie du bord au cas non linéaire.

Nous pourrons également effectuer l'étude de la stabilisation de l'approximation par éléments finis des équations d'évolution du second ordre pour des feedbacks non bornés en nous basant sur [96].

Bibliographie

[1] C. Abdallah, P. Dorato, J. Benitez-Read, and R. Byrne. Delayed positive feedback can stabilize oscillatory systems. In *ACC' 1993*, pages 3106–3107. San Francisco, 1993.

[2] F. Ali Mehmeti. A characterization of a generalized C^∞-notion on nets. *Integral Equations Operator Theory*, 9(6) :753–766, 1986.

[3] F. Ali Mehmeti. *Nonlinear waves in networks*, volume 80 of *Mathematical Research*. Akademie-Verlag, Berlin, 1994.

[4] F. Ali Mehmeti, J. von Below, and S. Nicaise, editors. *Partial differential equations on multistructures*, volume 219 of *Lecture Notes in Pure and Applied Mathematics*, New York, 2001. Marcel Dekker Inc.

[5] K. Ammari, E. M. Ait Ben Hassi, S. Boulite, and L. Maniar. Feedback stabilization of a class of evolution equations with delay. Technical report. submitted.

[6] K. Ammari, A. Henrot, and M. Tucsnak. Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string. *Asymptot. Anal.*, 28(3-4) :215–240, 2001.

[7] K. Ammari and M. Jellouli. Stabilization of star-shaped networks of strings. *Differential Integral Equations*, 17(11-12) :1395–1410, 2004.

[8] K. Ammari and M. Jellouli. Remark on stabilization of tree-shaped networks of strings. *Appl. Math.*, 52(4) :327–343, 2007.

[9] K. Ammari, M. Jellouli, and M. Khenissi. Stabilization of generic trees of strings. *J. Dyn. Control Syst.*, 11(2) :177–193, 2005.

[10] K. Ammari and M. Tucsnak. Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force. *SIAM J. Control Optim.*, 39(4) :1160–1181, 2000.

[11] K. Ammari and M. Tucsnak. Stabilization of second order evolution equations by a class of unbounded feedbacks. *ESAIM Control Optim. Calc. Var.*, 6 :361–386, 2001.

[12] W. Arendt and C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.*, 305(2) :837–852, 1988.

[13] C. Baiocchi, V. Komornik, and P. Loreti. Ingham-Beurling type theorems with weakened gap conditions. *Acta Math. Hungar.*, 97 :55–95, 2002.

[14] H. T. Banks, K. Ito, and C. Wang. Exponentially stable approximations of weakly damped wave equations. In *Estimation and control of distributed parameter systems (Vorau, 1990)*, volume 100 of *Internat. Ser. Numer. Math.*, pages 1–33. Birkhäuser, Basel, 1991.

[15] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5) :1024–1065, 1992.

[16] A. Bátkai. Second order cauchy problems with damping delay, 2000. Thesis, Tübingen.

[17] A. Bátkai and B. Farkas. On the effect of small delays to the stability of feedback systems. In *Evolution equations : applications to physics, industry, life sciences and economics (Levico Terme, 2000)*, volume 55 of *Progr. Nonlinear Differential Equations Appl.*, pages 83–94. Birkhäuser, Basel, 2003.

[18] A. Bátkai and S. Piazzera. Damped wave equations with delay. In *Topics in functional differential and difference equations (Lisbon, 1999)*, volume 29 of *Fields Inst. Commun.*, pages 51–61. Amer. Math. Soc., Providence, RI, 2001.

[19] P. Bégout and F. Soria. An interpolation inequality and its application to the stabilization of damped equations. *J. Differ. Equations*, 240(2) :324–356, 2007.

[20] J. von Below. A characteristic equation associated to an eigenvalue problem on c^2-networks. *Linear Algebra Appl.*, 71 :309–325, 1985.

[21] J. von Below. Classical solvability of linear parabolic equations on networks. *J. Differential Equations*, 72(2) :316–337, 1988.

[22] J. von Below. Sturm-Liouville eigenvalue problems on networks. *Math. Methods Appl. Sci.*, 10(4) :383–395, 1988.

[23] J. von Below. Parabolic network equations, 1993. Habilitation Thesis, Eberhard-Karls-Universität Tübingen.

[24] U. Brauer and G. Leugering. On boundary observability estimates for semi-discretizations of a dynamic network of elastic strings. *Control Cybernet.*, 28(3) :421–447, 1999. Recent advances in control of PDEs.

[25] N. Burq. Contrôlabilité exacte des ondes dans des ouverts peu réguliers. *Asymptot. Anal.*, 14(2) :157–191, 1997.

[26] N. Burq and M. Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2) :443–471, 2004.

[27] T. Caraballo, J. Real, and L. Shaikhet. Method of Lyapunov functionals construction in stability of delay evolution equations. *J. Math. Anal. Appl.*, 334(2) :1130–1145, 2007.

[28] T. Carleman. Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat., Astr. Fys.*, 26(17) :9, 1939.

[29] J. W. Cassals. *An Introduction to Diophantine Approximation.* Cambridge Univ. Press, Cambridge, UK, 1966.

[30] C. Castro and S. Micu. Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method. *Numer. Math.*, 102(3) :413–462, 2006.

[31] C. Castro, S. Micu, and A. Münch. Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square. *IMA J. Numer. Anal.*, 28(1) :186–214, 2008.

[32] T. Cazenave and A. Haraux. *Introduction aux problèmes d'évolution semi-linéaires*, volume 1 of *Mathématiques & Applications (Paris) [Mathematics and Applications]*. Ellipses, Paris, 1990.

[33] R. Dáger. Observation and control of vibrations in tree-shaped networks of strings. *SIAM J. Control Optim.*, 43(2) :590–623, 2004.

[34] R. Dáger and E. Zuazua. Controllability of star-shaped networks of strings. In *Mathematical and numerical aspects of wave propagation (Santiago de Compostela, 2000)*, pages 1006–1010. SIAM, Philadelphia, PA, 2000.

[35] R. Dáger and E. Zuazua. Controllability of star-shaped networks of strings. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(7) :621–626, 2001.

[36] R. Dáger and E. Zuazua. Controllability of tree-shaped networks of vibrating strings. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(12) :1087–1092, 2001.

[37] R. Dáger and E. Zuazua. Spectral boundary controllability of networks of strings. *C. R. Math. Acad. Sci. Paris*, 334(7) :545–550, 2002.

[38] R. Dáger and E. Zuazua. *Wave propagation, observation and control in 1-d flexible multi-structures*, volume 50 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

[39] R. Datko. Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM J. Control Optim.*, 26(3) :697–713, 1988.

[40] R. Datko. Two examples of ill-posedness with respect to time delays revisited. *IEEE Trans. Automat. Control*, 42(4) :511–515, 1997.

[41] R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.*, 24(1) :152–156, 1986.

[42] R. Datko, J. Lagnese, and M. P. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.*, 24(1) :152–156, 1986.

[43] H. O. Fattorini. *Infinite-dimensional optimization and control theory*, volume 62 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[44] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations : the linear case. *Adv. Differential Equations*, 5(4-6) :465–514, 2000.

[45] E. Fridman and Y. Orlov. On stability of linear parabolic distributed parameter systems with time-varying delays. CDC 2007, December 2007, New Orleans.

[46] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[47] R. Glowinski. Ensuring well-posedness by analogy : Stokes problem and boundary control for the wave equation. *J. Comput. Phys.*, 103(2) :189–221, 1992.

[48] R. Glowinski, W. Kinton, and M. F. Wheeler. A mixed finite element formulation for the boundary controllability of the wave equation. *Internat. J. Numer. Methods Engrg.*, 27(3) :623–635, 1989.

[49] R. Glowinski, C. H. Li, and J.-L. Lions. A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls : description of the numerical methods. *Japan J. Appl. Math.*, 7(1) :1–76, 1990.

[50] R. Glowinski and J.-L. Lions. Exact and approximate controllability for distributed parameter systems. In *Acta numerica, 1995*, Acta Numer., pages 159–333. Cambridge Univ. Press, Cambridge, 1995.

[51] K. P. Hadeler. Delay equations in biology. In *Functional differential equations and approximation of fixed points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978)*, volume 730 of *Lecture Notes in Math.*, pages 136–156. Springer, Berlin, 1979.

[52] J. Hale and S. Verduyn Lunel. *Introduction to functional differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[53] A. Haraux. Quelques propriétés des séries lacunaires utiles dans l'étude des vibrations élastiques. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XII (Paris, 1991–1993)*, volume 302 of *Pitman Res. Notes Math. Ser.*, pages 113–124. Longman Sci. Tech., Harlow, 1994.

[54] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. The Clarendon Press Oxford University Press, New York, fifth edition, 1979.

[55] C. Huang and S. Vandewalle. An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays. *SIAM J. Sci. Comput.*, 25(5) :1608–1632, 2004.

[56] J. A. Infante and E. Zuazua. Boundary observability for the space semi-discretizations of the one-dimensional wave equation. *M2AN*, 33 :407–438, 1999.

[57] A. E. Ingham. Some trigonometrical inequalities with applications to the theory of series. *Math. Z.*, 41(1) :367–379, 1936.

[58] E. Isaacson and H. B. Keller. *Analysis of numerical methods*. John Wiley & Sons Inc., New York, 1966.

[59] T. Kato. Linear and quasilinear equations of evolution of hyperbolic type. *C.I.E.M.*, II ciclo :125–191, 1976.

[60] T. Kato. *Abstract differential equations and nonlinear mixed problems.* Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, 1985.

[61] V. Komornik. *Exact controllability and stabilization.* RAM : Research in Applied Mathematics. Masson, Paris, 1994. The multiplier method.

[62] V. Komornik and P. Loreti. *Fourier series in control theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[63] J. E. Lagnese. Recent progress and open problems in control of multi-link elastic structures. In *Optimization methods in partial differential equations (South Hadley, MA, 1996)*, volume 209 of *Contemp. Math.*, pages 161–175. Amer. Math. Soc., Providence, RI, 1997.

[64] J. E. Lagnese and G. Leugering. *Domain decomposition methods in optimal control of partial differential equations*, volume 148 of *International Series of Numerical Mathematics.* Birkhäuser Verlag, Basel, 2004.

[65] J. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. *Modeling, analysis and control of dynamic elastic multi-link structures.* Systems & Control : Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

[66] S. Lang. *Introduction to Diophantine approximations.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[67] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations : continuous and approximation theories. I*, volume 74 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2000. Abstract parabolic systems.

[68] I. Lasiecka, R. Triggiani, and P.-F. Yao. Inverse/observability estimates for second-order hyperbolic equations with variable coefficients. *J. Math. Anal. Appl.*, 235(1) :13–57, 1999.

[69] G. Lebeau and L. Robbiano. Stabilisation de l'équation des ondes par le bord. *Duke Math. J.*, 86(3) :465–491, 1997.

[70] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics].* Masson, Paris, 1988.

[71] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1) :1–68, 1988.

[72] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.

[73] K. Liu, Z. Liu, and B. Rao. Exponential stability of an abstract nondissipative linear system. *SIAM J. Control Optim.*, 40(1) :149–165, 2001.

[74] Z. Liu and S. Zheng. *Semigroups associated with dissipative systems*, volume 398 of *Chapman & Hall/CRC Research Notes in Mathematics.* Chapman & Hall/CRC, Boca Raton, FL, 1999.

[75] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM J. Control Optim.*, 34(2) :572–600, 1996.

[76] G. Lumer. Connecting of local operators and evolution equations on networks. In *Potential theory, Copenhagen 1979 (Proc. Colloq., Copenhagen, 1979)*, volume 787 of *Lecture Notes in Math.*, pages 219–234. Springer, Berlin, 1980.

[77] L. Miller. Controllability cost of conservative systems : resolvent condition and transmutation. *J. Funct. Anal.*, 218(2) :425–444, 2005.

[78] A. Münch. A uniformly controllable and implicit scheme for the 1-D wave equation. *M2AN Math. Model. Numer. Anal.*, 39(2) :377–418, 2005.

[79] M. Negreanu and E. Zuazua. A 2-grid algorithm for the 1-d wave equation. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 213–217. Springer, Berlin, 2003.

[80] M. Negreanu and E. Zuazua. Uniform boundary controllability of a discrete 1-D wave equation. *Systems Control Lett.*, 48(3-4) :261–279, 2003. Optimization and control of distributed systems.

[81] S. Nicaise. Diffusion sur les espaces ramifiés, 1986. Thèse de Doctorat, Université de Mons.

[82] S. Nicaise. Spectre des réseaux topologiques finis. *Bull. Sci. Math. (2)*, 111(4) :401–413, 1987.

[83] S. Nicaise. *Polygonal interface problems*, volume 39 of *Methoden und Verfahren der Mathematischen Physik [Methods and Procedures in Mathematical Physics]*. Verlag Peter D. Lang, Frankfurt am Main, 1993.

[84] S. Nicaise. Stability and controllability of an abstract evolution equation of hyperbolic type and concrete applications. *Rend. Mat. Appl. (7)*, 23(1) :83–116, 2003.

[85] S. Nicaise and C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.*, 45(5) :1561–1585, 2006.

[86] S. Nicaise and C. Pignotti. Stabilization of the wave equation with variable coefficients and boundary condition of memory type. *Asymptot. Anal.*, 50(1-2) :31–67, 2006.

[87] S. Nicaise and J. Valein. Quasi exponential decay of a finite difference space discretization of the 1-d wave equation by pointwise interior stabilization. Technical report. submitted to Adv. Comput. Math.

[88] S. Nicaise and J. Valein. Stabilization of second order evolution equations with unbounded feedback with delay. Technical report. submitted to ESAIM Control Optim. Calc. Var.

[89] S. Nicaise and J. Valein. Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks. *Netw. Heterog. Media*, 2(3) :425–479, 2007.

[90] S. Nicaise, J. Valein, and E. Fridman. Stability of the heat and of the wave equations with boundary time-varying delays. Technical report. submitted to Discrete Contin. Dyn. Syst.

[91] S. Nicaise and O. Zaïr. Identifiability, stability and reconstruction results of point sources by boundary measurements in heteregeneous trees. *Rev. Mat. Complut.*, 16(1) :151–178, 2003.

[92] Y. Orlov and E. Fridman. On exponential stability of linear retarded distributed parameter systems. IFAC Workshop on TDS, September 2007, Nantes.

[93] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Math. Sciences*. Springer-Verlag, New York, 1983.

[94] H. Queffélec. *Les nombres. Problèmes anciens et actuels.* Mathémathèmes. Ellipses, 2000.

[95] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. *J. Funct. Anal.*, 226(1) :193–229, 2005.

[96] K. Ramdani, T. Takahashi, and M. Tucsnak. Uniformly exponentially stable approximations for a class of second order evolution equations—application to LQR problems. *ESAIM Control Optim. Calc. Var.*, 13(3) :503–527, 2007.

[97] R. Rebarber. Exponential stability of coupled beams with dissipative joints : a frequency domain approach. *SIAM J. Control Optim.*, 33(1) :1–28, 1995.

[98] R. Rebarber and S. Townley. Robustness with respect to delays for exponential stability of distributed parameter systems. *SIAM J. Control Optim.*, 37(1) :230–244, 1999.

[99] W. Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[100] D. L. Russell. Controllability and stabilizability theory for linear partial differential equations : recent progress and open questions. *SIAM Rev.*, 20(4) :639–739, 1978.

[101] I. H. Suh and Z. Bien. Use of time delay action in the controller design. *IEEE Trans. Automat. Control.*, 25 :600–603, 1980.

[102] L. R. Tcheugoué Tébou and E. Zuazua. Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity. *Numer. Math.*, 95(3) :563–598, 2003.

[103] L. R. Tcheugoué Tébou and E. Zuazua. Uniform boundary stabilization of the finite difference space discretization of the 1-d wave equation. *Advances in Computational Mathematics*, 26 :337–365, 2007.

[104] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1978.

[105] M. Tucsnak and G. Weiss. How to get a conservative well-posed linear system out of thin air. II. Controllability and stability. *SIAM J. Control Optim.*, 42(3) :907–935, 2003.

[106] J. Valein and E. Zuazua. Stabilization of the wave equation on 1-D networks. Technical report. submitted to SIAM J. Control Optim.

[107] T. X. Wang. Stability in abstract functional-differential equations. II. Applications. *J. Math. Anal. Appl.*, 186(3) :835–861, 1994.

[108] T. X. Wang. Exponential stability and inequalities of solutions of abstract functional differential equations. *J. Math. Anal. Appl.*, 324(2) :982–991, 2006.

[109] G. Q. Xu, D. Y. Liu, and Y. Q. Liu. Abstract second order hyperbolic system and applications to controlled networks of strings. *SIAM J. Control Optim.*, 47(4) :1762–1784, 2008.

[110] G. Q. Xu, S. P. Yung, and L. K. Li. Stabilization of wave systems with input delay in the boundary control. *ESAIM Control Optim. Calc. Var.*, 12(4) :770–785, 2006.

[111] E. Zuazua. Some problems and results on the controllability of partial differential equations. In *European Congress of Mathematics, Vol. II (Budapest, 1996)*, volume 169 of *Progr. Math.*, pages 276–311. Birkhäuser, Basel, 1998.

[112] E. Zuazua. Boundary observability for the finite-difference space semi-discretizations of the 2-d wave equation in the square. *J. Math. pures et appl.*, 78 :523–563, 1999.

[113] E. Zuazua. Controllability of partial differential equations and its semi-discrete approximations. *Discrete Contin. Dyn. Syst.*, 8(2) :469–513, 2002. Current developments in partial differential equations (Temuco, 1999).

[114] E. Zuazua. Optimal and approximate control of finite-difference approximation schemes for the 1D wave equation. *Rend. Mat. Appl. (7)*, 24(2) :201–237, 2004.

[115] E. Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2) :197–243, 2005.